CONCEPTUALISING PROCEDURAL FLUENCY AS A SPECTRUM OF PROFICIENCY

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Abstract

This paper draws on our experiences of using Kilpatrick et al's (2001) strands of mathematical proficiency as a framework for analysing learner responses to an orally administered numeracy instrument. Our research revealed a range of methods leading to accurate answers in response to questions. Overlap of categorising learner methods/responses as both procedural fluency and conceptual understanding resulted especially when considering the indicators of flexibility and efficiency in relation to procedural fluency. Thus categorising all methods which resulted in accurate answers as equal indicators of procedural fluency was problematic. This discomfort resulted in our consideration of the elaboration of a procedural fluency spectrum. In this paper we elaborate on our notion of this spectrum and illustrate the spectrum through the use of examples.

Introduction

Kilpatrick, Swafford and Findell's (2001) strands of mathematical proficiency have been widely used in South African mathematics education circles (Adler, Ball, Krainer, Lin, & Novotna, 2005; Schäfer, 1999; Venkatakrishnan & Graven, 2006). The highly influential document '*Adding It Up'* is recommended reading for many post graduate mathematics education students. The power of this work is that it provides a rich and elaborated notion of mathematical proficiency – an idealised notion that as teachers we can work towards developing in our learners. This paper draws on our experiences of using Kilpatrick et al's (2001) strands of mathematical proficiency as a framework for analysing learner responses to an orally administered numeracy instrument. Converting an idealised version of mathematical proficiency into a tool that one might use in order to support ones analysis of learner data requires much work. In this paper we share with you some of the work that we have done in this respect.

When coding and analysing an oral interview, overlap of learner methods/responses as both procedural fluency and conceptual understanding arose, especially when considering the dimensions of flexibility and efficiency in relation to procedural fluency. We felt that categorising all methods leading to accuracy as equal indicators of procedural fluency would be problematic. Following a procedure or method without evidence of understanding surely cannot be considered in the same way as adopting an appropriately chosen method flexibly and efficiently. This resulted in our consideration of the development of a procedural fluency spectrum.

Our spectrum ranges from restricted / constrained procedural fluency towards elaborated and fully flexible fluency. It became evident that as one moved to the upper end of the spectrum where flexibility and efficiency were high, conceptual understanding was increasingly intertwined with procedural fluency and the distinction between these strands became progressively blurred. Indeed the inter-relationship between these strands was progressively more visible at this upper end of the spectrum. In this paper we will elaborate on our notion of a spectrum of procedural fluency. We will illustrate this spectrum of procedural fluency through the use of examples from our oral interview instrument as well as to draw on the range of learner responses in order to illuminate the need for the spectrum in assessing learners' levels of procedural fluency.

The context of our study

The aims of the SA numeracy Chair are twofold and focus on a dialectical relationship between development and research. In development terms the Chair teams are expected to improve the quality of numeracy teaching at primary level and to improve learner performance in primary schools as a result of quality teaching and learning. In research terms we are to provide leadership in growing research which looks towards finding sustainable solutions to the many challenges in numeracy education in the broader Grahamstown area and beyond. Our research community consists of full time and part time doctoral and masters students researching in the field of numeracy education under the supervision of the Chair (first author, Mellony Graven). The second author Debbie Stott is a full time doctoral fellow and intern in the Chair researching the nature of student learning within Grade 3 after-school maths clubs. Debbie is also responsible for the design and setting up of the clubs.

Our Chair works with fifteen schools in the broader Grahamstown area in the Eastern Cape. Our teacher development program is called the Numeracy Inquiry Community of Leader Educators (NICLE). 57 numeracy teachers (ranging from Grade 1 to 6) participate in fortnightly workshop sessions focused on issues and challenges in numeracy teaching. Guest speakers are a regular feature of NICLE sessions. School visits are also part of the work of the Chair. Given the many negative perceptions and poor performance of learners in numeracy in our country (Bloch, 2009; Fleisch, 2008; Taylor, Fleisch, & Shinder, 2008) we chose to supplement the NICLE program with learner focused activities.

The aim of learner focused activities within the Chair is to foreground the importance of numeracy as well as to create a 'maths is fun' ethos in schools. However, given the enormous challenges teachers face when most of their learners do not have the mathematical foundations to be learning at the Grade level at which they are, we decided that a more focused and regular learner intervention is required. The range of our Grade 3 learner performance on a USAID[5] (Brombacher, 2011) standardised numeracy instrument

[5] Developed by Research Triangle International in the United States. These have been adapted by Aarnout Brombacher for African contexts.

indicated scores of between 25.2% and 72.2%. However, only two of our schools received above 38% indicating that in all but two of our schools learners are performing, on average, below the level required for learning in their grade. Thus a supplementary approach to NICLE 'teacher development' is 'direct learner focused interventions' in the form of after-school maths clubs. The conceptualisation of such clubs have been elaborated in previous work (Graven, 2011; Graven & Stott, 2012)

Learner clubs are therefore a key aspect of our Chair work. They provide a field of influence *and* an empirical field (for research) that enables a powerful dialectical research and development cycle that involves working *directly* with learners rather than via the teachers.

From August to November 2011 we established a pilot club with ten regularly participating Grade 3 learners who met every Wednesday after school. During this time we ran 14 sessions with the club focused on strengthening learners' basic number work, fluency in addition (and number bonds to 10), place value and problem solving. Additionally we focused on developing more positive participatory socio-mathematical norms (Cobb, Wood, Yackel, & McNeal, 1992; Yackel & Cobb, 1996) and mathematical discussion and argumentation through collaborative work. While our pilot club was only operational for two terms, results from the USAID written four operations test and oral interviews showed a significantly greater increase in performance for club learners than non-club learners in their class. This increased improvement as compared to non-club learners was however only part of the picture. What was even more remarkable than the improved accuracy on assessments for club learners was the increased efficiency with which our club learners now responded to questions. We noted how many club learners were first to complete the four operations written instrument and also had moved on from concrete methods of solution in the oral interviews. Indeed 8 of the 10 club learners in their math class of 37 learners were finished the USAID four operations test before the allocated time had elapsed. Only 10 learners in the whole class had managed to finish ahead of the allocated time and so 80% of these early finishers were club learners.

While the methods learners used were captured in the oral interview instruments and could sometimes be seen on the scripts of the written instruments (e.g. if they drew dots for 23 + 18), our coding of responses in terms of accuracy (i.e. correct or incorrect) failed to reveal this increased flexibility and fluency and thus failed to capture this key aspect of their learning. This led to a redesign of instruments for our clubs as well as the need to consider coding systems that would enable tracking learner movement in terms of procedural fluency.

Instruments for gathering data on learner proficiency

In 2012 our Chair has established another five learner clubs. For these clubs we have used some of the instruments in the pilot (USAID four operations instrument) but have also introduced a new oral interview instrument which is far more comprehensive than our earlier instrument. For this instrument we drew on the influential work of Askew and his team in the *Effective Teachers Of Numeracy* study conducted in England in the nineties (Askew, Brown,

Rhodes, Johnson, & Wiliam, 1997). Thanks to their permission and providing us with the instrument we were able to select and adapt various items they used in assessing learners numeracy proficiency. However the work of Bob Wright and his colleagues in Australia also provides excellent opportunities and examples of assessment items that enable one to gauge learner progress through various stages of numeracy development. These items are available in their book '*Teaching Number: Advancing children's skills and strategies*' work (Wright, Martland, Stafford, & Stanger, 2006). We thus assembled an instrument that combined questions (sometimes adapted) from both these key works.

Additionally we designed instruments aimed at gaining insight into learners' mathematical identities and dispositions as well as instruments focused on assessing speed and accuracy of operations. It is beyond the scope of this paper to discuss all of our instruments and we thus only focus on a few questions from our orally administered instrument in order to illuminate our conceptualisation of a spectrum of one of Kilpatrick et al's (2001) strands, namely, procedural fluency.

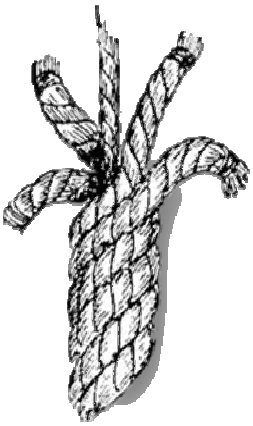
The discussion that follows emanates from our initial analysis of learner data on these interview questions and our struggles with relating questions and learner responses to Kilpatrick et al's holistic conceptualisation of the strand of procedural fluency. For our purposes of analysing student learning it is essential that our method of analysis records and reveals learner progress *over time*. Thus giving a learner a tick or a cross for procedural fluency on each question item would not be useful in terms of tracking learning and progression in these proficiencies in relation to their participation in the clubs over time.

Theoretical framing

The work of the South African Numeracy Chair at Rhodes University is to work towards improving numeracy proficiency among learners. In order to do this however one needs a clear conceptualisation of what such proficiency entails. In our Chair we base our notion of numeracy proficiency on Kilpatrick et al's (2001, p116) definition of mathematical proficiency. This definition comprises five intertwined and interrelated strands:

- Conceptual Understanding – comprehension of mathematical concepts, operations and relations
- Procedural Fluency – skill in carrying out procedures flexibly, accurately, efficiently and appropriately
- Strategic Competence – ability to formulate, represent, and solve mathematical problems
- Adaptive Reasoning – capacity for logical thought, reflection, explanation and justification
- Productive Disposition – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

They emphasise that mathematical proficiency cannot be achieved by focusing on only one or two of these strands. The metaphor and graphic they use is that of a rope. In our Chair we combined elements of Askew et al's (Askew et al., 1997) and Kilpatrick et al's (2001) definitions as follows in guiding our work:



- Simply defined: the ability to process, communicate and interpret numerical information in a variety of contexts. But overlaid with strands of numeracy proficiency:

1. Understanding numeracy concepts
2. Computing fluently (practically, mentally and procedurally)
3. Applying concepts to solve problems (in creative and inventive ways)
4. Reasoning logically (in creative and inventive ways)
5. Engaging with mathematics - seeing it as sensible, useful and doable (enjoyment, passion)

While Kilpatrick et al (2001) provide a powerful conceptualisation of mathematical proficiency the elaboration of each of the strands (with the inevitable overlaps that emerge from the interrelationships) does not support the evaluation of learner *progress* in mathematical proficiency. Indeed a fully conceptualised and idealised form of mathematical proficiency requires all five strands to be present and for each of them to be fully present in relation to their elaborated definitions. However while the fully elaborated notion of mathematical proficiency is something we work towards in our Chair we need to be able to assess different levels of proficiency in different strands across different points in time so that we can assess learner numeracy progression over time. Thus the fully embodied idealised version of a strand needs to be unpacked so that various levels of attainment might be identifiable.

Assessing learner progress in mathematical proficiency requires one to be able to assess the extent to which a learner may or may not have mastered a particular aspect of mathematics and a particular strand or interrelationship of strands at different points in time. Thus to consider that a student might be either procedurally fluent or not procedurally fluent is less useful than to gauge the extent to which they are mastering the fluency and to find ways to move them along the spectrum from what might be a restricted form of procedural fluency towards fully elaborated procedural fluency. Similarly, Adler and Venkat (2011) found the need to elaborate on a range of types of procedural teaching to allow for a more nuanced understanding of these approaches to teaching mathematics.

Thus our spectrum extended from restricted/constrained procedural fluency towards elaborated[6] and fully developed flexible fluency. It became evident that as one moved to the upper end of the spectrum where flexibility and efficiency were high, conceptual understanding was progressively intertwined with procedural fluency and the distinction between these strands became increasingly murky. The interrelationship between these strands was increasingly visible at this upper end. Indeed we believe that Kilpatrick et al's choice of conceptual understanding as the first strand to be defined was strategic, rather than random, as this enabled each subsequent strand to be defined in terms of their relationship to this strand.

In the next section we will elaborate on our notion of a spectrum of procedural fluency. While we deal with procedural fluency, in this paper, as a strand on its own, this should not imply that we see the strand as separate from other strands or that we are falling prey to the false dichotomy often set up between procedural fluency and conceptual understanding. We will illustrate the spectrum of procedural fluency through the use of examples from our oral interview instrument as well as to draw on the range of learner responses in order to illuminate the need for the spectrum in assessing learners' levels of procedural fluency as well as to provide the opportunity for tracking learner progress along the spectrum.

Analysis of learner responses – the emergence of a spectrum

Following the administration of our interviews with learners in two of our Grade 3 clubs we began to reflect on how we might now code and record, in summary form, responses that were noted on learner interview forms. Detailed notes had been taken by each interviewee (the authors and other Chair staff) and the challenge now was to find a means of capturing the richness of the range of responses in a way that would allow us to notice shifts in responses over time. Thus for a question the final answer of a student might be the same over time but the method of solution and the efficiency with which the solution is found could be greatly different.

As we looked at each question, we discovered that each would, in almost all cases be categorised as a question engaging procedural fluency. Taking it a step further, we realised that if the learner answered the question correctly, regardless of the method they used to answer it, we were tempted to say they had achieved procedural fluency. Yet, the method they used may have contradicted the notions of efficiency and fluency, even while accuracy had been achieved. This struck us as problematic.

Reflecting on the definition of procedural fluency, we could see that by categorising a question as procedural fluency and simply marking the learner response to questions as

[6] While these terms resonate with Bernstein's (1966) use of the terms in his restricted and elaborated codes we do not use them in the Bernstein sense as relates to class differences and linguistic capital. However there is some similarity in terms of what restricted and elaborated codes allow learners to access and what constrained/restricted levels of procedural fluency and fully developed and elaborated levels enable access to learning in relation to the other strands of proficiency.

correct or incorrect, meant that we were dis-regarding several key elements of the strand, namely appropriateness, flexibility and efficiency.

> *Procedural fluency refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently (Kilpatrick et al, p. 121).*

Our first realisation was that even while we had selected a range of questions for our interview relating to the range of strands of proficiency – whether a question in fact provided information about learner proficiency in a particular strand related more to *how* the learner answered the question than to the question itself. Thus a method of solution might have suggested basic procedural fluency in terms of accuracy of calculation even when the question was designed to elicit and assess adaptive reasoning.

If we look at the question example below (task 10 from the interview), we can see that learners would be able to arrive at correct answers to these questions using a variety of methods, some of which are more efficient or appropriate than others.

By way of an example, for question part 'a) add 10 to 92' we found that the learners answered the question using these possible methods:

- Less efficient ways would be to: draw the sum on paper (e.g. concrete representation of the number by drawing 10 and 92 dots and recounting), or count on 92 from 10 using fingers (or counters)
- A slightly more efficient way may be to use a standard vertical algorithm such as:

  $$\begin{array}{r} 92 \; + \\ 10 \\ \hline \\ \hline \end{array}$$

- More efficient[7] ways would be to count on 10 from 92 in ones or by using fingers
- The most efficient method would be to mentally add 10 to 90 and then add 2 (or 10 to 92 with knowledge of the pattern of adding tens) thereby involving conceptual understanding of commutativity, patterns and place value.

Similarly, it can be seen, that the less efficient methods would become even less efficient ways of answering Task 10 part b and d.

[7] We are aware that whether the standard algorithm or counting on from 10 is more efficient, depends on each individual learners skill at using these methods and thus aware of the subjectivity we have used in placing them in this list

Task 10: Adding / subtracting with tens

[Ask orally]	Note Given Answer	Correct
(a) Add 10 to 92		
(b) Add 10 to 294		
(c) Take 10 away from 50		
(d) Take 10 away from 700		

Let's look at another example. Task 9 from the interview involves the placing of successive strips of dots in each part of the question. Thus for example for part e 'add another 30 to make 74' the interviewer will place an additional three strips of ten dots onto the previous laid out strips in part e and ask 'Now how many dots altogether?'

Task 9: Counting with incrementing tens

[Use pink strip cards. Show strip (a) then add others for steps b to e. Ask] *How many dots are there altogether?*

	Note Answer & How Answered	Correct?
(a) The 'four dot' strip		
(b) Add a 'ten dot' strip to the right		
(c) Add another 10 to make 24		
(d) Add another 20 to make 44		
(e) Add another 30 to make 74		

Again, this could be categorised as a procedural fluency type question, and the learner could be said to be fluent if he/she were to answer the elements correctly. But the issue arises again as to the spectrum of efficiency and flexibility.

- Less efficient ways would be to count the dots each time a new strip is added including the 4 dot strip
- Slightly more efficient ways would be to physically count the first 10 strip in ones or twos and then count each strip as they are laid down
- Slightly more efficient again, would be to physically count the first 10 strip in ones or twos and then mentally add 10 each time a strip is laid down.
- The most efficient method is to mentally count the dots on the 10 strip and then count in groups of 10 e.g. 44 + 30 is 74 (knowing 40 and 30 is 70)

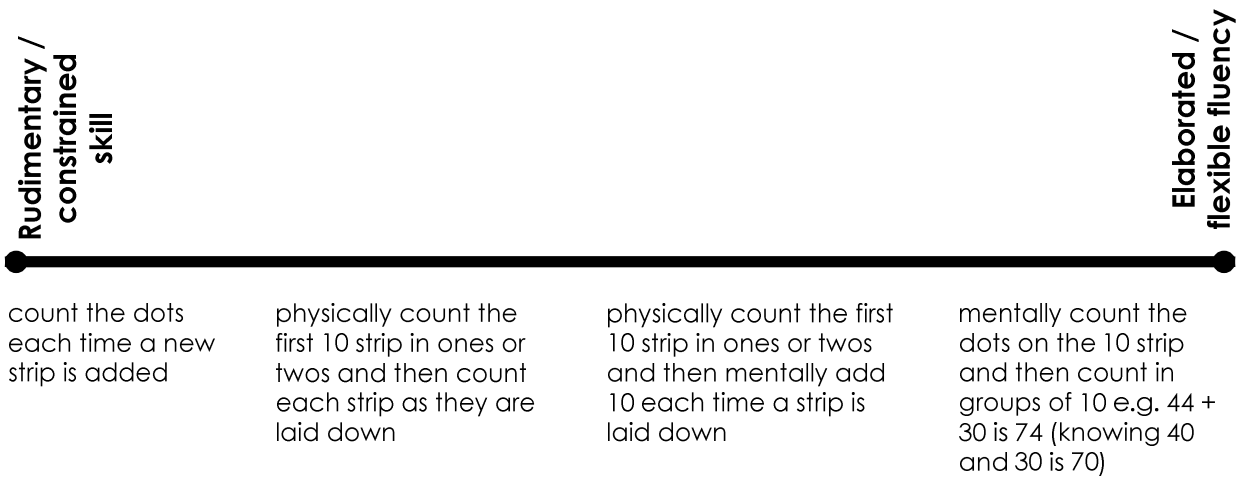
We can also see that the less efficient methods are not really appropriate for working with bigger numbers especially as we get in part e and 74 dots.

Indeed the difficulty was that some learners each time counted in ones from the beginning even in part e to get 74, accurately and correctly. To say they have no procedural fluency would do no justice to fact that they performed the skill of counting in ones accurately. On the other hand, to give it more value than simply acknowledging a basic procedural skill (of one-to-one counting) would be problematic. Thus we saw the need for a spectrum that

indicates that as one moves along the spectrum one is working with increasing levels of cognitive demand, mathematical complexity and levels of abstraction.

At the one end of the spectrum we have restricted procedural skill that could be described as rudimentary or constrained, whilst at the other end we have procedural fluency that is elaborated and fully flexible. For any given question, there may be points in between these two ends as illustrated in the examples given above.

A sample spectrum for the task 9 example above could look like this:



Using this same process, we have examined each question in the interview and have drawn up a spectrum for each question based on the range of responses we recorded from the learners.

Conclusion

Much work needs to be done if using Kilpatrick et al's (2001) notion of mathematical proficiency is to be transformed as a framework for assessing learner progression in terms of mathematical proficiency over time.

In this paper we have shown how we have adapted one of Kilpatrick et al's (2001) five strands of proficiency into a spectrum, namely, procedural fluency. This adaptation has enabled us to assess learner levels of procedural fluency so as to interpret the nature of their progress over time. Indeed we are exploring the creation of spectrums for each of the five strands and will interrogate the value of these through using the spectrums to analyse and summarise our data. We hope to share these findings in subsequent writing.

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