A FEW REMARKS ON THE TEACHING OF INEQUALITIES TO GRADE TEN STUDENTS

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We remark on the teaching of linear inequalities in the pedagogic situations of schooling, reflecting on the treatment of the topic in a popular grade 10 textbook, in an example of classroom teaching of inequalities to Grade 10 students, and in an example of a grade 10 student's work on inequalities in a mid-year examination. What we find is that schooling uses a trio of implicitly-defined operations for working on the inequality symbols in expressions. Those operations are auxiliary to the operations described in the field axioms for the reals. We speculate that the production of such operations is a pedagogic effect of a general recontextualising strategy of school mathematics; namely, the replacement of the study of propositions with lists of computational rules.

INTRODUCTION

What we wish to do in this paper is to generate a few remarks on the effects of the type of recontextualising of content evidenced in the treatment of inequalities by Pike *et al*. on Grade 10 teachers and students. The particular content we focus on is that of the treatment of inequalities in Grade 10. Pike *et al.* is merely a useful (and popular) example of the type of recontextualising of mathematics contents to schooling that we are interested in exploring.

To start off, let's recall that the resources for working with inequalities stem from four axiomatic properties of the real numbers concerned with order:

(Order 1) The set of real numbers contains a subset of positive real numbers $(\exists \mathbb{R}^+ \middle| \mathbb{R}^+ \subseteq \mathbb{R})$, the properties of which are laid out in the three properties that follow.

(Order 2) The sum of two positive real numbers is positive. $(\forall x, y \in \mathbb{R}^+, (x+y) \in \mathbb{R}^+)$

(Order 3) The product of two positive real numbers is positive. $(\forall x, y \in \mathbb{R}^+, (x \cdot y) \in \mathbb{R}^+)$

(Order 4) Given any real number x, $x \neq 0$, either x is positive or $-x$, but not both. $(\forall x \in \mathbb{R} \middle| x \neq 0, [(x > 0) \vee (-x > 0)])$

Strictly speaking, we can use just two axioms if we like (see Beckenbach & Bellman, 1961, Chapter 1, and Stewart & Tall, 1977, Chapter 9).

Now consider the list of rules for working with inequalities as published in the most recent edition of the Grade 10 textbook of the very popular textbook series *Classroom Mathematics* (see Figure 1).

Rules for inequalities

1. Adding or subtracting any number on both sides of an inequality has no effect on the sign of the inequality:
   For example $2 < 3$ and $2 + 5 < 3 + 5$ and $2 - 3 < 3 - 3$
2. Multiplying or dividing both sides of an inequality by a positive number has no effect on the sign of the inequality:
   For example $2 < 3$ and $2 \times (3) < 3 \times (3)$ and $\frac{2}{5} < \frac{3}{5}$
3. Multiplying or dividing both sides of an inequality by a negative number reverses the inequality sign:
   For example $6 < 9$ but $6 \times (-3) > 9 \times (-3)$ and $\frac{6}{-3} > \frac{9}{-3}$
4. Taking reciprocals in an inequality reverses the inequality sign:
   For example $2 < 3$ but $\frac{1}{2} > \frac{1}{3}$
5. Never multiply or divide an inequality by a variable unless you know whether the variable is positive or negative.

Figure 1: Rules for working with inequalities (Pike *et al.*, 2011: 97)

The Rules listed in Figure 1 focus very specifically on the left-right orientation of inequality symbols, dispensing information about the 'reversing' effects of various operations on the orientation of the symbol. What is not made clear by the Rules is that the orientation of the inequality symbol is, ultimately, to be decided by the relation between the expressions to which it refers. That the calculus for determining 'reverse' effects offered by Pike et al. is designed to focus attention rather myopically on immediate calculations can be seen in their claim that: "Taking reciprocals in an inequality reverses the inequality sign: For example $2 < 3$ but $\frac{1}{2} > \frac{1}{3}$." So what happens when we 'take reciprocals' for the true statement, $-2 < 3$? Is it the case that $-\frac{1}{2} > \frac{1}{3}$? The process of 'taking reciprocals' in the sense proposed by Pike et al. is not an operation in its own right, yet they refer to it in a manner suggesting that it is. Rather, the specific effect of 'taking reciprocals' in the example is the outcome of multiplying both 2 and 3 by $\frac{1}{2\times 3}$, and so is, generally, subject to their Rules 2 and 3. The point to be emphasised here is that the left-right orientation of the inequality symbol is always determined by the expressions we are dealing with in each instance.

It appears to be the case that what we have in Rule 4 of Pike et al. is their attempted recontextualising of the well-known proposition stating that if $a, b \in \mathbb{R}^+$ such that $a < b$, then it is the case that $\frac{1}{a} > \frac{1}{b}$. If $0 < a < b$, then $xa < xb$ for $x \in \mathbb{R}^+$. Choosing $x = \frac{1}{a \cdot b}$ is possible since, by Order 3, $a \cdot b > 0$, and $a \cdot b > 0 \Rightarrow \frac{1}{a \cdot b} > 0$. We can

therefore write $\frac{1}{a \cdot b} \cdot a < \frac{1}{a \cdot b} \cdot b$, from which it follows that $\frac{1}{b} < \frac{1}{a}$; or, $\frac{1}{a} > \frac{1}{b}$, if we like. It should be clear from the computations entailed in the proof of the proposition

$$\forall a, b \in \mathbb{R}^{+}, (a < b) \Rightarrow \left(\frac{1}{a} > \frac{1}{b}\right) \text{ ..... (1)}$$

that 'taking reciprocals' occurs nowhere as a operation, and that the arrival at the relations between the desired reciprocals emerges from the deployment of recognisable operations. Further, an examination of the axioms for the ordered field constituted by the reals with addition and multiplication, $(\mathbb{R}, +, \times)$, does not anywhere refer to 'taking reciprocals' as an operation.

By using the order properties of the reals, we can produce a series of arguments that allow us to deduce that, given non-zero $a, b, c \in \mathbb{R} | a < b$, if $c < 0$, then $ac > bc$; and if $c > 0$ then $ac < bc$. What these propositions permit us to do, without having to rehash the proof in each instance of a statement of the kind $a < b$, is to substitute that statement by another, like $ac > bc$, under certain conditions. The two propositions become recontextualised as 'rules' by the likes of Pike et al. and others, and are offered to students and their teachers without the benefit of arguments persuading them of the mathematical necessity of such 'rules'. In the absence of such arguments the 'rules' take on the feel of operations on the inequality symbol.

INEQUALITIES: THE RULES OF PIKE ET AL.

Rules 1 to 5 of Pike et al. all refer to the effects of various operations on the inequality symbols, as though arithmetic operations applied to the expressions related by an inequality symbol have a collateral effect on the inequality symbol, either changing its left-right orientation, or leaving its left-right orientation unaltered. From one perspective, nothing of the sort happens, and the language of the Rules should be treated as merely a loose but convenient shorthand for the mathematical processes that guarantee the results. Perhaps such is the case for a mathematics adept, but it doesn't follow that the same is true for the neophyte and for many of their school teachers. We believe that it is, rather, the case that the Rules prescribe under which operational conditions the left-right orientation of the symbols in the collection $\{<, >, \leq, \geq\}$ can be altered, as though one was taking a particular symbol and rotating it through 180°. Actually, the collection of symbols should, at least, be the set $\{<, >, \leq, \geq, \not<, \not>, \not\leq, \not\geq\}$, but the last four are rarely used in school contexts, so we'll exclude them here.

Following on from the second aspect of the particular recontextualising of (1) we see in Pike et al., the third is that "taking reciprocals" is the content-specific name given to the displacement of proposition (1) and its replacement by something that is suggested as an

operation. We believe that this feature of the recontextualising of proposition (1) is an instance of a general recontextualising principle that is endemic to the pedagogic treatment of mathematics in school settings: propositions tend to be recontextualised as operations or procedures. It may even be the case that definite descriptions (definitions) are also subject to the same principle in school mathematics settings these days.

What begins to emerge now is that the Rules define and call into play a series of auxiliary operations that enable teachers and their students to generate the required operational effects on the representation of the relations of inequality between expressions, but without having to directly invoke the relevant propositions. The Rules use binary operations that take an element of $\{<,>,\leq,\geq\}$ along with a bit of information about either addition, subtraction, multiplication or division of terms, as inputs, which then generate an element of $\{<,>,\leq,\geq\}$ once again, as output. Using additive and multiplicative inverses, we can treat cases of division as multiplication, and cases of subtraction as addition, since the reals with addition and multiplication form a field, $(\mathbb{R},+,\times)$. We will, therefore, treat the set of operations defined by the Rules as referring to addition and multiplication, so that we have two operations to cover the effects of addition and subtraction on the one hand, and of multiplication and division on the other. We have also seen that Rule 4 is really an instance of multiplication by the product of the reciprocals of the related expressions, so that we do not need an additional operation to cover 'taking reciprocals'. However, since the latter is presented as a distinct operation by Pike et al., we will describe its operational features as well.

We refer to the auxiliary inequality operation that is sensitive to the multiplication of the terms in an expression of inequality by some value as $O_{\times}(symbol,sign)$, where its arguments, symbol and sign, refer to an element of $\{<,>,\leq,\geq\}$ and an element of $\{+,-\}$, respectively. The value of $O_{\times}(symbol,sign)$ is an element of $\{<,>,\leq,\geq\}$. For example, $O_{\times}(\leq,-)$ has $\leq$ and $-$ as its arguments, and its value is $\geq$; that is, $O_{\times}(\leq,-) = \geq$. The domain for $O_{\times}(symbol,sign)$ is the set obtained by taking the cross product $\{<,>,\leq,\geq\}\times\{+,-\}=\{(<,+),(<,-),(>,+),(>,-),(\leq,+),(\leq,-),(\geq,+),(\geq,-)\}$. Let $D=\{(<,+),(<,-),(>,+),(>,-),(\leq,+),(\leq,-),(\geq,+),(\geq,-)\}$ and $C=\{<,>,\leq,\geq\}$, then we can say that $O_{\times}:D\rightarrow C$. The elements of $O_{\times}$ can be listed as a set of ordered triples,

$$O_{\times}=\{(<,+,<),(<,-,>),(>,+,>),(>,-,<),(\leq,+,\leq),(\leq,-,\geq),(\geq,+,\geq),(\geq,-,\leq)\},$$

where the final member of each triple is the outcome of the operation, given the first two members of the triple as input. This covers Rules 2 and 3 of Pike et al.

We can define the auxiliary operation sensitive to the addition of terms in an expression of inequality to some value in a similar way, as $O_{+}:D\rightarrow C$. The sets D and C are defined as before and we can, once again, express the operation in full as a set of triples:

$$O_+ = \{(<,+,<),(<,-,<),(>,+,>),(>,-,>),(\leq,+,\leq),(\leq,-,\leq),(\geq,+,\geq),(\geq,-,\geq)\}.$$

This covers Rule 1.

Now for Rule 4. As we noted earlier, Rule 4 is faulty as it is stated in Pike et al. (2011). Since the rule is a recontextualising of proposition (1), we should explicitly restrict the domain for 'taking reciprocals' to $\mathbb{R}^+$. We'll refer to the auxiliary inequality operation as O_r. Now, because of the restriction of the domain for 'taking reciprocals' to $\mathbb{R}^+$, the domain for O_r is obtained by taking the cross product $\{<,>,\leq,\geq\}\times\{+\} = \{(<,+),(>,+),(\leq,+),(\geq,+)\}$. The codomain of O_r is $\{<,>,\leq,\geq\}$. As with the other inequality operations, we can represent O_r as a set of ordered triples: $O_r = \{(<,+,>),(>,+,<),(\leq,+,\geq),(\geq,+,\leq)\}$. If we let $D^+ = \{(<,+),(>,+),(\leq,+),(\geq,+)\}$, with $C = \{<,>,\leq,\geq\}$, then we can write $O_r : D^+ \to C$.

The auxiliary operations $O_\times$, O_+ and O_r are what implicitly emerge as inequality-specific pedagogic effects of a recontextualising strategy that replaces the study of propositions with the learning of lists of computational rules of the type prescribed by Pike et al. (2011). Since $D^+ \subset D$, each of the inequality operations, $O_\times$, O_+ and O_r, is a subset of the cross product $D \times C$, which is shown in Figure 2. It should be clear that the Rules cannot select any of the sixteen elements of $D \times C$ listed in the bottom four rows of the collection shown in Figure 2.

$(<,+,<),(<,+,>),(<,-,<),(<,-,>),$
$(>,+,<),(>,+,>),(>,-,<),(>,-,>),$
$(\leq,+,\leq),(\leq,+,\geq),(\leq,-,\leq),(\leq,-,\geq),$
$(\geq,+,\leq),(\geq,+,\geq),(\geq,-,\leq),(\geq,-,\geq),$
$(<,+,\leq),(<,+,\geq),(<,-,\leq),(<,-,\geq),$
$(>,+,\leq),(>,+,\geq),(>,-,\leq),(>,-,\geq),$
$(\leq,+,<),(\leq,+,>),(\leq,-,<),(\leq,-,>),$
$(\geq,+,<),(\geq,+,>),(\geq,-,<),(\geq,-,>)$

Figure 2: The collection of elements that constitute $D \times C$.

The examples provided by Pike *et al.* to explicate their Rules are instances in which the Rules apparently work, as is clearly seen in the example supplied in support of their bogus Rule 4, and the student is required to extend the Rules inductively to all similar cases—as they illegitimately do for their Rule 4. One of the effects on mathematics as constituted in pedagogic situations that accept the sort of treatment of content offered by Pike *et al.* is the problematic displacement of deductive argument by inductive generalisation, but that's not

an issue we wish to pursue in this paper, so we'll set that aside for the moment.

INEQUALITIES: A TEACHER EXPLAINS

The research literature informs us that it is often the case that students operate on inequalities as if they were equations (Kieran, 2004; Tsamir & Bazzini, 2004; Vaiyavutjamai & Clements, 2006; Ellerton & Clements, 2011). Vaiyavutjamai & Clements (2006: 131) describe their findings by saying that, when solving inequalities,

> the most common incorrect answer was the answer that would be obtained by regarding the inequality as an equation. With this strategy an inequality would be solved by treating it as an equation, except that the symbol used in the original statement of the inequality would be repeated throughout the setting out.

They also found that learners often thought that there was only one answer. They had been asked to solve for *x* and three years of learning to solve equations had told them that when one solved for *x* the answer was a number. Ellerton & Clements (2011: 399) found that "most of the prospective middle-school mathematics teachers treated inequalities as equations that happened to have inequality signs rather than equal signs." Equations were often found to serve as a prototype in the algorithms for solving linear inequalities (Tsamir & Bazzini, 2004).

This is all hardly surprising given that the rules for the transformation of expressions elaborated in schooling tend to be focused on the syntactical resources required to solve series of standard problems rather than on studying the implications for syntactical operations that flow from the mathematical relations expressed in statements. The same set of rules for working with equations is employed along with an auxiliary set of rules for 'operating' on the inequality symbol. As much is clear from the example of the teaching of inequalities we discuss now.

The data referred to in this section is part of a corpus of data collected over the past four years from an on-going research and development program involving five schools. The general problematic within which the research and development project is located is that of the constitution of school mathematics. Building on earlier work that discussed the resources used by grade 10 students to transform equations (Gripper, 2011a), and students' use of the notion of equality (Gripper, 2011b; Davis & Gripper, 2012), we turned our attention to the use of the notion of inequality. The specific focus is on the procedures for solving inequality problems that are taught to grade 10 students. To that end, we discuss an instance of a teacher explaining to his students how to solve the inequality $-5x+2\leq 12$.

First, consider an extract from the transcript from the lesson.

Teacher:	What is it that I need to get rid of which is? … A term, which is with the *x*?
Students:	Five … maybe negative five.
Teacher:	Okay. There is negative five *x* plus two, less than or equal to twelve. What must I have on this side alone? [Referring to the LHS of the expression.]
Students:	*x*.
Teacher:	*x*. Now what is together with the *x* that I need to get rid of?
Students:	Minus five. … Minus two.

Teacher: The plus two first. I need to get rid of the plus two by doing what?

Students: Minusing.

Teacher: I must subtract the two. So I have the negative five *x*. I still have the plus two, so I must subtract the two. What I do on the left hand side, I have inequality … what must I do on this side?

Student: The same.

Teacher: I must do the same. Meaning what? I must subtract two. Here I have negative five *x*. What is two minus two?

Students: Nought.

Teacher: And I still have my less than or equal to. What is twelve minus two Nancy?

Nancy: It is ten.

Teacher: It is ten. Therefore I have ten. Next. What do I do next?

Student: Divide. Divide sir. Divide by five sir.

Teacher: Is it five?

Student: Minus.

Teacher: Negative five. Minus five. Okay. Now, when we have inequalities what happens to my inequality when I divide by a negative number?

Student: The sign will change.

Teacher: The sign will change. The less than or equal to becomes a more than or greater than or equal to. Do you all understand that? I indicate that by divide by negative five. … Negative five *x* divide by negative five. Then … This is now important. … So normally we divide … The moment you do that your sign changes. Okay guys. So it is less than becomes greater than or equal to, ten divided by negative five. What happens here? Negative five cancels. *x* and … There is still greater than or equal to … negative two.

There are a couple of interesting features of the teacher's method that tell us something about the way in which mathematical expressions are regarded in such pedagogic situations. First, the procedure is focused on the *place* occupied by the inequality symbol in the expression. In terms of the rules taught, whatever appears in that place takes on a particular form subject to whatever is happening computationally. The sensitivity to the place occupied by the inequality symbol indexes the use of a particular template for the solution of problems of that type (Brodie & Berger, 2010; Dowling & Brown, 2000; Johnson & Davis, 2010).

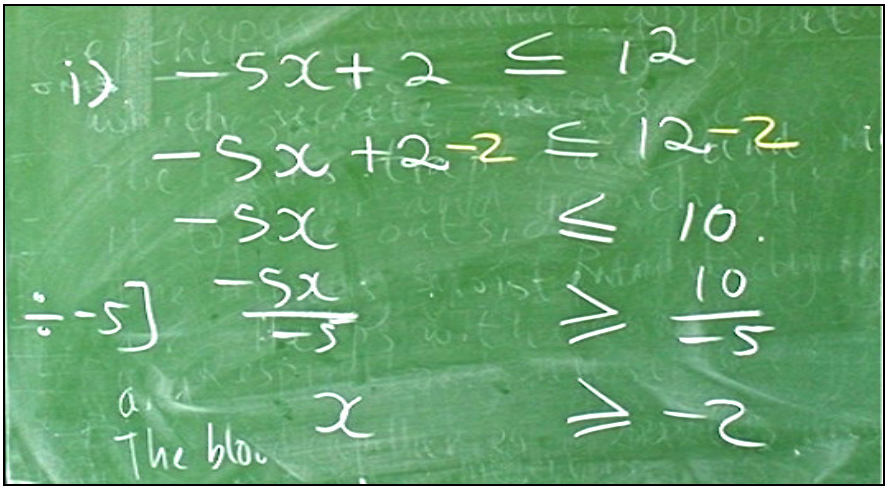

Figure 3: Teacher's solution for *x* in $-5x+2 \leq 12$.

The procedure for transforming the expression $-5x+2 \leq 12$ is pretty much the same as would be used for the solution of equations, which is not unusual, as we remarked earlier. What is of particular interest here, however, is the way in which the inequality sign is treated as a symbol that changes its left-right orientation as determined by rules of the type listed in Pike *et al*. (2011).

With reference to the subtraction of 2 from both sides of the inequality expression, the teacher concludes: "*And I still have my less than or equal to*". In this way the teacher points out that the "less than or equal to" has not changed. What this tells us is that the operation $O_{+}(symbol, sign)$ is in use.

With reference to division by -5, the teacher, echoing his student and elaborating on the student's response, says: "The sign will change. *The less than or equal to becomes a more than or greater than or equal to.* Do you all understand that?" This is a very clear expression of the use of the operation $O_{\times}(symbol, sign)$.

What we have done here is make explicit the operational features of computational resources being taught by a particular teacher to his students for the purpose of working with expressions of inequality. We do, however, believe that the teacher's strategy is not an idiosyncratic feature of his practice but an example of a more general approach to the topic, as is clearly suggested by texts like Pike *et al.* (2011) which is disseminated very widely across South African schools.

INEQUALITIES: A STUDENT EXPLAINS

With the aim of sharpening the focus of our study of the constitution of mathematics in the pedagogic situations of schooling, we conducted a series of interviews with twenty top grade 10 students across three schools about their performances on the 2010 mid-year mathematics examinations. We interviewed the two top-performing students in each of ten

grade 10 classes about their thinking in selected problems in their 2010 June exam. We chose a correct solution and one where they had made an error. For that purpose. we have selected an example of an examination script where a student, who we refer to as Lucy, has produced a fully correct solution (Figure 4), as per the teacher's memorandum of marking (Figure 5).

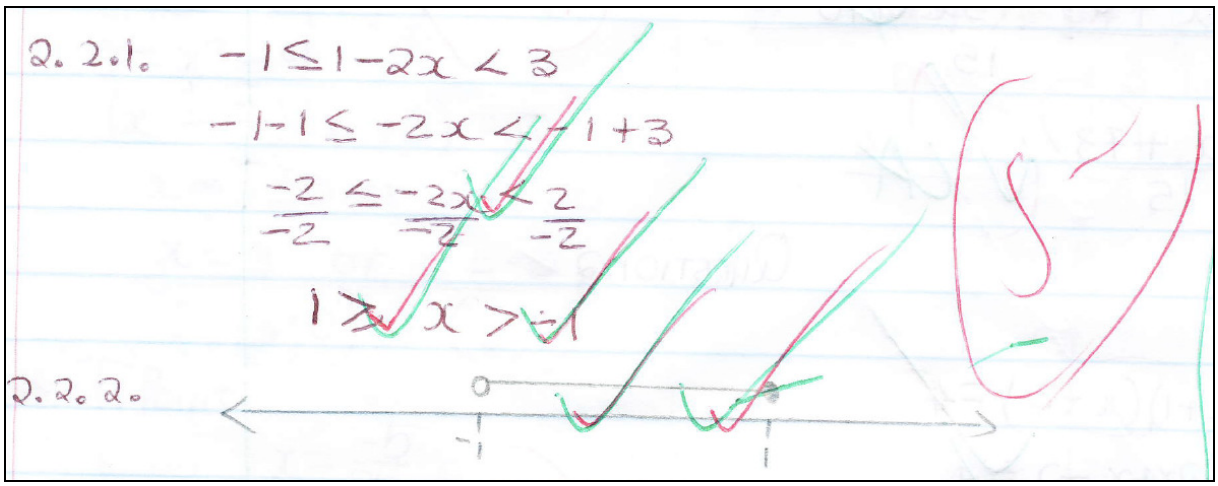

Figure 4: Lucy's solution to the inequality problem, Question 2.2.

The question Lucy worked on reads as follows:

2.2.1 Solve for x where -1 ≤ 1 - 2x < 3 (3 marks)

2.2.2 Represent your answer to QUESTION 2.2.1 on a number line. (2 marks)

Lucy described her working in the following manner:

Lucy:	When *x* is negative my inequality sign changes …
Interviewer:	Why didn't you change the sign on this line?
Lucy:	Here we wanted to get rid of the coefficient …
Interviewer:	Okay … first?
Lucy:	First. Then we changed the sign when we got *x*. Same applies here. Positive two divided by negative two, then we got negative one.

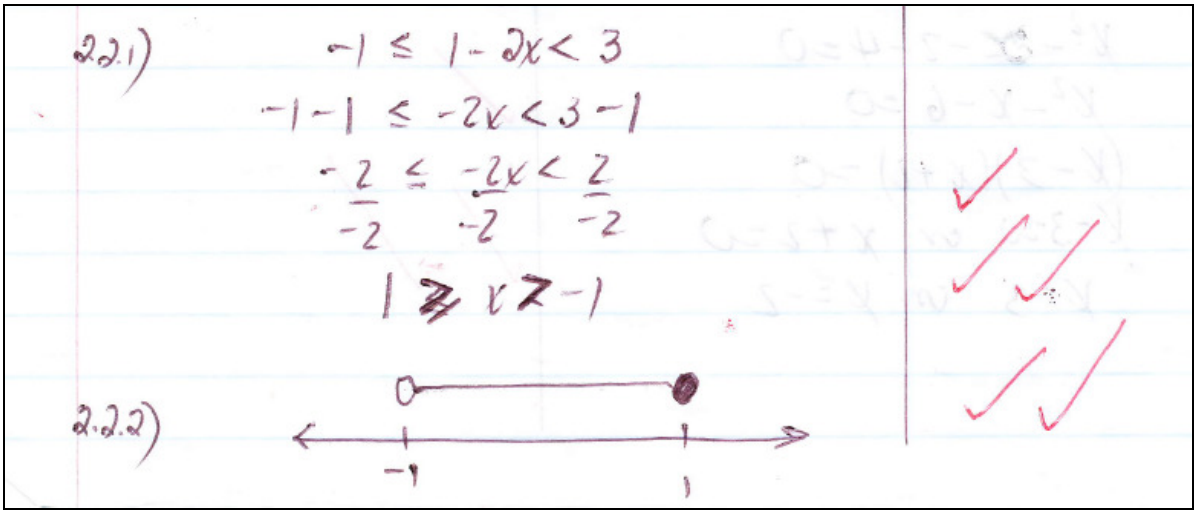

Figure 5: Teachers Memo for Question 2.2

Jaffer & Davis (2012: 111), with reference to students' solutions to the inequality problem discussed here, observed that students "were performing operations in very localised fashions and failing to use the global features of expressions as resources for regulating their operational activity." Lucy, too, fails to grasp the global meaning of the expression when she divides by negative two and produces an incorrect expression that is, nevertheless, marked as correct by her teacher. The teacher appears to encourage his students to think in the same way and almost forgets to reverse the inequality symbol, as required by the rule that has been taught (which is an application of $O_{\times}$). This suggests that the readings of expressions related as inequalities in the pedagogic situations of schooling are often not aligned with implications that flow from the order axioms for the real numbers.

The teacher's error and its repetition by one of his top-performing students suggests that the operations $O_{\times}$ and O_{+} are often applied in school settings in a manner that dislocates them momentarily from the very relations that they are intended to realise.

CONCLUDING REMARKS

When we use operations like $O_{\times}$, O_{+} and O_{r} to transform expressions we are using operations not usually associated with the field of real numbers. $O_{\times}$, O_{+} and O_{r} are part of a fairly extensive system of auxiliary operations and operation-like manipulations that we believe are routinely used in school mathematics, and which appear to be necessitated by two, inter-related features of school mathematics. The first, as we have already indicated, is that mathematics tends to be recontextualised to schooling in a manner that favours lists of rules over the study of propositions. The other feature, which requires an extended discussion not possible here, is that whatever emerges in school as mathematics is obliged to register some computational fidelity to mathematics in general. As a result, whatever does emerge in schooling as mathematics is implicitly designed to realise a reasonable degree of

structure-preservation with respect to the axiomatic foundations of the content, even if only at a local level, and the fashioning of operations of the type we encountered here is one way in which teachers and their students attempt to achieve that goal.

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