LONG DIVISION:

IN SEARCH OF THE PRIMARY SCHOOL MATHEMATICS HOLY GRAIL

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This theoretical paper analyses the treatment of long division in the South African Curriculum and Assessment Policy Statement (CAPS) for intermediate phase mathematics (2011) against some key theoretical features which are documented in mathematics education literature. It concludes that there is clear research-based progress evident in the CAPS curriculum in relation to this topic, but there remains more that can be done to support learners' efficient long division written calculation procedures. It therefore provides additional guidelines to teachers, including examples of an 'efficient chunking' written procedure for long division which may be of use to primary mathematics teachers implementing the intermediate phase CAPS.

NOTES ON TERMINOLOGY

For the purpose of this paper the key terms are defined to mean the following:

CAPS: Department of Education, 2010, Curriculum and Assessment Policy for Mathematics Intermediate Phase. Pretoria: Department of Education.

Clue board: A written recording of multiplication calculations for the same whole number multiplier, which may be written as a list of number sentences, in a table, or using a flow diagram. Examples of clue boards for the multiplier 17 are shown in Figures 1a-c.

10 x 17 = 170
20 x 17 = 340
5 x 17 = 85
3 x 17 = 51

Figure 1a: Number sentence clue board

X 17	
10	170
20	340
5	85
3	51

Figure 1b: Tabulated clue board

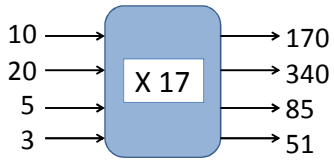

Figure 1c: Flow diagram clue board

Input for a clue board: The input for the multiplicand recorded on a multiplication clue board. In the examples presented in Figures 1a-c the inputs shown are 10, 20, 5 and 3. The outputs are the products written in the clue board, which for Figures 1a-c are 170, 340, 85 and 51.

Short division: Any written calculation for division where the dividend is whole number with at least 2-digits and the divisor is a single digit whole number.

Long division: Any written calculation for division where the dividend is a whole number with at least 3-digits and the divisor is a whole number with at least 2-digits, and

Traditional long division algorithm: The formal condensed written procedure for recording long division as shown in Figure 2.

```
      26 r 3
17 ⟌ 445
     -34
      105
     -102
        3
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Figure 2: Traditional long division algorithm for 445 ÷ 17

The traditional long division algorithm may be referred to as the "bus shelter algorithm" in some mathematics education literature.

Multiply and subtract table: A table used to keep track of multiplication and subtraction steps in a written calculation for division as shown in a worked example of 375 ÷ 8 in the CAPS (p109):

Multiply	Subtract
40 × 8 = 320	375 – 320 = 55
6 × 8 = 48	55 – 48 = 7

Table 1: Number sentence clue board

MOTIVATION

In my work with South African primary schools I am struck by how often I am asked by teachers for help with long division. The questions are sometimes very direct: "What's the best way to teach long division?". The questions are sometimes posed to be about engaging with curriculum change: "Does CAPS say we can stop teaching long division, now?" At times the question appears desperate: "Does it really matter if a learner can't do long division when they get to high school?" My short answers to these questions: "No there is no magic bullet, but we know a lot more about different ways of approaching this topic"; "It depend what you mean by 'long division', but learners are still expected to divide large numbers in CAPS" and; "Learners can use calculators in high school, so they are unlikely to need to ever 'do long division'. But there are things they should understand about divisibility that are

very important mathematically" respectively; are probably not very helpful. It is for this reason a theoretical essay on long division as it currently appears in CAPS was considered potentially useful.

Before embarking on this paper however, it is opportune to briefly reflect on how much time seems to be spent in primary schools in search of the long division holy-grail. Long division appears to be the pinnacle of the primary school mathematics experience. Learners who can clear the long division high jump are seen as having a rosy mathematical future ahead of them. These are our best athletes. And the high jump bar tends to be set as the very condense – and mathematically elegant - long division algorithm. If learners can master this – then surely they will cope with the high school mathematics field events? At least that seemed to have been the thinking in many primary school mathematics curricular around the world. At least, until the curricula were shifted by mathematics reforms; and South African has been no exception. I therefore write this paper with both a healthy scepticism of the value of teaching formal written procedures at all, and a healthy pragmatism that this remains an area where numerous primary teachers have been vocal in requesting support.

So this paper sets out to answer these questions: What does the current South Africa's intended curriculum expect in relation to long division at primary school? How can these expectations be understood in relation to key theoretical shifts in mathematics education internationally? And what does this mean for a primary school teacher focusing on this topic with their classes?

It should be noted that this paper does not consider the (very important) connections between division of whole numbers with work on common fractions, decimal fractions and percentages. Nor does it consider the various uses of division in problem contexts, including for example, connection to measurement contexts, financial contexts, ratio and rate problems, multiplicative comparison (including scale and reduction factors), equivalent groups, and Cartesian products. It focuses only on the bare calculation of division of whole numbers.

THEORETICAL FRAMEWORK

There are several theoretical features that are relevant to an analysis of the intended curriculum relating to long division, which are discussed in this section, with reference to the literature that informed each one. It is not my intention to imply that this is an exhaustive list. What is presented here represents the main theoretical features that have informed my analysis of the treatment of long division in the South African Intermediate Phase CAPS.

Feature One: Shift from sets-based conceptions to counting based conceptions of number

The turn of the century saw major shifts in mathematics learning theory. One such shift as from arithmetic as a 'set-based conception' to a 'counting-based conception' of number (Yackel, 2001, pp21-30).

Images can support calculation strategies based on counting, which has been explored in Dutch mathematics education research which focused on the progression from a beaded string, to a number line and then to an 'empty number line' (Beishuizen, 1999 cited in Anghileri, 2006, p74). The empty number line is a rough sketch of a line (either drawn or imagined) which does not have regular intervals marked on it (as is the case for a ruler) but orders only the numbers involved in a calculation on a line. Calculations are imagined as hops (of ones or units) and jumps (reflecting groups of units) either forwards or backwards on the line, as shown in Figure 3:

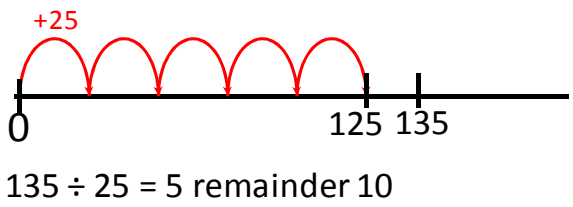

Figure 3: Empty number line image for long division example 135 ÷ 25

Feature Two: Shift from recall or facts to developing number sense

Another shift was from expecting learners to recall arithmetic facts, to expecting children to confidently navigate through a landscape of number concepts and relationships (Fosnot & Dolk, 2001) and develop number sense (Anghileri, 2006). Anghileri (2006) explains that

In society today there is general acceptance that 'drill and practice' of taught routines will not prepare children for life in a technological society, and that teaching approaches need to focus on the links that demonstrate the logical structure underlying numbers and number operations. (p1)

This includes a deliberate shift to align written recording more closely with mental calculation strategies:

Rather than being shown how to do written calculations, children are to be encouraged to work mentally, to observe patterns, to predict results and to talk about the connections that they can make. (p2)

Part of this trend is to avoiding a digit-wise conception of number in favour of a more holistic approach to number. Fuson (1992) reflects on the traditional approaches that emphasis place value partitioning of numbers or writing numbers in expanded notation (185 is 1 hundred 8 tens and 5 units). She finds that when this partitioning is introduced too early for children, before they have established their own mental calculation strategies, it leads to misunderstandings and mistakes. As such calculation methods for arithmetic emphasise a holistic concept of the numbers, where "numbers are kept as wholes in any partition" (Anghileri, p41) and teachers are to avoid procedures that treat individual digits within a number independently. It is a result of this observation that the concept of 'chunking' emerges. Numbers can be partitioned in many ways and expanded notation is only one option. Chunks can make use of adding and subtracting (for example 36 is 30 + 6 it is also

32 + 4 and 20 + 16) or multiplying and dividing (for example 36 is double 18 or half of 72). (Aghileri p41)

Feature three: Two models of division

For the context of this paper, I refer only to 'two models for division'. However given their inverse relations, these are two models for multiplication as well.

The two models of division: the 'quotitive' and 'partitive' models which have formed the basis of analyzing division operations of whole numbers (Anghileri, Beishuizen, & Putten, 2002) are fundamental to the teaching and learning of division. The former refers to division as equal groupings (how many sevens in 28?), while the latter refers to division as sharing (28 shared between seven). In this regard, primary school teachers' knowledge on this distinction of two division models has been found to be lacking by both Ball (1990) and Ma (1999). What is particularly interesting about Ma's study is that her work contrasts teacher knowledge in China to the United States of America. She found that the Chinese teachers had a good grasp of the two models, and could flexibly move between them as required. However American teachers were not taught the two models explicitly, and so lacked this basic understanding of the division concept. This made is particularly difficult for teachers to explain and work with division which requires the quotitive model. In particular American teachers were not able to explain what a calculation like $3 \div \frac{1}{2}$ meant, while the Chinese teachers were able to use the quotitive model to consider how many halves made up 3.

It is significant that the partitive model tends to be the dominant model, and division as equal sharing is usually more familiar to young children (Frobisher 1999 and Zazkis 1998). Ruthven has analysed of children's written recordings strategies which are conducted for two purposes: recording; and schematisizing, which may each be associated with informal, and formal or taught procedures respectively. He noted that there can be conflict between computational procedures and context structure, explaining that the partitive model for understanding division seems to dominate children's understanding of division, while taught written procedures tend to privilege the quotitive model (Ruthven, 1998). The shift in division model (from partitive to quotitive) is seldom made explicit to learners, and it perhaps not surprising that so many learners find the written division calculation algorithms confusing.

Feature four: The shift from division as repeated subtraction, to division as chunking and then to efficient chunking

Written calculation methods of long division has been the specific focus of a mathematics education study by Angheleri, Beishuizen and van Putten (2002) which contrasted teaching approaches in England to those in the Netherlands. The study sought to analyze the informal strategies used by children and the progression from informal strategies to more structured and efficient procedures that resulted from the different teaching approaches in the two countries.

The study revealed that Dutch learners were coached to use 1, 2, 4, 8 and 10 as the inputs for their multiplication clue boards. In this way they could use knowledge of doubling and

multiples of ten to generate their multiplication clue boards. The researchers found that the most commonly used strategy amongst Dutch pupils was the 'identification and use of large chunks usually in a structured procedure of repeated subtraction', and was 51% successful. In contrast the most commonly used strategy was the 'traditional algorithm' for English pupils, which was only 25% successful. Improvements between individual scores in the two tests were also considered between the two countries. 69% of Dutch pupils improved their score between the two tests, while almost half of the English pupils showed no improvement or deterioration.

Anghileri et al conclude from this study that learning is most effective when written methods build on pupils' intuitive understanding in a progressive way. They argue that the Dutch approach - involving repeated subtraction using large chunks builds progressively on children's intuitive strategy and retaining whole numbers at all stages - is preferable to the English approach of teaching the traditional algorithm.

This study provided clear evidence to support a shift in teaching approach from the traditional algorithm for division, to one that is more closely aligned to children's intuitive strategies – which allows learners to track the chunks that they subtract from the dividend. Further is outlined a possible 'efficient chunking' written procedure where learners can generate a clue board using the inputs 1, 2,4 and 8 and 10 which relies only on secure doubling, and on confidence in multiplying by ten, to create a clue board from which learners can select sensible chunks to subtract from the dividend.

WHAT DOES THE CURRENT SOUTH AFRICA'S INTENDED CURRICULUM EXPECT IN RELATION TO LONG DIVISION AT PRIMARY SCHOOL?

The South African CAPS for Intermediate Phase mathematics locates division of whole numbers within the Number Operations and Relationships content area, of which whole numbers is a sub section. As with other content areas, whole number division progresses by year with the increase in number range; as well as the changes in calculation techniques (CAPS p8). With regard to the former, in Grade 4 learners are expected to divide "at least whole 3-digit numbers by 1-digit numbers"; Grade 5 learners divide "at least whole 3-digit numbers by 2-digit numbers" and by Grade 6 learners divide "at least whole 4 digit numbers by 3-digit numbers" (CAPS p9)

The CAPS framework presents the topics of multiplication and division at times together as a single topic, and at times separately. Both concepts are revisited in three terms of each year of the Intermediate Phase. The teaching guidelines provide the following reasons for teaching the two topics together: "using inverse operations to check solutions" and "we almost always use multiplication to solve division" (CAPS p48). The calculation techniques that learners are expected to use include: estimation, breaking up and breaking down numbers, rounding off and compensating, doubling and halving, using a number line and using multiplication and division as inverse operations (CAPS p40).

The concepts and skills for the whole number division topics are presented in Table 2 at the end of this paper. The CAPS includes "clarification notes or teaching guidelines" for each topic in each grade in a section titled: Clarification of content. What follows is a description of the main features of these sections in relation to the division topics for each grade.

In Grade 4 the CAPS emphasises that "it is important that learners understand that they change any division statement into a [equivalent][18] multiplication statement" (p42). Emphasis is placed on the commutative property of multiplication and using arrays to visualise multiplication and division. The teaching guidelines for the first CAPS topic on multiplication and division outline the following:

There are two kinds of problems that result in division. It is important that learners experience both of these, namely:

Problems involving sharing: 6 learners share 32 sweets [equally]. How many sweets does each learner get?

Problems involving grouping: Samkele has one large packet with 32 sweets. How many smaller packets can she make with 6 sweets each.

Some problems and calculations should have a remainder, and some should not. (p42)

By the second CAPS topic on multiplication and division for Grade 4, learners are expected to round off to the nearest 10, estimate their answers and judge the reasonableness of their solutions. They are expected to use multiplication to check division problems and they are introduced to the distributive property of multiplication and division over addition and subtraction with reference to breaking up arrays (p48). The calculation techniques for division are supposed to "build on what learners know for multiplication to do division" and specific guidance is given on how teachers should shift from past practice:

In the past learners have sometimes been taught to write out the whole times table, which they are encouraged to work out by repeated addition. It is better not to limit learners' division ability to repeated addition. Rather let them work with useful and easily remembered multiplication facts, especially multiples of 10, and then doubling and halving (p49)

The written technique of using a multiplication "clue board" for division is introduced. The example is 75 ÷ 4, and the clue board has inputs 10, 20, 5, 4 and 3. A multiply and subtract table is used for recording the steps in the division problem. Finally learners are expected to check their answers by multiplying (p50). In the second term of Grade 4, the work on multiplication focuses on keeping track of their written calculations using brackets, when multiplying using either the distributive property or factorising ("breaking down numbers into factors to multiply") (p59). The division topic in term 2 of Grade 4, focuses on sharing and grouping problem contexts, estimating and judging the reasonableness of solution. The teaching guidelines are explicit about avoiding calculations techniques that treat each digit separately, with this statement:

With all calculations in Grade 4, learners are not encouraged to treat the digits separately, but rather to consider the numbers as a whole and keep in the mind the value of the parts of the number. (CAPS p64)

The caution about learners not progressing beyond using the whole times table for division (which is worked out using repeated addition) is repeated, and the following additional point is made:

[18] [Author addition shown in brackets]

At other times in the past, Grade 4 learners were encouraged to divide using repeated subtraction of the divisor. Most Grade 4 learners got lost in the extensive repeated subtraction of the divisor when dividing 3-digit by 1-digit numbers. When dividing 3-digit by 1-digit numbers, it is preferable for learners to work with easily remembered multiplication facts of multiples of 10 and doubling and halving. These large groups of numbers can then be subtracted from the number being divided into. In this way learners do fewer subtractions and are more likely to arrive at the correct answer (p64).

An example of 375 ÷ 8, using a multiplication clue board with inputs 10, 20, 30, 40, 5, 6 and 3, and a layout showing multiplication and subtraction steps, with final checking of the solution by multiplying is given (CAPS p65). The remaining multiplication and division topics in Grade 4, provide no further guidance to teachers, and encourage more practice of the work done in terms two and three.

The CAPS teaching guidelines for Grade 5 the multiplication and division topic opens with a consolidation and revision of what was done in Grade 4. Once again emphasis is placed on context free calculations and solving problems in contexts where learners are encouraged to judge the reasonableness of solutions, and use the inverses to check solutions. The teaching guidelines list the following problems types as important: "sharing, grouping, treating groups as units, rate and ratio" (p106). The same exemplar question (375 ÷ 8) as was presented for Grade 4, is repeated. The clue board has the same inputs, and the multiply and subtract steps are presented in two columns (although the borders for the table cells are not included). The cautions about not using approaches that treat digits independently, not writing out the whole times table using repeated addition, and not encouraging the use of repeated subtraction of the divisor which are presented in the grade 4 teaching guidelines are appear again in Grade 5. A worked example for short division is presented which makes use of multiplication clue board using multiples of ten; and other multiplication facts. The worked example uses a table method for recording the multiplication and subtraction steps and includes how to check the solution using multiplication, and addition of the remainder. It also includes rounding off and estimating to check the reasonableness of the solution (p109). When division is revisited again in Grade 5, the teaching guidelines which were provided for short division in grade 4, are repeated for long division:

When dividing 3-digit by 2-digit numbers, it is preferable for learners to work with easily remembered multiplication facts of multiples of 10 and then doubling and halving. These large groups of numbers can then be subtracted from the number being divided into. In this way learners to fewer subtractions and more likely to arrive at the correct answer (p132).

A worked example showing a long division approach for 442 ÷ 17 is provided. This again uses a number sentence clue board with inputs 10, 20, 30, 5, 2, 3 and 6. A multiply and subtract table is presented for keeping track of each step in the calculations. There is a remainder of 0 for this example, and how to check the division calculation by multiplying is shown (p133). No further clarification is offered.

HOW CAN THESE EXPECTATIONS BE UNDERSTOOD IN RELATION TO KEY THEORETICAL SHIFTS IN MATHEMATICS EDUCATION INTERNATIONALLY?

In general terms, the CAPS seems to draw on *Feature One: The shift from a set-based conception of number to a counting based conception of number.* This is evident in the absence of definitions or approaches to multiplication and division which refer to sets and set notation. It is also evident in the inclusion of counting forwards and backwards in specified

intervals in the specification and clarification of content area. However, how the concept of division relates to counting forwards or counting backwards is not explored, or provided in the clarification notes. Division as repeated subtraction is mentioned, but only in relation to discouraging learners from using repeated subtraction of the divisor, when more efficient techniques can be used. Although "using a number line" is listed as a calculation technique with regard to concepts and skills for multiplication and division, no guidance is given to teachers on how to this can be done.

With regard to *Feature Two: Shift from recall or facts to developing number sense* the CAPS makes explicit mention of the need to avoid written calculation methods that use a digit-wise conception of numbers, and it emphasises the need to encourage all calculation methods (not just those for multiplication and division) to use whole number partitions. However, there is no clear explanation of the distinction between 'breaking down' a number using addition or subtraction, and 'breaking down' a number using multiplication or division (factorising). While the teaching guidelines for multiplication do present examples using factorising, there are no examples of this kind for division calculations. Although doubling and halving is included as a calculation technique, teachers or learners are not given any guidance or examples of how factorising (which may make use of doubling and halving) may be appropriate for division, as in this example:

$$
\begin{aligned}
704 \div 8 &= (352 \times 2) \div 8 && \textit{(halve 704)} \\
&= (176 \times 2 \times 2) \div 8 && \textit{(halve 352)} \\
&= (88 \times 2 \times 2 \times 2) \div 8 && \textit{(halve 176)} \\
&= (88 \times 8) \div 8 \\
&= 88
\end{aligned}
$$

Similarly, although CAPS includes examples of how to partition numbers in ways other than expanded notation, and to use this for multiplication; no examples on how this can be used for division are provided. An example, like the one below, may have supported teachers and learners to make the connection between multiplication and division calculation techniques more clear:

$$
\begin{aligned}
400 \div 13 &= (130 + 130 + 130 + 10) \div 13 \\
&= (130 \div 13) + (130 \div 13) + (130 \div 13) + (10 \div 13) \\
&= 10 + 10 + 10 + (10 \div 13) \\
&= 30 \text{ remainder } 10.
\end{aligned}
$$

The distributive property of division over addition and subtraction is included as an explicit focus, but is unfortunate that the connection between this property and its application to long division is missing.

With calculators only introduced as an explicit calculation technique in Grade 6, the CAPS appears to have ignored the shifts that are evident in the technological society. There is no direction provided on how to use a calculator to support learners to "to observe patterns, to predict results and to talk about the connections that they can make" in relation to division (Anghileri, 2006, p2). It may be argued that this type of work is encouraged in the Patterns Functions and Algebra content area of the CAPS, however the connections to this are not made explicit when focusing on the division topics. It is also true, that in its broad

conceptualisation CAPS includes deliberate emphasis on mental calculations, with 10 minutes a day allocated to mental mathematics. However, there is little guidance provided to teachers on how to align the mental calculation techniques with the approaches required for long division calculations. MAs such, most of the guidance for the division topics is directed towards "showing learners how to do written calculations" (Anghileri, 2006, p2).

In relation to *Feature three: Two models of division,* the CAPS is explicit about the two types of problems which give rise to division calculations, mentioning both equal sharing and grouping contexts. Examples of both problem types are also provided. This is a welcome improvement for the South African curriculum. However it is unfortunate that the CAPS does not make explicit that the two models for division and fundamental to understanding different approaches to written calculations for long division; nor does it make clear that the written calculation technique for long division presented in the CAPS, makes use of division as grouping (where division as sharing is the more likely dominant model of division for children). As such the two models for division are described only in relation to problem contexts, and not in relation their implications for approaches to bare division calculations.

The CAPS shows substantial awareness of *Feature four: The shift from division as repeated subtraction, to division as chunking and then to efficient chunking.* This is evident in the explicit introduction of the concept of a multiplication clue-board, and the direct caution to avoid repeated subtraction of the dividend as this is an inefficient method that is prone to errors. It refers to the need to subtract large groups (it does not name them as 'chunks') and encourages the use of easily remembered multiplication facts making use of doubling and halving and multiples of ten. It provides a clear presentation for both short and long division (which does not require adjustment as the number ranges increase). The multiplication and subtraction table, and column layout is repeated. So the CAPS provides worked examples which reflect the chunking method for long division. However, it stops short of providing examples which make use of efficient chunking which was the more effective Dutch approach that specifically referred to a clue board which has the inputs 1, 2, 4, 8 and 10 as its inputs.

WHAT DOES THIS MEAN FOR A PRIMARY SCHOOL TEACHER FOCUSING ON THIS TOPIC WITH THEIR CLASSES?

The CAPS shows evidence of awareness of the key theoretical shifts in mathematics education internationally. There is clear research-based progress evident in the CAPS curriculum in relation to the treatment of long division. However in many instances insufficient guidance is provided to teachers to make the implications of these research shifts explicit.

The following teaching guidelines may therefore be useful to educators in South Africa teaching the topic of long division within the current intended curriculum:

1. Learners' (and teachers') early introduction to division frequently refers only to a sharing model. This is evident in their sense of halving as dividing into two equal groups (and their reticence to see counting the pairs, as an equivalent halving process). So be explicit that there are two models for division, and provide learners' with time and opportunity to partition numbers into equal groups, and to share out numbers into equal groups for them to full internalise the two division models, their

mathematical equivalence, and their difference in terms of problem contexts. This connects with work done on the commutative property of multiplication (which reveals the equivalent two models for multiplication).

2. Connect multiplication to repeated addition and to counting patterns. This can be done through work in Patterns, Functions and Algebra; but should be explicitly linked to division and multiplication. Make use of the empty number line to visualise the counting patterns, and repeated addition, and then use the same empty number line concept for work with multiples and factors, and division and multiplication mental mathematics techniques.
3. Make it clear that division is the inverse of multiplication (multiplication is the reverse if the division function). Division calculations can be checked using an equivalent multiplication number sentence. Every multiplication number sentence can be re-written as an equivalent division number sentence, and vice versa. Also division can be done either by counting backwards from the number, or by counting forwards from zero to reach the number as a target. Many learners find using counting forwards from zero easier that counting backwards.
4. Encourage learners to build on the techniques for division that they use in mental calculations. This relies on quick recall of easily remembered facts especially those involving doubling and halving and multiplication by ten. Learners also benefit from knowing their timetables, and can add these know facts into their multiplication clue boards as required (although this is not a requirement for the efficient chunking method).
5. Support learners to become completely secure when building doubling and times ten multiplication clue boards. The inputs are 1, 2, 4, 8 and 10, and the outputs are obtained by doubling and knowing how to multiply by ten. This gives learners a good sense of how a multiplication function behaves (they have five easily generated inputs and outputs for this function).
6. Once learners are comfortable with the doubling and times ten multiplication clue board, they should become secure with creating clue boards for multiples of ten the inputs for which are 10, 20, 40, 80 and 100. Learners use their knowledge of multiplying by 10, to adapt the doubling and times ten clue board.
7. Connect the multiplication clue-boards to work done in the Patterns, function and algebra content area where the equivalent representations of function machines or flow diagrams, tables of input and output values and families of number sentences are explored. Learners' multiplication clue boards can be created as families of number sentences, using a table or a flow diagram.
8. When working with the distributive property of multiplication/division over addition/subtraction make it clear that this implies that it is possible to break up a number using adding and subtracting and then multiply or divide each part. Breaking up numbers using adding and subtracting makes the division calculation easier

(encourage learners to use multiples of ten, doubles, and halves where necessary). Breaking up can be done using expanded notation – but often there are other, better ways to break up a number that make the calculation ever easier. How best to break up a number depends on the particular numbers and the required calculation.

9. When working with the associative property make it clear that numbers can also be broken up using multiplication or division. When numbers are broken up using multiplication this is called factorising. Factorising can help with division calculations, as you can use doubling and halving to simplify the big numbers.

10. Encourage learners to become fluent in using an 'efficient chunking' algorithm which works for both short and long division. This algorithm should only be used when division calculation cannot be done using mental approaches.

    The following provides an example of efficient chunking which makes use of a multiply and subtract table:

    445 ÷ 17

    First create a doubling and times ten clue board (using number sentences, a table, or flow diagram):

    1 × 17 = 17

    2 × 17 = 34 (this is double 17)

    4 × 17 = 68 (this is double 34)

    8 × 17 = 136 (this is double 68)

    10 × 17 = 170 (to multiply by 10 you write the number with a 0 in the units column)

    Second, it may be necessary to create another clue board for the multiples of 10, with inputs 10, 20, 40, 80 and 100 that builds on the above. Some learners may not require this step.

    Third, create a multiply and subtract table by choosing sensibly sized chunks from the clue board.

Multiply	*Subtract*
20 × 17 = 340	*445 – 340 = 105*
4 × 17 = 68	*105 – 68 = 37*
2 × 17 = 34	*37 – 34 = 3*

    Fourth, add together the inputs that you have used in the multiply column, and include the remainder from what is left in the subtract column:

    445 ÷ 17 = 20 + 4 + 2 remainder 3

    = 26 remainder 3

It is of course possible to use 'efficient chunking' to add to reach a target (rather than subtract to reach 0). The following provides an example of efficient chunking which makes use of a multiply and add table:

400 ÷ 13

First create a doubling and times ten clue board (using number sentences, a table, or flow diagram):

1 × 13 = 13

2 × 13 = 26 (this is double 13)

4 × 13 = 52 (this is double 26)

8 × 13 = 104 (this is double 104)

10 × 13 = 130 (to multiply by 10 you write the number with a 0 in the units column)

Second, it may be necessary to create another clue board for the multiples of 10, with inputs 10, 20, 40, 80 and 100 that builds on the above. Some learners may not require this step.

Third, create a multiply and add table by choosing sensibly sized chunks from the clue board.

Multiply	*Add*
20 × 13 = 260	*260*
10 × 13 = 130	*260 + 130 = 390*
	390 + 10 = 400

Fourth, add together the inputs that you have used in the multiply column, and include the remainder from what you had to add in order to reach the target, from the add column:

400 ÷ 13 = 20 + 10 remainder 10 = 30 remainder 10

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ACKNOWLEDGEMENTS

This paper was made possible by a funding grant provided by Apex Hi, administered by Tshikululu Social Investment, for the Focus on Primary Maths research project which has The Grove Primary and Capricorn Park primary schools as its locus of classroom-based research.

ANNEXURE: TABLE 2

Table 2: CAPS ‘concepts and skills’ for Intermediate Phase division topics

	Grade 4	**Grade 5**	**Grade 6**
Number range for calculations:	Division of at least whole 3-digit by 1 digit numbers	Division of at least 3-digit by 2-digit numbers	Division of at least 4-digit by 3-digit numbers.
Calculation techniques	Use a range of techniques to perform and check written and mental calculations of whole number including: Estimation Breaking up and breaking down numbers Rounding off and compensating Doubling and halving Using multiplication and division as inverse operations	As for Grade 4.	As for Grade 4 and: Using a calculator
Number range for counting, ordering and representing place value of digits	[Count forwards and backwards in 2s, 3s, 5s, 10s, 100s between 0 and at least 10 000] [Order compare and represent numbers to at least 4-digit numbers] [Represent odd and even numbers to at least 1 000] [Recognise the place value please value of digits in whole numbers to at least 4-digit numbers] [Round off to the nearest 10, 100 and 1 000]	[Count forwards and backwards in whole number intervals to at least 10 000] [Order compare and represent numbers to at least 6-digit numbers] [Represent odd and even numbers to at least 1 000] Recognise the place value of digits in whole numbers to at least 6-digit numbers. Round off to the nearest [5], 10, 100 and 1 000.	Order compare and represent numbers to at least 9 digit numbers. Represent prime numbers to at least 100 Recognise the place value of digits in whole numbers to at least 9-digits. [Round off the nearest 5, 10, 100, 1 000, 100 000, and 1 000 000]
Number range for multiples and factors	Multiples of 1-digit numbers to at least 100	Multiples of 2-digit numbers to at least 100 Factors of 2-digit numbers to at least 100	Multiples of 2-digit and 3-digit numbers. Factors of 2-digit and 3-digit whole numbers Prime factors of numbers up to at least 100.
Multiplication facts	Multiplication of whole numbers to at least 10 x 10 Units by multiples of 10 Units by multiples of 100.	Multiplication of whole numbers to at least 10 x 10 Units by multiples of 10 Units by multiples of 100. [Units by multiples of 1 000 Units by multiples of 10 000]	Multiplication of whole numbers to at least 10 x 10 Units by multiples of 10 Units by multiples of 100. [Units by multiples of 1 000 Units by multiples of 10 000]
Properties of whole numbers	Recognise and use the commutative, associative and distributive properties of whole numbers	As for Grade 4, and: 1 in terms of its multiplicative property	As for Grade 5, and: 0 in terms of its additive property

	Grade 4	Grade 5	Grade 6
Solving problems	Solve problems in contexts involving whole numbers [in] financial contexts and measurement contexts Solve problems involving whole numbers including grouping and equal sharing with remainders; comparing two or more quantities of the same kind (ratio) and comparing two or more quantities of different kinds (rate)	As for Grade 4.	As for Grade 4.

Note: Additions in square brackets do not appear in the CAPS 'clarification notes' for topics on division, however they appear as part of the CAPS 'specification and clarification of content' on pp8-10. They are included here for ease of reference to allow comparison between grades.