PATTERNS, FUNCTIONS AND ALGEBRA IN THE SOUTH AFRICAN PRIMARY MATHEMATICS CURRICULUM: TOWARDS MORE DETAIL FOR SOUTH AFRICAN TEACHERS

Nicky Roberts

University of Witwatersrand

This theoretical paper seeks to highlight the key concepts relating to the mathematics education literature on 'early algebra' and provide some additional guidelines to South African primary school teachers implementing the Curriculum and Assessment Policy Statements (CAPS) for primary school mathematics. It includes short easy-to-read guidelines, with related examples, and suggested lines of questioning, which may be useful to teachers and curriculum planners for primary mathematics in South Africa.

NOTES ON TERMINOLOGY

For the purpose of this paper the key terms are defined to mean the following:

Algebra: The study of general properties of numbers, and generalisations.

Early algebra: The teaching of arithmetic in the early grades, with a deliberate focus on generalising arithmetic, generalising towards the idea of function, and using modelling as a language of mathematics.

CAPS: Department of Education, 2010, Curriculum and Assessment Policy for Mathematics Intermediate Phase. Pretoria: Department of Education.

Input: Something that is put in or given. Inputs go into a function machine, the rule changes them; and they become outputs. In a number or shape pattern, the position of the number or shape in the pattern (1st, 2nd, 3rd…position) is the input that is used to describing the rule that generates the pattern (the outputs).

Function: A mathematical rule that changes each input to give only one output. For example '+10' is a function. If the input is 6, the only possible output is 16. But 'find a factor' is not a function. If the input is 6, the outputs can be 1, 6, 2 or 3. This is not a function.

Function machine / flow diagram: A picture with inputs and outputs to show a mathematical rule.

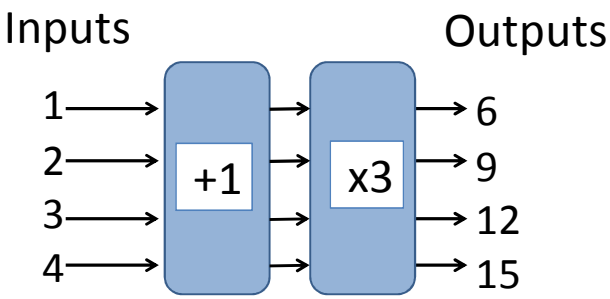


Figure 1: Function Machine

Pattern: A regular structure of shapes or numbers which is created by repeating a rule.

Output: The result of a rule acting on an input. In a shape pattern the outputs are the shapes in the pattern. In a number pattern the outputs are the numbers in the pattern.

Relation: The way in which two or more objects or variables are connected to each other. A relation is not necessarily a function.

RNCS: Department of Education, Revised National Curriculum Statement of South Africa, which preceded CAPS.

Rule: A mathematical reason for how a pattern is formed. It can be simple with only one step, like add 3, or it can have two or more steps. A rule can be written as a number sentence. It can be about numbers, for example, "input + 3 = output"; or about transformations (reflect, translate, rotate, enlarge or reduce, change colour), for example: "rotate input clockwise by a right angle = output"

MOTIVATION

The South African intended curriculum for mathematics in primary schools has included a learning outcome (or content area) focus on 'Patterns, functions and algebra' for some time. However, as most primary school teachers have not been taught this themselves while at primary school, and have frequently completed their professional training, before this was a curriculum focus, there remains a great deal of confusion regarding why this is a content area, and how to approach teaching it.

This paper sets out to be an accessible starting point for South African primary school teachers considering the theoretical realm of 'early algebra'. It provides a synopsis of the theoretical background to the mathematical literature on this content area, and some easy to read teaching guidelines how it can be better integrated in mathematics classrooms by teachers.

As the target range for this paper includes Foundation Phase teachers, the numbers used for examples are deliberately whole numbers less than 100 and multiplication and division examples deliberately reflect only doubling and halving. Intermediate Phase teachers should reflect on the ideas using the relevant number types and ranges for their Grades.

THEORETICAL BACKGROUND

Algebra in primary school mathematics curricula around the world

Consideration of algebra in the primary curriculum has had a long history in several countries. See for example: Bodanskii, (1991) in the Soviet Union, Warren, (2004) in Australia, Amerom, (2003) in the Netherlands, Davis, (1985) and a plethora of other mathematics educators in the USA including Carraher, Schlieman, Kaput, Blanton, Carpenter, and Fuson. This arose from concerns in the transition from primary to secondary school and the 'cognitive gap' between arithmetic and algebra (Herscovics & Linchevski, 1994). There was increasing evidence that the way in which arithmetic was being taught at the primary schools was not supporting the algebraic reasoning expected at secondary-school level. As a result there have been calls to introduce algebra into the primary curriculum. This 'early algebra', however, is not intended to be 'algebra early': simply a matter of replacing arithmetic with algebra or introducing algebra to primary grades (Carraher et al. 2007). Rather, 'early algebra' is an approach to teaching and learning arithmetic which focuses on the structure and patterns underlying arithmetic. As an example, the observation that 'five plus two' results in the same answer as 'two plus five', and that this

happens with all pairs of numbers reveals the commutative property of addition. So the advocates of early algebra are not expecting the primary-school learners to be aware that a + b = b + a (for all whole numbers a,b), but they are expecting familiarity of the commutative property of addition within arithmetic. As such, this notion was deliberately referred to as 'early algebra' or 'algebraic reasoning' to distinguish it from 'algebra' as understood at the secondary-school level.

The last approximately fifteen years have seen substantial mathematics educational research into precisely what is meant by 'early algebra', and a growing sophistication of its interpretation and use. Kaput and Blanton (1999) refer to "algebrafying primary mathematics". In this conception 'algebra' is considered to be generalized arithmetic. Amerom (2003) described early algebra as being when "arithmetic activities are infused with algebraic meaning, for the purpose of bringing out their algebraic character". In a recent publication, Kaput provided a detailed framework for considering the various elements of algebra by referring to two aspects and three strands:

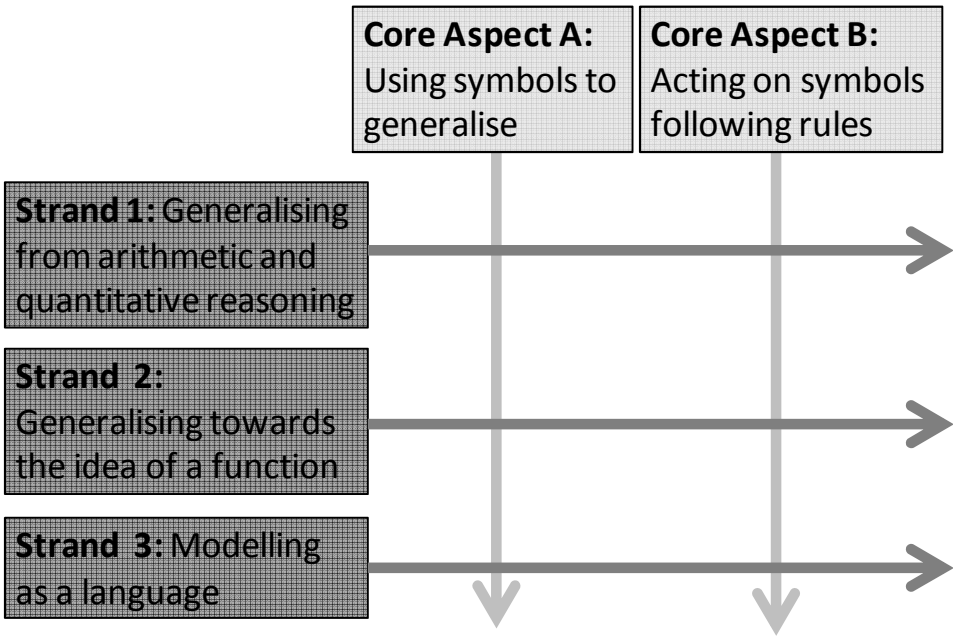


Figure 2: Kaput's Core Aspects and Strands of Algebra

The two core aspects are evident in Kaput's explanation that algebraic reasoning is taken to be "symbolisation activities that serve purposive generalisation" (or using symbols to generalise), and simultaneously to be "reasoning with symbolised generalisations" (or acting on symbols, following rules) (Kaput et al. 2007). These are represented in the above figure as *Core Aspect A,* and *Core Aspect B,* respectively. Kaput explains that it is generally assumed in mathematics curricula that *Core Aspect A* usually precedes *Core Aspect B*, where the latter has traditionally been the main focus of abstract algebra as experienced in many secondary schools. The first two strands in this categorisation of early algebra consider the two types of generalising that are at the core of algebraic thinking: generalising arithmetic; and generalising towards the idea of function. The third strand refers to modelling processes where situations are understood and interpreted using algebraic reasoning and language. Kaput identified an aspect of algebraic modelling as 'algebrafying an arithmetic problem', where the constraints of a particular problem are relaxed to explore its more general form. Kaput explains that there are overlaps between the strands and provides useful illustrative examples relevant to the early grades for each strand.

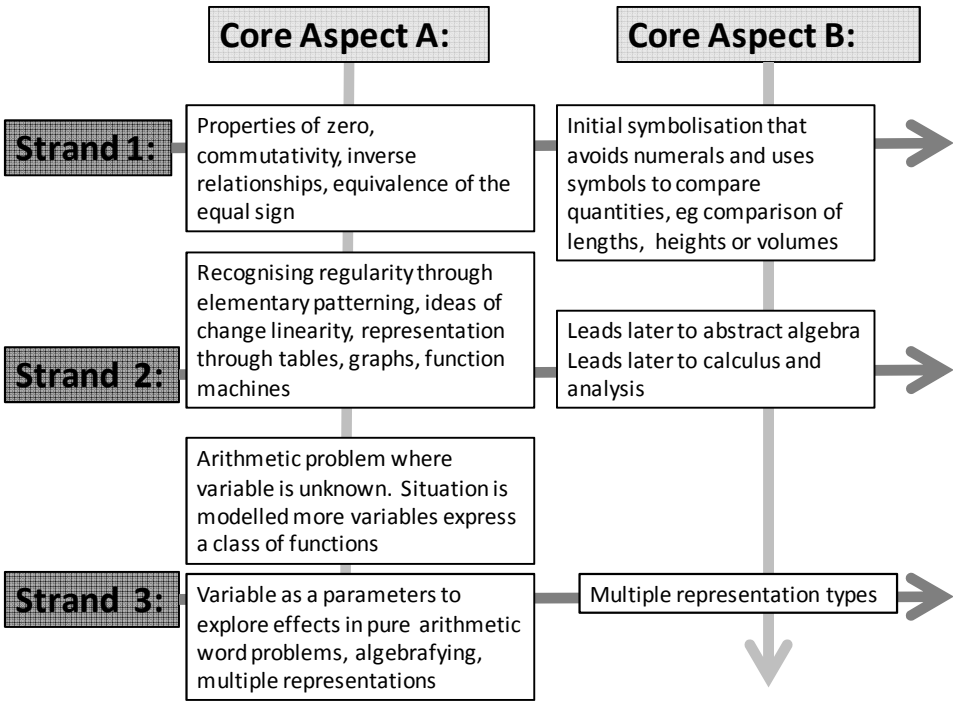


Figure 3: Kaput's early primary mathematics examples pertaining to each aspect and strand in 'early algebra'

These examples are particularly helpful in making the framework relevant to the early grades, while demonstrating the links this has with later grades in primary and secondary school.

Published research on early algebra in South Africa

There are several South African papers published in local practitioner journals and conference proceedings that focus specifically on algebra at the primary mathematics level in South Africa. Some of these papers focus on equivalence and structure of numeric and algebraic expressions (Liebenberg et al. (1999a), Liebenberg et al. (1999b), Liebenberg et al. (1998)), generalising (Linchevski et al. (1998), Samson et al (2008), Sasman et al. (1998); Sasman et al. (1999a) and Sasman et al. (1999b)), patterns (Samson (2008), and Samson (2006)) and the distributive property in arithmetic and algebra (Vermeulen et al. (1996)). Many of these papers involved MALATI researchers. It is clear that MALATI approaches had a fairly substantial influence on the revised national curriculum statements as both generalized arithmetic and generalising towards the idea of a function, as described in the MALATI publications, is clearly evident in the RNCS and CAPS content area 'patterns, functions and algebra' from grade four onwards. However all of the papers on algebra in primary schools focus on grade four and beyond. There appears to be no published research on algebra (as generalising) that focuses specifically at the Foundation Phase in South Africa at this time.

More recently Vermeulen (2007) conducted an investigation into the extent to which the South African curriculum (RNCS) supports the development of elementary algebra in South Africa. Although Vermeulen mentions that elementary algebra is usually relevant to grades seven to nine, he does include all three primary-school phases in his analysis. Vermeulen defined three aspects that he considers should be incorporated in a curriculum that promotes successful learning of elementary algebra. These are mapped to the 'strands' defined by Kaput in this figure:

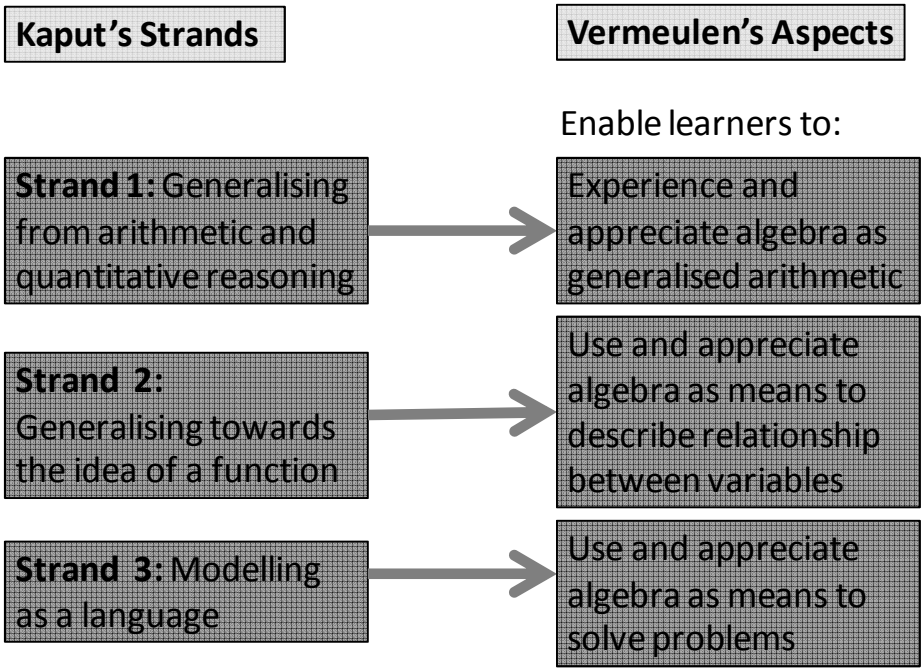


Figure 4: Mapping of Kaput's 'Strands' to Vermeulen's 'Aspects'

The author of this paper has also published a comparison of the treatment of algebra in the South African and English curricula, focusing particularly on the Foundation Phase (see Roberts (2010)), the main thesis of which is that the South African intended curriculum (Revised National Curriculum Statement) includes algebra by name, but it does not deal with it in much substance, while the English intended curriculum includes algebra in more substance, although does not mention it by name. The 2010 paper highlighted the need for more detailed guidance for South African teachers on the treatment of early algebra in the primary curriculum. This 2012 paper is intended as a contribution towards this goal.

How is the concept of 'early algebra' used in this paper?
So, for the purpose of this paper, 'early algebra' is taken to mean generalising and using modelling as language of mathematics in the early grades of primary school. This generalising has two main manifestations: generalising from arithmetic and quantitative reasoning; and generalising towards the idea of a function. As such, this paper takes 'early algebra' to include the following three key aspects in the teaching and learning of mathematics in the early grades.

Aspect one: Generalising arithmetic, which has two subsections: generalizing about properties of numbers and operations; and generalising about particular number properties and relationships.

Examples relating to generalising the exploration of properties of numbers and operations include:

- Properties of zero;
- Properties of multiples and factors: Exploring the properties of odd and even numbers is an example of exploring the properties of numbers that are multiples of two (evens) and those that are not (odds). Exploration of other patterns and relationships pertaining to other multiples, in different contexts such as, for example, in the 100 square, is another example of this type of activity.
- Commutative property: The notion that addition and multiplication can be conducted in any order. Algebraically this is $a + b = b + a$; and $a \times b = b \times a$,

- Associative property: The notion that how numbers are grouped together for addition and multiplication does not matter. Algebraically this is
  $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$ and
- Distributive property: Adding and then multiplying is equivalent to multiplying and then adding. Algebraically this is $a \times (b + c) = (a \times b) + (a \times c)$. The commutative and distributive laws are being applied when recognizing that multiplication is repeated addition (and that division is repeated subtraction). To view three times four as 'four plus four plus four' requires conceptualizing $3 \times 4 = (1 + 1 + 1) \times 4 = 4 + 4 + 4$. Considering division as sharing or as equal grouping also draws on these properties. 'Ten divided by five' is either 'how many equal groups of five make up ten?' ($10 = 5 + 5 = 2 \times 5$) or 'how can ten be shared equally between five?' ($10 = 2 + 2 + 2 + 2 + 2 = 5 \times 2$) and
- Inverse relationships: Inverse relationships refer to any inverse relationships, however at primary-school level this commonly applies to addition as the inverse of subtraction and multiplication as the inverse of division. A special case of the latter is doubling as the inverse of halving, which tends to be introduced very early on in primary-school mathematics.

Examples of generalising about particular number properties and relationships emphasise exploring equivalence of particular arithmetic expressions. Here the equal sign is used to show equivalence and not as an instruction to compute an answer. This is distinct from point one, above, as it relates to properties and relationships for particular numbers and not to properties of numbers and operations in general. Examples include:

- Expressing seven in multiple ways shows the equivalence of the equal sign: $7 = 3 + 4 = 3 + (2 \times 2) = 7 + 0 = 3 + 3 + 1 = (3\times 2) + 1 = 10 - 3$, etc. This draws on particular properties of numbers and relationships but it is not a specific exploration of these properties in general terms.
- Identifying unknown values in a simple linear equation (or 'number sentence'). Where such questions are posed as $3 + 2 = \square$, the equal sign is interpreted as an instruction to compute. However problems such as $3 + \square = 5$ or $5 = \square + 3$, draws on the equivalence of the equal sign. It is important to realise that arithmetic approaches which encourage 'partitioning' or 'breaking down' and 'building up' numbers draw on these properties of particular numbers and operations.

Aspect two: Generalising towards the idea of a function. This includes recognising regularity in elementary patterns (for example, in numbers and shapes), ideas of change including linearity, and representation through tables, graphs and 'function machines'. This is the main focus on the 'Patterns, functions and algebra' content area in CAPS at the Intermediate phase. It includes investigations of the relationship between two variables including the various representations of functions using families of number sentences, function machines, and input and output tables.

Aspect three: Modelling as a language for mathematics. A model is something that is used to illustrate or explain a mathematical idea or problem solving process. It can be a drawing or story. It can be a process that it shown or described. It can use concrete objects, representations of objects or just words. Modelling can be presented by a teacher. The teacher can also create opportunities for learners to explain their mathematical reasoning. This may make use of different means of communication (talking, writing, drawing, imagining enacting, showing) etc. This includes the potential to investigate any word problem context which is 'algebrafied' by relaxing the parameters of the problem and exploring the patterns that emerge.

GUIDELINES FOR PRIMARY SCHOOL TEACHERS FOCUSING ON EARLY ALGEBRA

This section provides some simple preliminary teaching guidelines for South African primary teachers on how to approach the teaching of early algebra in the primary mathematics curriculum. They are presented as short easy-to-read suggestions, with examples and recommendations for lines of questioning which may be used by learners. The suggestions are organised against the framework for understanding early algebra, and are relevant to all content areas, not just the 'Patterns, functions and algebra' content area as defined in CAPS.

Guidelines for generalising arithmetic

This aspect of early algebra is located mainly in the 'Numbers, operations and relationships' content area in CAPS. However this aspect is also relevant when working on Geometry, Measurement and Data Handling as these are the patterns that underlie all number work.

1. *Be deliberate about, and explore when something sometimes happens and when it always happens in mathematics.*

   Example 1: There are 4 hot dogs on the table. There are 4 legs on the dog. Are there always 4 hot dogs, or only sometimes? What about the legs on a dog? Is that sometimes or always? What always comes in a group of 4?

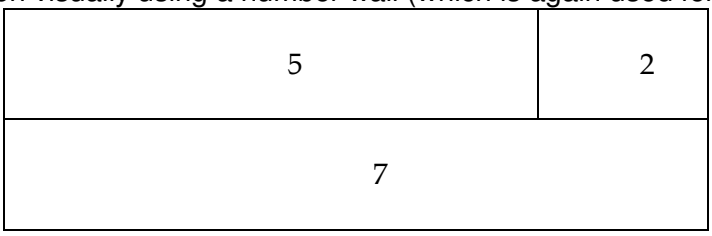
   Example 2: When you add 3 and 5, which are both odd numbers, your get 8 which is an even number. Does that always happen? Do you always get an even number when you add two odd numbers together?

   Example 3: When you draw in a diagonal of a square you get two equal triangles. Does that sometimes happen or always happen? What happens with other quadrilaterals? What happens when you draw in all the diagonals?

2. *Look for patterns in groups of number sentences*

   Example 1: Families showing the relationship between + and –, for example:
   5 + 2 = 7
   2 + 5 = 7
   7 – 2 = 5
   7 – 5 = 2

   This can be seen visually using a number wall (which is again used for fractions):



   Example 2: Families or triples using x and ÷, for example:
   5 × 2 =10
   2 × 5 = 10
   5 = 10 ÷ 2
   10 ÷ 2 = 5…

Example 3: Multiplication as two ways of repeated addition, for example:
$3 \times 4 = 3 + 3 + 3 + 3 = 4 + 4 + 4$

Example 4: Division as repeated subtraction and division as equal grouping/ sharing. For example $12 \div 3$ means how many 3's in 12 AND how do I share 12 equally into 3 groups?

3. *Look for patterns in sequences of sums:*

Example 1: A sequence of multiply by ten:
$3 \times 10 =$
$4 \times 10 =$
$5 \times 10 = \ldots$
"A number" $\times 10 = ?$

Example 2: A sequence of add 11:
$1 + 11 =$
$2 + 11 =$
$3 + 11 =$
$4 + 11 =$
$5 + 11 = \ldots$
"A number less than 10" + 11 =
$20 + 11 =$
$21 + 11 =$
"A number more than 10 less than 100" + 11 =

You can use "a number" or a very big number (that is outside the calculation range of the learners) to help to get the learners to generalise the pattern. What happens with a thousand, or with a million, or with a hundred million?

4. *Ask about (observe, describe, talk about) how special numbers (0, 1, 5, 10 and multiplies of 10) behave.*

Example 1: Properties of 1:
- What happens when you add 1 to any number?
- What happens when you subtract 1 from any number?
- Divide any number by 1?
- Multiply any number by 1?
- What kind of sums are guaranteed to (always) give me an answer of 1?
- 1 = …?
- How do I make 1's?
- Why are 1's helpful to me?

Example 2: Properties of 0:
- What happens when you add 0 to any number?
- What happens when you subtract 0 from any number?
- What happens when you divide any number by 0?
- What happens when you multiply any number by 0?
- What kind of sums are guaranteed (always) give me an answer of 0?
- 0 = …?
- How do I make zeros?
- Why are zeros helpful to me?

Example 3: Properties of 10 (and multiples of ten)

- What is the significance of ten? What makes tens special?
- Tens are used for all work on place value
- Tens are used for all work on money and then decimals more generally.
- What happens when you +10 to any number?
- What happens when you – 10 from any number?
- What happens when you multiply any number by 10 ?
- What happens when you divide any number by 10?
- Does that always happen? How can you be sure?

Example 4: Other special numbers:

- 2: odds and evens, doubles, alternating pattern
- 5: two fives are ten (and 2 and 10 are special, so five is special)
- 4: the double doubles
- 8: the double double doubles
- Other primes (in addition to 2 and 5): 2, 3, 5, 7,11…
- The composites with many factor pairs…: 12,24,64,36

5. *Expect and ask for descriptions of what is observed.*

- What do you notice?
- Can you tell me the story of what you notice?
- Can you draw a picture to explain what is going on?
- Explain to a friend what you see.
- Imagine we are about to get a visitor to this class. How would you tell them about what you have done and noticed?
- What words can help up with this description? Call out some useful words so that I can write them on the board for you to use when you are explaining or describing.

6. *Ask for explanations to show if something is always true (basis of mathematical proof).*
   - Does that always happen?
   - Are you sure it always happens?
   - What happens with other types of numbers? Have you tried odd numbers, or very big numbers, what about with fractions?
   - Do you think this is always true? How can you be sure?
   - Don't just give me more examples, convince me, so I know for sure.
   - Tell me a story to convince me this is always true

7. *Explore and talk about equivalence (what does the equal sign mean?)*

A number expression gives a plan of what is to be done mathematically. For example 4 + 5 is an expression. The process or plan is to add or combine 4 and 5. In this way an expression is like a phrase in English as just it gives a description (of what is to be done).
A number sentence, like an English sentence, has a verb (doing word). A number sentence compares two expressions. The verbs for comparing the expressions, that are used in the South African primary curriculum are: equals (=), not equals (≠), approximately (≈) bigger than (>), smaller than (<) and bigger than or equal to (≤) and smaller than and equal to (≤).

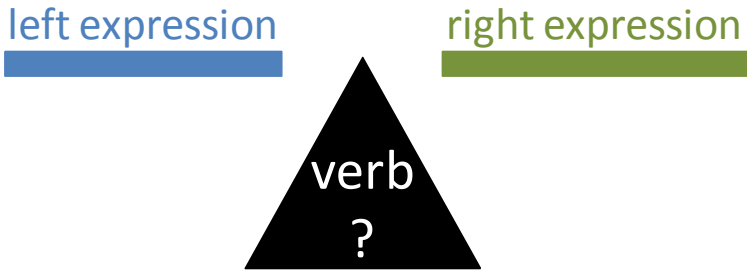


Figure 5: Number sentences compare two expressions

Equals means the same as. The equals sign is like a balance. It shows when two expressions have the same value. Like English sentences, number sentences can be true or false.

If you subtract the same thing from each side of the equals sign the balance stays the same. If you add the same thing to each side the equals sign, the balance stays the same.

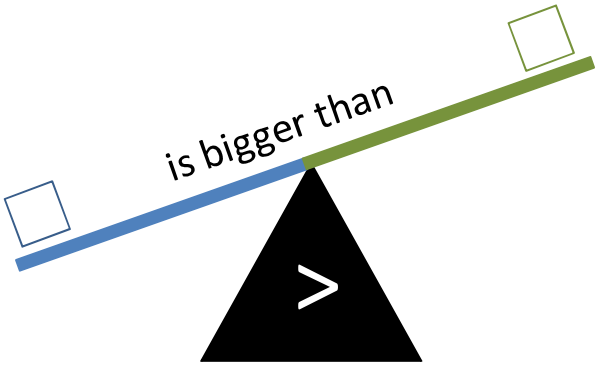


Figure 6: More than as a balancing scale

When using this image of a balance, point out that it is mass scale. Heavier things tip the balance down. Some learners find it hard to imagine that bigger number are lower down than the smaller numbers, so the connection to mass must be made clear.

Ensure that the unknown of a number sentence appears on both the left and right side of equal signs.

> Example 1: Avoid only asking problems like 4 + 5 = □ and rather also include problems like □ + 4 = 9 or 9 - □ = 4.

Encourage learners to see that you can have a very simple expression (like just one number) on the left side, and a more complicated expression on the right side.

> Example 2: Pose questions like: 35 = ? and see how many different number sentences learners can create.

Remember that equivalence (being the same) refers to both number expressions and to number sentences:

- *The expression* 4 + 5 *is equivalent to the expression* 10 – 1. This can be written as a number sentence: 4 + 5 = 10 – 1.
- *The number sentence* 90 × 10 = 9 *is equivalent to the number sentence* 90 ÷ 9 = 10. The two number sentences show same relationship, and reflect the inverse relationship between × and ÷.

Provide opportunities for learners to test and explore whether every addition number sentence can be written as an equivalent subtraction number sentence and vice versa. Emphasise that adding and subtracting fit together, they are inverses. Adding can be used to check subtracting, and vice versa

Example 3: Is 3 + 4 = □ the same as □ - 4 = 3 and the same as □ - 3 = 4? Does that work with other numbers? What about big numbers? What about fractions?.

Provide opportunities for learners test and explore whether every multiplication number sentence can be written as an equivalent division number sentence and vice versa. Emphasise that multiplying and dividing fit together, they are inverses. Multiplying can be used to check dividing, and vice versa.

Example 4: Is 12 ÷ 4 = □ the same as □ × 4 = 12? Does that work for other numbers? What about odd numbers? What about big numbers? What about fractions?

When working with inverse functions it is helpful to use the "I am thinking of a number", game.

Example 5: I am thinking of a number. When you double my number you get 22. What is my number?

Guidelines about generalising a rule or function

Generalising a rule or a function is the focus of the current 'Patterns, function and algebra' content area in the CAPS. The CAPS provides guidance to teachers on investigative work involving problems contexts that can be explored, and the representations of functions which include families of number sentences, input and output tables, and function machines.

1. *Expect learners to describe a number pattern in detail:*

Learners can complete this simple paragraph for a number pattern:

The pattern starts at …
It goes up / down. It increases / descreases.
The jumps are the same / different
It is a …. pattern

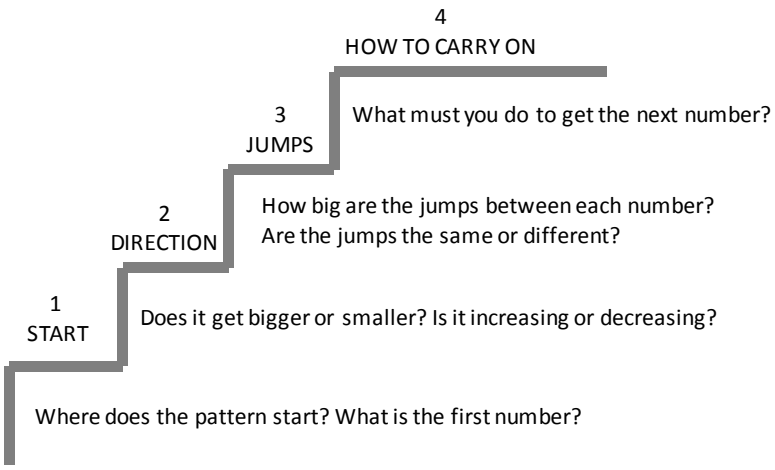


Figure 7: Steps for describing a number pattern

Remember patterns can be finite (they end) or infinite (never ending).
If the pattern is finite, it is useful to say so and say clearly where it ends.

Example 1: Describe the pattern: 6; 8; 10; …

The pattern starts at 6.
It goes up/ increases.
The jumps are the same. It is a jump of 2 each time.
To get the next number you add two.
It is a counting on in 2s pattern, starting at 6.

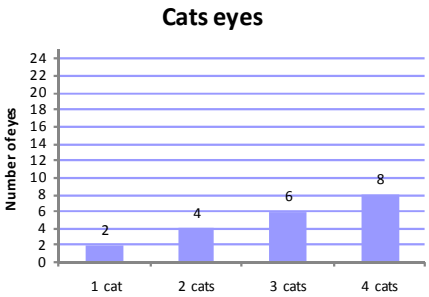
Example 2: Describe the pattern: 100, 50, 25
The pattern starts at 100.
It goes down / decreases.
The jumps are not the same. The jumps get smaller each time.
To get the next number you halve.
It is a halving pattern, starting at 100.
It is finite. It ends at 25.

2. *Look at and talk about the operations or functions (what is being done), not just the numbers*

- What is the rule making something change?
- Function machines / flow diagrams, tables, patterns
- What is always being done to the numbers going into the machine to create the new numbers?
- Link this to graphical representations used in data handling (graphs)
- Plot bar graphs as another representation of functions (connecting to data handling bar graphs)

We don't connect patterns, functions and algebra to data handling… Why not?

Cats eyes = double the cats



Cats legs = four times the cats

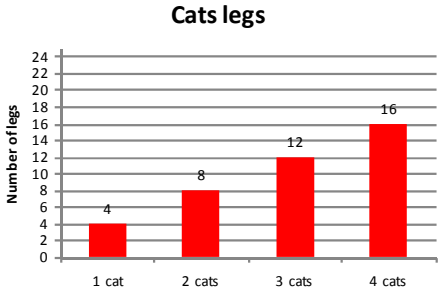


Figure 8: Connections to data handling

Example 1: What does the function '+10' look like? How is it different from the function '×10'?

Example 2: I am thinking of a rule. If you give me 3, my rule changes it to 1. If you give me 4, my rule changes it to 2. If you give me 10, my rule changes it to 8. Give me other numbers and I will tell you how my rule changes your inputs. What is my rule?

3. *Ask about (observing, describing, talking about) how operations (+,-,×,÷) behave.*

Example 1: Commutative property

- Have you noticed that when you add 5 to 3 it is the same as adding 3 to 5?
- Does that work with other numbers?
- Does that always happen with +?
- Does that work with other operations?

Example 2: Associative property

- My rule is add 2, then add 5 and then add 3
- Your rule is add 5, then add 2 then add 3
- Are our rules the same or different?
- Does that work with other numbers?
- Does that always happen with +?
- Does that work with other operations?
- What about the rules: Add 5, then subtract 2 then add 3; and the rule: Add 5 then add 3 then subtract 2? Are those rules the same or different?

Example 3: Distributive property

- Double 20
- Double 3
- Double 23
- What do you notice?
- Does that work with other numbers? What about "Double 7"? Can I double 2 and double 5 and add them together to get double 7?
- Does that always work with doubling?
- Doubling is × 2, what about other multiplication rules? Does it work for × 3, or × 4?
- Does this work with other operations?
- What about working with subtraction instead of addition? Double 10, double 3, double 7, double (10 – 3). What do you notice?
- Does that work with other numbers? What about "Double 7"? Can I double 2 and double 5 and add them together to get double 7?

4. *Set a problem context which requires investigation of a certain function, and then use different representations of the function (flow diagram, table, bar graph, family of number sentences)*

   Example 1: Imagine (or give a picture of) a pride of 34 lions.

   - How many eyes? How many ears? How many tails? How many legs?
   - What about a pride of 10 lions? Or 20 lines, or 30 lions, or 100 lions?
   - What does this show us about the eyes, ears and leg functions?
   - Summarise the information in a table, flow diagram, or bar chart

5. *Connect work done in the 'Patterns, functions and algebra content' area to work in other content areas (like Measurement, Geometry and Data handling).*

   Example 1: What is the rule for working out the minutes past the hour (output), if you know where the long hand is pointing on an analogue clock (input)?

   Example 2: What is the rule for converting units of length? mass? capacity? time? money?

   Example 3: Explore functions for calculating perimeter and area of different shapes, and different sized shapes.

Example 4: If I have 1 triangle then there are 3 sides, 2 triangles give 6 sides. What about 3, 4, 5 triangles? How many sides? What is the rule for finding the total number of sides from triangles? What is the triangles touch each other? What happens now?

Guidelines about modelling as a language of mathematics

1. *Model mathematics concepts, problem solving approaches, and calculation techniques and encourage learners to model their understanding and problem solving*

Modelling concepts and encouraging learners to model their understanding and problem solving processes engages them in mathematical thinking and reasoning. It is helpful to be aware of the different ways that you can model. You can: talk, show, draw, write number sentences, write English sentences, use concrete equipment, use a number line, sketch an empty number line, use a hundred square, a number or fraction wall, fraction chains, tables, graphs, function machines, input and output tables, act out or role play, imagine a situation or problem context, etcetera.

2. *Know and make explicit the basic models for the basic operations.* These include:

- Addition: combining AND increase/change
- Subtraction: take away AND compare AND decrease/change
- Multiplication: 4 taken 3 times, AND 4 groups of 3, … AND scale factor, AND percentage increase AND Cartesian products, multiplicative factors (including scale factor for enlargements) AND arrays AND area models
- Division: Equal sharing AND grouping… AND fractions, AND decimals, AND ratio, AND rate, AND % decrease, AND multiplications factors (including scale factors for reduction)

3. *Provide opportunities for learners to use concrete objects, draw or imagine objects or processes, and move between the representations (concrete, iconic and symbolic) all the time.*

There seems to be a pervasive sense that the job of primary school teachers is to move children from the concrete, to the iconic to the abstract realm. The abstract realm is seen as the mathematical destination. However, in order to model mathematical situations learners and teachers must constantly move between the iconic, symbolic and abstract realms. They need to simultaneously reflect on the problem context (real or imagined concrete situation) describe this using words and drawings and work with these ideas using the abstract symbolic realm of digits, numerals and symbols. But most primary teachers seem to push to the symbolic or abstract as a destination and do not use modelling.

The job of primary school teachers is to help children be aware of AND constantly move between the concrete, iconic and symbolic to understand mathematics and explain their reasoning.

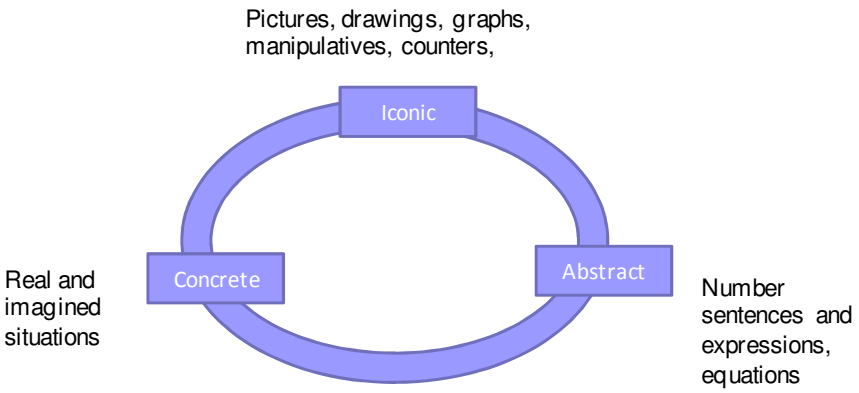


Figure 9: The symbolic is not a destination

When moving from the abstract to the concrete, ask learners to imagine or invent problem contexts that might give rise to the abstract mathematical calculation.

Support learners to model a mathematical problem and communicate this using all the different representations available to them to explain this process. This means teachers must ask for explanations: How did you work that out? Explain how you did that? Is your way the same was as … or different? Is there another way of doing this?

This means that the starting point for a mathematical problem can vary. It can be:

- Concrete: These beads need to shared equally between Ayanda, Tessa and Fatima; or
- Iconic: Look at this picture of 20 beads, and draw circles, to show how to share them equally 3 groups
- Abstract: 20 ÷ 3. What problem could this expression describe?

The role of the teacher is to move children between these realms – in both directions.

4. *'Algebrafy' any word problem and turn it into an investigation, to model the process of solving the problem, when one or more of the parameters is relaxed.*

   Example 1: A tin of beans costs R9. How many tins can I buy for R100? How much change will I get?
   By relaxing the price parameter, learners can investigate and model the process of working with this problem. What happens if the tin of beans costs R8, or R7 or R6? Investigate…
   You can also relax the amount of money. What happens if I have R200, or R300 or R400 to buy the beans? Investigate…
   What happens if I must have at least R15 change, so that I can buy bread on the way home? How does this change the problem?

CONCLUSION

This paper is by no means comprehensive as an introduction to the field of early algebra. There is much in the mathematics education literature on this field which has been omitted. There is no attempt to provide a theoretical framework of approaches to mathematics teaching and learning, or to teacher knowledge in South African with regard to early algebra, neither is there any critical analysis of its treatment in the current intended curriculum. Nevertheless, it is hoped that this paper provides a relatively accessible starting point for South African primary school teachers considering the theoretical realm of 'early algebra'

with practical guidelines, including concrete examples and suggested lines of questioning, for teachers.

In sum, it is hoped that teachers will be aware that early algebra is an approach to teaching and learning arithmetic which focuses on the structure and patterns underlying arithmetic. Early algebra is the subject of research and curriculum reform in many countries around the world (including South Africa). There are three main aspects to early algebra:

1. Generalising arithmetic;
2. Generalising towards the idea of a function; and
3. Modelling as language of mathematics

In South Africa, the 'Patterns, functions and algebra' content area in the CAPS tends to emphasise the second aspect of early algebra. It focuses on investigating numeric and geometric patterns making use of function machines and inputs and output tables. But there are also components of the first aspect of early algebra in this content area as well as in the 'Number, operations and relationships' content area of the CAPS. Here the focus is on equivalence, inverse relationships, and properties of numbers and operations. The last aspect of early algebra (modelling as a language of mathematics) is also important and is evident particularly in approaches to problem solving and encouraging learners to explain their reasoning and compare calculation techniques. All three aspects of early algebra are important and can be seen to underlie all the content areas in Foundation and Intermediate Phase CAPS.

There is clearly a need for greater specification, guidelines and resources to teachers working with early algebra. Most teachers have never been taught using early algebra approaches themselves at primary level and struggle to understand its relevance. This paper is a small contribution towards this need. Annexure one details relevant vocabulary for early algebra concepts, which may be an additional useful resource for teachers.

REFERENCES

Amerom, B. A. V. (2003) Focusing on Informal Strategies When Linking Arithmetic to Early Algebra, *Educational Studies in Mathematics*, 54(1), pp. 63-75.

Bodanskii, F. (Ed) (1991) *The Formation of an Algebraic Method of Problem Solving in Primary School Children,* (Reston, V.A, NCTM.).

Carpenter, T. P., Hiebert, J., Fennema, E., Fuson, K., Wearne, D., Murray, H., Olivier, A. & Human, P. (1996) Problem Solving as a Basis for Reform in Curriculum and Instruction: The Case for Mathematics, *Educational Researcher*, May, pp. 12-21.

Carpenter, T. P., Hiebert, J., Fennema, E., Fuson, K., Wearne, D., Murray, H., Olivier, A. & Human, P. (1997) *Making Sense: Teaching and Learning Mathematics with Understanding* (Portsmouth, Heineman).

Carraher, D. W., Schliemann, A. D. & Schwartz, J. E.(2008), Early Algebra is Not the Same as Algebra Early, in: J. Kaput, D. Carraher & N. Blanton (Eds) *Algebra in the Early Grades* (Taylor & Francis), pp. 235-272.

Davis, R. (1985) ICME-5 Report: Algebraic Thinking in the Early Grades, *Journal of mathematical behavior*, 4, pp. 195-208.

Herscovics, N. & Linchevski, L. (1994) A Cognitive Gap between Arithmetic and Algebra, *Educational Studies in Mathematics*, 27(1), pp. 59-78.

Kaput, J. & Blanton, M. (1999) Algebraic Reasoning in the Context of Elementary Mathematics, *Annual Meeting of the American Education Research Association,* (Montreal, Quebec, Canada).

Kaput, J., Carraher, D. & Banton, M. (2007) *Algebra in the Early Grades* (Routledge).

Liebenberg, R. E., Linchevski, L., Sasman, M. C. & Olivier, A. (1999b) Focusing on the Structural Aspects of Numerical Expressions, in: J. Kuiper (Ed) *Seventh Annual Conference*

of the Southern African Association for Research in Mathematics and Science Education, (Harare, Zimbabwe), pp. 249-256.
Liebenberg, R., Linchevski, L., Olivier, A. & Sasman, M. (1998) Laying the Foundation for Algebra: Developing an Understanding of Structure, *4th Annual Congress of the Association for Mathematics Education of South Africa (AMESA)* (Pietersberg, AMESA).
Liebenberg, R., Sasman, M. & Olivier, A. (1999a) From Numerical Equivalence to Algebraic Equivalence: Vol. 2. (pp. 173-183). Port Elizabeth: Port Elizabeth Technikon., *Fifth Annual Congress of the Association for Mathematics Education of South Africa (AMESA)* (Port Elizabeth Technikon, AMESA), pp. 173-183.
Linchevski, L., Olivier, A., Sasman, M. & Liebenberg, R. (1998) Moments of Conflict and Moments of Conviction in Generalising, in: A. Olivier & K. Newstead (Eds) *Twenty-second International Conference for the Psychology of Mathematics Education* (Stellenbosch, South Africa), pp. 215-222.
Roberts, N. (2010). Comparison on two levels of the content treatment of 'early algebra' in the intended curricula in South Africa and England. In P. Andrews & M. Joubert (Eds) *Research proceedings British Congress of Mathematics Education BCME - 7 in collaboration with the British Society for Research into Learning Mathematics,* Manchester, 30 (1), 167-174.
Samson, D. (2006) Number Patterns, Cautionary Tales and Finite Differences, *Teaching and Learning Mathematics*, 3, pp. 9-15.
Samson, D. (2008) Approaches to Generalising Quadratic Sequences, *Learning and Teaching Mathematics*, 6, pp. 9-15.
Sasman, M. C., Linchevski, L. & Olivier, A. (1999a) The Influence of Different Representations on Children's Generalisation Thinking Processes., in: J. Kuiper (Ed) *Seventh Annual Conference of the Southern African Association for Research in Mathematics and Science Education* (Harare, Zimbabwe), pp. 406-415.
Sasman, M. C., Olivier, A. I. & Linchevski, L. (1999b) Factors Influencing Students' Generalisation Thinking Processes, *23rd Annual Conference of the International Group for the Psychology of Mathematics Education (PME)* (Haifa, Israel).
Sasman, M., Linchevski, L., Olivier, A. & Liebenberg, R. (1998) Probing Children's Thinking in the Process of Generalisation, *Fourth Annual Congress of the Association for Mathematics Education of South Africa* (Pietersburg: University of the North, South Africa), pp. 210-218.
Vermeulen, N. (2007) Does Curriculum 2005 Promote Successful Learning of Elementary Algebra?, *Pythagoras*, 66 (December, 2007), pp. 14-33.
Vermeulen, N., Olivier, A. & Human, P. (1996) Students' Awareness of the Distributive Property, *Twentieth International Conference for the Psychology of Mathematics Education (PME 20)* (València, Spain)

ACKNOWLEDGEMENTS

This paper was made possible by a funding grant provided by Apex Hi, administered by Tshikululu Social Investment, for the Focus on Primary Maths research project which has The Grove Primary and Capricorn Primary as its locus of research.

ANNEXURE 1: VOCABULARY FOR PATTERNS, FUNCTIONS AND ALGEBRA

The following are important words that learners need to be able to interpret and use for early algebra:

Foundation Phase		**Intermediate Phase**	**Types of numbers**	**Words needed from Senior Phase onwards**
Observe **Look** for **Explain** **Describe** **Tell** me **What**'s happening? **Why** is it happening? **Show** me **Convince** me Are you sure? **Justify** **Equal** **More than** **Less than** **Sum** **Product** **Difference** **Quotient** **Plus** **Add** **Minus** **Subtract** **Share equally** **Divide** **Group** **Multiply** **Times** **Increase** **Decrease** **Double** **Half** **Halve**	**Same** **Different** **Always** **Sometimes** **True** **False** **Pattern** **Change** **Stay the same** **Constant** **Rule** **Operation** **Function** (mapping with only 1 output for each input) **Opposite** **Undo** or **reverse** a rule **Output** **Input** **Function machine** **Flow diagram** **This** one, number, operation, sum, problem… **All…** **Any…**	**Relationship** **Reciprocal relationship** **Inverse** **Equivalent** **Particular** **Unique** **General** **Commutative property** **Associative property** **Distributive property** **Ratio** **Rate** **Quotient** **Cube** **Square** **Position**	**Whole numbers** (0,1,2,3…) **Natural numbers** (1,2,3,…) **Odd** **Even** **Prime** **Multiple** **Integers** (…-2,-1,0,1,2,3…) **Fraction** **Half, quarter, third, eighth, tenth,** **Common fraction** **Proper fraction** **Improper fraction** **Mixed fractions**	**Domain** (input values) **Range** (output values) **Relation** (more than one output for each input) eg "What number times itself gives me 4?" Answer is +2 or -2. "the factors of 6" **Sequence:** 3; 6; 9;… **Series:** 3; 3+6; 3+6+9; … **Rational numbers** (can be written as a fraction, which includes all whole numbers) **Term** (the numbers in a sequence or series)