EXPLAINING EXPLAINING

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Explanation is central to the task of teaching. Gaea Leinhardt has written that "Instructional explanations are recognizable as being a part of the instructional landscape by teachers, students and observers". It is hard to conceive of mathematics teaching without it. One can surmise whether an explanation has been successful in achieving its intended purpose, or otherwise, by the response of the intended audience – usually a student, or students. Yet analysing our own explanations, or teaching someone else 'how to explain', is problematic. This paper examines some examples of explanation, from my own teaching and drawing on videotapes of other teachers, with the aim of identifying some key components of what might be called a mathematics explanation repertoire.

ORIENTATION

In most countries, primary school children are taught, at some point, a written algorithm for subtraction. The actual algorithm taught changes over time, and from place to place, but basically it will be one of two types, with minor variations in the way it is recorded.

Figure 1: Two written subtraction algorithms

In most of the UK, the first of these algorithms has come into fashion since the 1960s, and been predominant since the 1980s. It is called the *decomposition* algorithm, because the minuend (85 here) is decomposed into 70+15, so that both 30 and 7 can be subtracted to leave 40+8. The text books usually demonstrate this with Dienes multibase apparatus, beginning with 8 longs ('tens') and 5 units. One of the longs is then traded for – in effect, decomposed into – 10 units.

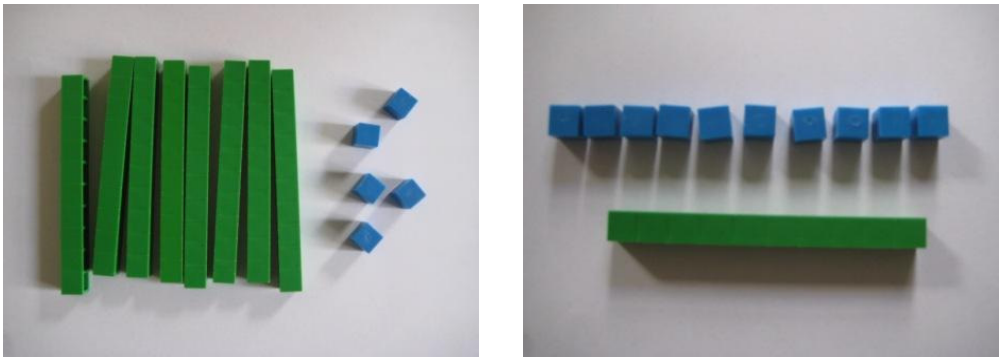

Figure 2: 8 tens and 5 units: a 'long' is decomposed into 10 units

Since the 1970s, this subtraction algorithm has gradually displaced the second one (shown on the right in Figure 1) and for some time has been taught in most primary schools in the

UK. For this reason, teacher educators include it in their methods courses at university departments of education, so that prospective teachers understand *why* it works. But what I most enjoy is when someone in the class – usually, but not always, an older student, or one from overseas – says that they do it differently. I invite them to the board, and they demonstrate the second method, invariably adding that they don't know why it works. In effect, they ask me to *explain* it.

Now, *I* know why the second method works, in the sense that I am satisfied that it will always give the right answer. The subtraction 85-37 has been replaced by (80+15)-(40+7). That is, the minuend (85) and the subtrahend (37) have both been increased by 10, so the difference remains the same. In fact, this algorithm is named the *equal-addition* algorithm. Formally, (85+10)-(37+10) = 85+10-37-10 = (85-37)+(10-10) = 85-37. But my explanation looks a bit different. I ask two students to come to the front of the class.

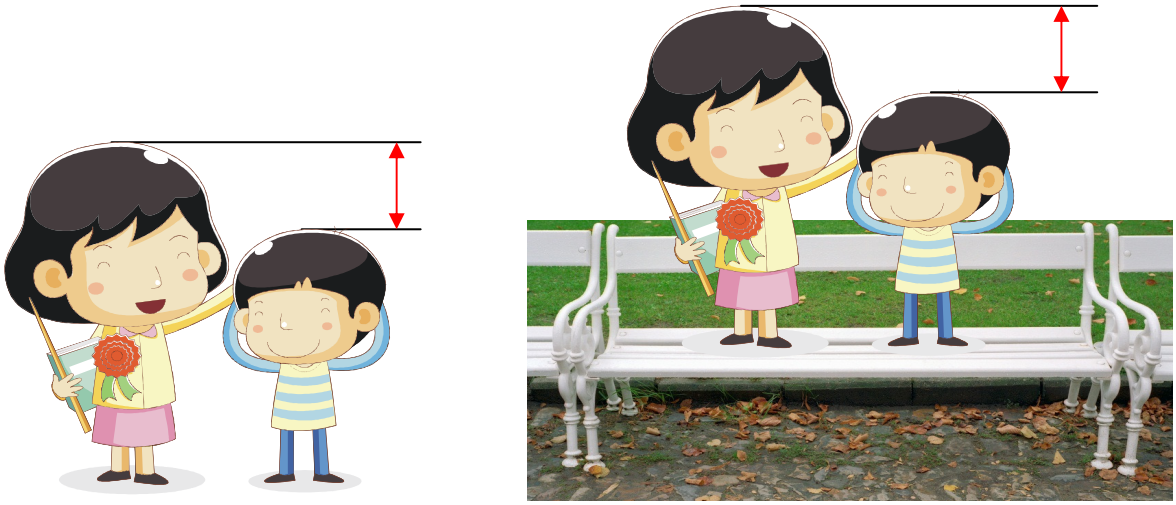

Figure 3: Alison and Bob – difference unchanged by equal addition

I'll call them Alison and Bob. I ask them to stand next to each other (Figure 3, left) and for the students to estimate the difference in their heights. I then place two identical chairs next to each other and ask Alison and Bob to step up onto them (in Figure 3, right, they are standing on the same bench). I ask the class to estimate the difference in their heights now. The 'explanation' draws on this analogy: the difference in the two heights is unchanged when they are both 'taller' by the same amount – the height of the bench. In the same way, the difference between two numbers remains the same when the same number is added to both.

More recently, I have drawn on an idea of Ian Sugarman's (2007), which begins with my asking (usually written on the board): "Write down a subtraction that is different from 643-267, but which has the same answer". The idea is that they propose things like 743-367, 646-270 and so on, and then once again the equal-addition, equal-difference connection is made and related back to the subtraction algorithm. To deviate for a moment, two common responses to this activity are of interest. First, there are always some students who actually perform the subtraction to get the difference 376, and then proceed to invent subtractions like 380-4 with the desired difference. This is an interesting case, I think, of what Kevin Collis (1975) referred to as difficulty with "lack of closure"; that is a preference for finding 'the

answer' rather than an equivalent expression. The other interesting responses are of the kind: "643-267 = 646-270 = 676-300 = 376 … so why do we need any of the vertical algorithms anyway!?" This was in fact the whole point of Sugarman's article.

Before I move on, please note that I am making no particular claims about how successful or convincing these explanations prove to be in the eyes of my students. I merely note what I do when this particular call to explain arises. I could give numerous other explanation-examples, and so could every mathematics teacher.

EXPLANATION: THE MISSING CODE?

In 2002 I undertook some empirical research into mathematics teachers' knowledge, in collaboration with colleagues in Cambridge. For details, see Rowland, Huckstep & Thwaites, 2005, or the book Rowland, Turner, Thwaites & Huckstep, 2009. Our approach to investigating the relationship between teacher knowledge and classroom practice was to observe and videotape novice teachers teaching. The participants were 12 graduate prospective ('trainee') elementary school teachers in our university faculty of education. We observed and videotaped two mathematics lessons taught by each participant. In the analysis of these videotaped lessons, we identified aspects of trainees' classroom actions that seemed to be informed by their *mathematics* subject matter knowledge or their *mathematical* pedagogical content knowledge (Shulman, 1987). We realised later that most of these related to choices made by the trainee, in their planning or more spontaneously. Each was provisionally assigned an 'invented' code, such as: 'choice of examples'; 'choice of representation'; 'adheres to textbook'; and 'decision about sequencing'. These were grounded in particular moments or episodes in the tapes. This provisional set of codes was rationalised and reduced (e.g. eliminating duplicate codes and marginal events) by negotiation and agreement in the research team. This inductive process generated 20[1] agreed codes: they are listed in Table 1, shown in the four 'dimensions' into which we grouped them later.

Dimension	Contributory codes
Foundation:	awareness of purpose; adheres to textbook; concentration on procedures; identifying errors; overt display of subject knowledge; theoretical underpinning of pedagogy; use of mathematical terminology.
Transformation:	choice and use of examples; choice and use of representation; use of instructional materials; teacher demonstration.
Connection:	anticipation of complexity; decisions about sequencing; making connections between procedures; making connections between concepts; recognition of conceptual appropriateness.
Contingency:	deviation from agenda; responding to students' ideas; use of opportunities; teacher insight during instruction.

Table 1: The Knowledge Quartet – dimensions and contributory codes

Later, when we stood back to look at this list of codes, we were surprised to realise that none of them made explicit reference to *explaining*. How could this be? Surely, whenever we

observe mathematics teaching, we are bound to witness some explanation? In her classic treatise on explanation, Gaea Leinhardt (2001, p. 338) expressed it like this: “Instructional explanations are recognizable as being a part of the instructional landscape by teachers, students and observers”. Part of the instructional landscape … so how could it be that none of us made claim to ‘explanation’ as a code when we analysed the 24 mathematics lessons? In fact, my puzzlement in the face of this conundrum began the train of thinking that led to this paper.

Ten years later, I have an answer to this conundrum, or perhaps a partial answer. I will reveal it later. The clue lies in examples of explanation above, and others, and the codes that we grouped (Table 1) under the Knowledge Quartet dimension ‘Transformation’.

REFLECTING ON EXPLANATION

You might have sensed, in my account of my explanations of the subtraction algorithms, is that there is no reluctance to explain, and no hardship in the act of explanation. What teacher can resist an opportunity to explain something to an attentive student or class? Candidates for pre-service teacher education sometimes talk at interview about wanting to “pass on” their knowledge to students. To expound something that we know to someone who does not know it, in a way that gives it meaning, to assist another person to understand it, somehow meets a very basic, fundamental need in the pedagogue. The sense of reward at the response “Oh, I see!”, and the light of insight in the eyes of the student, more than rewards our efforts, and the sense of satisfaction never dims.

When I started out as a mathematics teacher, although I never expressed it like this, I believed that the task ahead of me was to find the perfect explanation of every topic in mathematics, or at least of the topics that I would teach. Once I had found these perfect explanations, I would be properly equipped to do my job, my students *would* learn, and they *would* understand: I would have succeeded in sharing what I knew, and also the pleasure of knowing. Of course, it didn’t turn out like that. Over time I fine-honed my explanations, in the style of a one-man lesson study group, but the process turned out to be one of infinite regress. More often than not, there were students for whom my supposedly-perfect explanation misfired. Worse, an explanation that was a total success one year would go down like a lead balloon the next. My self-explanation for this, now, is that sometimes, if not most of the time, we cannot ‘pass on’ what we know and understand for ourselves ready-made and gift-wrapped. Surprise, surprise, students have to construct knowledge for themselves, and their constructions sometimes look different from our own; or even if they look the same, they only ‘work’ for the student because the edifice was their work, not the teacher’s. In a nutshell, as von Glasersfeld (1983, p. 66) says, “it appears that knowledge is not a transferable commodity and communication not a conveyance”. In fact this is very clearly exemplified by my experience (described earlier) of implementing Ian Sugarman’s “Write down a different subtraction with the same answer” approach to equal-addition. It seems to me that Sugarman’s approach is constructivist in its intent – the teacher does not ‘tell’ the students why equal-addition works, not even by analogy. S/he offers the student a

task, and asks them to make sense of it. As I remarked, this approach misfires when the student desires closure, so that their meaning-making, their cognitive construction, is not at all 'wrong', but is nevertheless incompatible with the one intended.

What is explanation?

In one of the most-cited articles in the whole of mathematics education, Richard Skemp (1976) points out that the word 'understanding' is used with at least two different meanings: whereas one entails "knowing both what to do and why", the other mere familiarity with "rules without reasons". My pejorative 'merely' serves to emphasise that the first meaning, which Skemp calls 'relational' understanding, is the one that educationists would usually associate with it. Yet the second, 'inferior' kind of understanding underlies the pupil's response "Oh, I see" to the clear exposition of a rule, an algorithm perhaps, for finding the answer to a standard type of problem.

I believe that the same potential confusion surrounds the verb "to explain", which has multiple meanings. In a book written for teachers, Wragg & Brown (1993) acknowledge this diversity of meanings in the classroom, but offer as an operational definition:

> Explaining is giving understanding to another (p. 3).

The harnessing of understanding to the definition is interesting, given Skemp's dichotomy, but Wragg and Brown proceed to indicate a variety of things that might be understood, such as concepts, procedures, purposes, cause and effect, processes and relationships. Skemp's distinction between relational and instrumental understanding is especially relevant to consideration of what it might mean to explain a *procedure*. In the case of fraction division, for example, "invert and multiply" is an explanation of sorts, but one in the category of 'rules without reason'. The distinction here is between explaining *how* and explaining *why*. In the case of a procedure (such as a subtraction algorithm), both kinds of explanation are necessary. Rather, the how explanation is essential to effective application of the procedure, and the *why* explanation supports meaningful learning and flexible application.

Chloe: subtracting near-multiples of 10

I mention here one of the lessons that we observed in the research study mentioned earlier. Chloe was teaching a class of 5 to 7-year-olds a particular strategy for subtracting near-multiples of 10, such as 9, 11, 19 and 20. The strategy was to subtract 10 or 20, and then adjust (one more or one less). The teachers' handbook presented this symbolically, for instance "70-11=59 because it is the same as 70-10-1". However, just as the lesson was about to begin, Chloe noticed a large 1– 100 square lying on a table in the classroom. She mounted it on a metallic vertical board at the front of the class, and proceeded to demonstrate the compensation strategies *spatially*, by reference to horizontal and vertical 'moves' on the grid. In particular, she demonstrated 70-19 by placing a magnetic counter on 70, at the right end of one of the rows, moving up two squares to 50, and then 'adding 1', necessitating a move to the beginning (51) of the next row. It would be fair to suggest, I think, that Chloe was aiming to explain *how* to perform this type of strategy, and also *why* it

works, by reference to the moves on the 100 square. In the event, the children found it difficult to choose the appropriate variant from the four possibilities (up/down, then left/right).

Instructional explanations and other explanations

I find the writing of Gaea Leinhardt on this topic, within and beyond the realm of mathematics, especially illuminating. In order to understand explanation in the context of teaching, Leinhardt says that we need first to distinguish it from other kinds of explanation. Leinhardt (2001) identifies four types, or families, of explanation: common explanations, disciplinary explanations, self-explanations, and instructional explanations. Drawing on Leinhardt, my account of these four types is as follows.

Common explanations are answers to why-questions, usually about everyday matters, such as: Why are you wearing a suit today? Why must I go to school? Common explanations are usually given orally, and build on a particular relationship. The success of the answer depends on the asker's evaluation, not on a set of externally-imposed 'rules'. By contrast, *disciplinary explanations* arise from enquiries embedded in a discipline, such as: Why does it get dark at night? What were the causes of the American Civil War? Why are the diagonals of a rectangle the same length? The purpose of these explanations is to establish normative truths in established domains subject to their rules of evidence and argument. The one giving the disciplinary explanation needs a good command of the relevant *syntactic* knowledge (Schwab, 1978). The third type identified by Leinhardt (2001), a *self-explanation*, is, as the name suggests, given to the self, often in response to inconsistency, puzzlement or seeming contradiction. The purpose of this explanation is to make sense of experience, and to give meaning to claims from outside. Self-explanations may be fragmentary, idiosyncratic, contextual; the criteria for success are self-determined.

In contrast to common, disciplinary and self- explanations, *instructional explanations are designed to teach*. They are pedagogical actions in response to questions (sometimes questions posed by the teacher him/herself). Leinhardt (2001, p. 340) states that they are intended to help students learn, understand, and use knowledge "in flexible and creative ways", and that "Such an explanation is complete when coherence exists among the critical components". Instructional explanations tend to be more extended, less formal, and with more redundant content, than disciplinary explanations, and to conform to general discourse conventions, with awareness of an 'audience'.

In the earlier discussion of subtraction algorithms, I gave one-line justifications for the two procedures, and these were disciplinary explanations, whereby the mathematical community can be confident in the efficacy of each procedure. Since (at some time, long-forgotten) I concocted these disciplinary explanations, they also served for me as self-explanations. The explanations with Dienes apparatus, and with students standing on chairs, are intended as instructional: they draw respectively upon a concrete *representation* and a carefully devised *analogy*, in order to render the explanations more accessible and more palatable. Both of these explanations are prepared in advance, are part of an instructional repertoire, and have been tried and tested many times. In the next section, I describe a more spontaneous explanation-response to an unanticipated, contingent question-situation.

JENNIFER: THE VOLUME OF A CONE

Our one-year graduate pre-service elementary teacher education program in Cambridge recruits around 200 students each year, organised into 7 seminar groups for many aspects of the course, and about two-thirds of the mathematics 'methods' component. Half of the year is given to practicum placements in schools. In my final seminar session earlier this year, students were working on tangram and pentomino tasks in pairs. As I stood observing one pair, Jennifer (as I shall call her) came across to me and said "I know it's not what we're working on today, but see if you can answer this: if you have a cone and a cylinder with the same base and height, what is the volume of the cone compared with that of the cylinder". In fact I hadn't had to think about that for some time, but I seemed to remember that it was a third. "Yes, it is", said Jennifer. "This came up in my last [school] placement, with a Year 5 class [age 10], but we thought it would be a half. So why is it a third?"

I paused for a moment, then suggested using a graduated cylinder, and measuring the displacement when the cone and the cylinder, in turn, were immersed in water (evidently suitable models were available at the time). Jennifer liked this idea. I did not however, because it offered no explanatory insight beyond the particularities of that cone and that cylinder. An empirical approach to this explanation falls outside the epistemological norms and expectations of mathematics. In other words, it's not a disciplinary explanation. I said to Jennifer that I'd like to think about it a bit longer, and she returned to the tangram/pentomino tasks. I was annoyed with myself for not having an answer for Jennifer – my lust to explain was not satisfied!

In fact I was confident that I could have reconstructed and presented her with an explanation from integral calculus. Consider the cylindrical slice of the cone distant *x* from the vertex, thickness *dx* (shown in side-elevation, Figure 4).

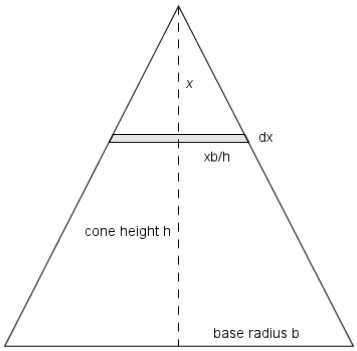

Figure 4: Side-elevation view of the cone, showing a slice parallel to the base

By similar triangles, its radius is bx/h, so its volume is $\pi(b/h)^2x^2$, and the volume of the cone is $\pi\frac{b^2}{h^2}\int_0^h x^2dx$, or $\pi\frac{b^2}{h^2}\frac{h^3}{3}=\frac{1}{3}\pi b^2h$

I judged that this was not what Jennifer was looking for, however, and sat and 'had a think' while the students busied themselves with the assigned tasks. After a few minutes I returned to Jennifer with rough sketches like the ones shown in Figure 5.

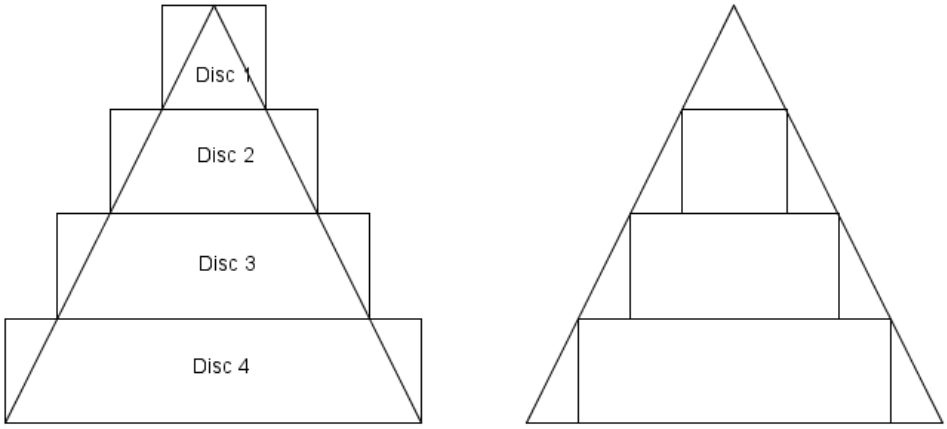

Figure 5: Side-elevation view.
Left figure: cone inside 4 stacked discs. Right figure: cone outside 3 stacked discs

Again these are side-elevation views of three-dimensional figures. Suppose the height of the cone is 4 and its base radius is also 4. Figure 5 (left) shows four discs, each of thickness 1, stacked as shown. Taken together, their volume is greater than that of the cone. In fact, the radii of discs 1, 2, 3, 4 are 1, 2, 3, 4, which follows from similar triangles, but looks very plausible anyway, and their volumes are $\pi 1^2$, $\pi 2^2$, $\pi 3^2$, $\pi 4^2$, making $(1+4+9+16)\pi$ or 30π altogether. Figure 5 (right) on the other hand, shows three discs stacked 'inside' the cone, so their total volume is less than that of the cone. In fact these three discs are the three smaller ones from Figure 5 (left), and their total volume is $(1+4+9)\pi$ or 14π. The average of these upper and lower estimates, 22π, seems like a reasonable first estimate of the volume of the cone.

In fact the cylinder with the same base and height as the cone has volume $\pi 4^3$, or 64π. To be honest, I was very surprised that my very crude estimate of 22π was so close to a third of the volume of the cylinder! Jennifer seemed quite pleased too, and asked if she could keep my rough sketches and scribbled calculations. I was only too happy to leave them with her. Five minutes later I could not resist returning to her with more scribbled calculations, this time for similar upper and lower volumes estimates for a cone now dissected into 10 slices. It made the calculations tidier if now the cone was 10 units high and its radius 10 units. The upper limit will be $\pi(1^2+2^2+\ldots+10^2)$, which is 385π. The lower limit (omitting the largest of the 10 discs) will be 285π, and the average of these two 335π. The volume of the corresponding cylinder is 1000π, and our estimate once again very close to one third of it. [As a difference, the cruder estimate of 22π with only 4 slices looks even closer to its 'actual' value of $21\frac{1}{3}$, but the *ratio* 335/1000 is closer to 1/3 than 22/64 is].

Again, Jennifer asked to keep my notes, and I was pleased for her to have them. I had the sense that what I had been able to offer her amounted to an instructional explanation as far

as Jennifer was concerned. Whether it would be helpful to her Year 5 class was another question, but I believe that sometimes explaining for others is preceded by explaining for ourselves, and that the two can be different. In fact, mensuration of cones and spheres only enters the curriculum very much later, around Year 10.

There remained one final demand for *self*-explanation on my part, resolved during a sleepless night soon afterwards. The sum ($1^2+2^2+ \ldots+10^2$) can be found using the formula $\frac{1}{6}n(n+1)(2n+1)$ for the sum of the first *n* squares, and the process previously applied to 4, 10 slices readily applied and extended to any number of slices – 100, 1000 … one would expect that even with 100, the process would yield an approximation to the volume of the cone extremely close to a third that of the corresponding cylinder.

In, fact, taking *n* slices, the quotient [volume of slices / volume of corresponding cylinder] is $\frac{\frac{1}{6}n(n+1)(2n+1)}{n^3} = \frac{1}{6}(1+\frac{1}{n})(2+\frac{1}{n})$, which can be seen to approach $^1/_3$ as *n* increases[2]. This self-explanation was important for me, but not for Year 5. I don't know about Jennifer. Only later did I recall that my 'slices' approach to Jennifer's problem recapitulated early Greek hints at what was to become integral calculus, and the limiting approach with *n* slices is more or less a Riemann integral.

KATE: EXPLAINING BY MEANS OF A GENERIC EXAMPLE

Earlier, I discussed the close relationship between explanation and understanding. In mathematics, it can also be difficult to distinguish between explanation and proof. The theorem, after all, states what *is* the case: the proof demonstrates why it must be true. De Villiers (1990) and others have drawn attention to the multiple purposes of proof, including verification and explanation. Verification relates to the function of proof which lends certainty to the truth of a mathematical claim. (This begs the meaning of 'truth', but this is not the place to discuss it.) The explanation role, on the other hand, is about understanding why something must be true, and giving insight of some kind. These two roles are easily matched, respectively, to Leinhart's categories of disciplinary explanation and instructional explanation. The first kind of proof is subservient to the edifice of mathematics, perhaps to the editor and reviewers of the mathematics research journal. The second kind recognises an obligation to the learner. Anyone who has studied mathematics at university will instantly recognise the distinction. One fruitful approach to the second kind of proof is by means of the generic example (Balacheff, 1988). Essentially, it is about 'seeing' *why* something is true in general, but through the medium of a *particular* example of it. Explanatory insight, or illumination, can sometimes be achieved by 'talking through' a *single* example, drawing attention to the *structure* of that example. If such a demonstration succeeds, the audience – colleagues, students, whoever – see why the same, or analogous, reasoning would work just as well in other cases. In a video made for the National Numeracy Strategy in England in the late 1990s, a group of 10 to 11-year-old children investigate the well-known 'Jailer Problem'[3] in small groups, before being brought together by Kate, their teacher. The solution turns out to hinge

on the fact that every square number has an odd number of factors. In fact, Kate explains this to the class by reference to a generic example, although no reference is made in the commentary to this aspect of her teaching and proof strategy. She points out that every factor of 36 has a distinct co-factor (i.e. 1↔36, 2↔18 and so on) – with the exception of 6, which remains alone, un-paired. So it must follow that 36 has an odd number of factors. Kate's choice of 36 is interesting – small enough to be contemplated with mental arithmetic but with sufficient factors to be non-trivial. Presented in algebraic generality (as a disciplinary explanation) the same argument would not be accessible to Kate's audience. Instead, her instructional explanation is, indeed, a proof that explains. For more about generic examples, see Rowland (2001); NRICH (2012).

THE MISSING CODE: DENOUEMENT AND CONCLUSION

As I wrote earlier, my colleagues and I were surprised, on reflection, that our list of codes relating to teachers mathematics-related classroom actions did not include a code for explanation. Now, I have laid the ground, or as much ground as space allows, for my own account – my explanation, in fact – of how this could have come about. In this paper, I have described some examples of explanation – my own, Chloe's, and Kate's. Looking back at these explanations, one can extract from them four characteristic ingredients or components, pedagogical devices that identify their instructional purposes and delineate what might be called an 'explanation repertoire'.

The synthesis is as follows: *instructional explanations in mathematics are characterised* by one or more of:

The use of representations: for example, my use of Dienes' multibase apparatus and Chloe's use of the 100 square, as different embodiments of the place value number system.

The use of examples: this is more-or-less universal in instructional explanations (e.g. Bills & Watson, 2008) and evident in all the example-explanations given earlier.

Reference to analogies: for example, in my explanation of the equal-addition algorithm.

Inductive, plausible reasoning: for example, in my 'volume of a cone' explanation which stood in for an otherwise-inaccessible proof by calculus.

Proof via generic examples: for example, Kate's justification of the jailer problem solution. In this case, the explanation moves beyond induction to demonstrating how a general proof would be structured.

It is also essential to my argument that (a) I could cite numerous additional examples from classroom practice, of each type; (b) I am confident that readers will recognise these same components in their own instructional-explanatory practices. At the same time, the five-item list is provisional, and open to enlargement and revision.

It should come as no surprise that the first three of these explanation strategies can be matched directly to Shulman's (1987) description of teachers' *pedagogical content knowledge* resources. Moreover, Leinhardt, Zaslavsky & Stein (1990) observed that:

> Explanations consist of the orchestrations of demonstrations, analogical representations, and examples. […]. A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases. (p. 6).

I would add that knowledge of the relevant *subject matter* is a *sine qua non*; in order to explain, for example, a procedure or to justify it, first one must know the procedure, or the justification, for oneself. This claim is inherent in what Shulman (1987) called 'comprehension', being the first stage of a six-point cycle of pedagogical reasoning.

In contrast with instructional explanations, given the rules of evidence for truth in the field of mathematics, *disciplinary* explanations are necessarily characterised by deductive reasoning. That is not to say that the above five characteristics are entirely absent in written discourse in mathematical communities, only that they are an embellishment in this context, and not a necessity. This can have well-recognised consequences for students as they make the transition from school-mathematics to lectures in university mathematics departments (e.g. Tall, 2008).

To conclude: once the five ingredients of mathematical explanation are disentangled from the narratives in which they are necessarily set, and brought into the light, the reason for the absence of 'explanation' as a code in our lesson analysis becomes apparent in the presence of the components of the Transformation dimension of the Knowledge Quartet, notably the choice and use of representations and of examples. It is also fair to remark that explaining *how* was commonplace in our 24-lesson database (corresponding to the 'teacher demonstration' code). However, explaining *why* was almost absent, although we have subsequently found it more often in our observations in secondary mathematics classes. In any case, our inadvertent deconstruction of explanation has almost certainly turned out to be more helpful and more useful to teachers planning and evaluating their own classroom explanations. The focus on representations and examples in particular has lent some transparency to this very familiar, but otherwise inscrutable, "part of the instructional landscape".

Acknowledgement and thanks

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Notes

1. In 2002 there were 18 in fact: two more were subsequently added in the light of new data.

2. This particular argument, for a cone with equal base-radius and height, is easily adapted to the general case, but the notation is then more 'messy' and outwardly complex – something I was deliberately avoiding when responding to Jennifer. I also note that the formula for the sum of consecutive squares is amenable to visual proof (Nelson, 1993): by generic example, in fact.

3. A certain prison has 100 prisoners in 100 cells, and 100 jailers. One night, when the prisoners are all locked away, the first jailer unlocks all the cells. Then the second jailer locks all the cells whose numbers are multiples of 2. Next, the third jailer changes the state of all the cells that are multiples of 3, and so on through to the 100th jailer. The jailers then fall asleep. Which prisoners were able to escape?

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