A DISCUSSION OF REFERENCES TO CANCELLATION IN THE MATHEMATICAL WORK OF A GRADE 10 STUDENT

Zain Davis

School of Education, University of Cape Town

An analysis of the use of cancellation in the work of a top-performing grade 10 student suggests that students (and possibly their teachers) routinely use operations and operation-like manipulations that are not described by the axioms for the field of real numbers. Such operations/manipulations require arguments and produce values that are not real numbers, but are used as computational resources by students (and perhaps even by their teachers) in the solutions to routine mathematics problems. Further, the absence of the explicit study and use of cancellation theorems in secondary schooling appears to be implicated in the realisation of cancellation in the form of one or more operations/manipulations rather than as merely a name for a certain class of deductions.

INTRODUCTION

In late-2010 we conducted a series of interviews with twenty top grade 10 students across three schools about their performances on the 2010 mid-year mathematics examinations (the two top-performing students in each of ten grade 10 classes). The student interviews form part of a more general study of what comes to be constituted as mathematics in the pedagogic situations of schooling in the context of South African schools. In this instance we were specifically interested in the describing the resources used by students as they went about solving problems in situations where they are unable to recruit resources from their peers, teachers and textbooks. Each student was interviewed individually and video records were made of each interview. We obtained copies of their examination scripts and confronted each student with two of the solutions that they had produced, one of which was a successful solution as determined by the teacher's memorandum of marking, and another which was an unsuccessful attempt at a solution. We first showed each student their successful solution of an examination problem and interviewed them on their production of the solution. We then showed them an unsuccessful solution attempt for a different problem and ask them to explain their working. Since the study was exploratory we had no strong preconceptions about what would emerge in the interviews. We have reported on particular features of what emerged in those interviews in some detail in a series of papers.

In this paper I wish to explore the computational effects of the idea of cancellation that emerged in the work of many of the students we interviewed. When we reviewed the students' examination scripts prior to interviewing them, cancellation was not indicated as an

obvious object of interest. It was only during the interviews that students' use of curious forms of cancellation became apparent, and it was only once the interviews had been transcribed and each of the four interviewers had read all the interview transcripts, that it became apparent that references to cancellation in schooling index the use a series of related ideas and computational resources, not all of which co-exist with mathematics in a consistent manner.

CANCELLATION IN AND OUT OF SCHOOL

Cancellation theorems abound in mathematics and are rather helpful propositions which enable us to simplify descriptions of mathematical relations by permitting the exclusion of particular bits of symbolic expressions of such relations. The group cancellation theorems are helpful illustrative examples of this point and are, implicitly, used routinely in school mathematics. Let's briefly consider the right-cancellation theorem, which tells us that if *a*, *b* and *c* are real numbers, then it must be the case that $a+c=b+c \Rightarrow a=b$. Since we are dealing with a group, $(\mathbb{R},+)$, there must exist an additive inverse for *c*, which we can indicate as $-c \in \mathbb{R}$. Furthermore, since we also have additive closure, additive associativity and additive identity available to us, it is the case that

$$\begin{aligned}(a+c)+(-c)&=(b+c)+(-c)\\ a+(c+(-c))&=b+(c+(-c))\\ a+0&=b+0\\ a&=b.\end{aligned}$$

So, having deduced that $a=b \text{ if } a+c=b+c$ when we are dealing with addition over the reals, we refer to the addition of *c* to *a* and of *c* to *b* in the original expression as having being cancelled. Usually we simply say that we cancel the *c*s in such a situation. Once we have this theorem available to us we can invoke it to replace the statement $a+c=b+c$ with the statement $a=b$, without any additional computations needing to be performed.

Now, an interesting thing about schooling in South Africa, (and probably elsewhere as well) is that many, though not all, school teachers would balk at the idea of their students solving the equation $x+2=7$ by writing the solution:

$$\left.\begin{aligned}x+2&=7 &&\text{..... (1)}\\ x+2&=5+2 &&\text{..... (2)}\\ x&=5. &&\text{..... (3)}\end{aligned}\right\}\text{A}$$

The statement (2) in solution A is the result of a transformation from (1) to produce the conditions for the right-cancellation theorem to apply, enabling us to deduce that $x=5$ by invoking the theorem. In fact, many teachers would read solution A as having a 'step' missing—specifically, one entailing the explicit subtraction of 2 from both sides of the equation. Teachers are more likely to expect their charges to write down something along the lines of the solution:

$$\left.\begin{array}{ll} x+2=7 & \ldots\ldots (1') \\ x=7-2 & \ldots\ldots (2') \\ x=5. & \ldots\ldots (3') \end{array}\right\} \text{B}$$

There are teachers and students who would justify $(2')$ in solution B by saying that they added -2 to both sides of equation $(1')$, and then there are others who would say that they took 2 over the equal sign and changed its sign from plus to minus (Gripper, 2011a, 2011b; Lima & Tall, 2008). Teachers and students who use the first procedure (entailing adding -2 to both sides of the equation) do something that shows elements of the proof establishing the cancellation theorem referred to earlier: -2 is the additive inverse of 2 and adding -2 to the equated expressions $x+2$ and 7 produces $x=5$. On the other hand, those who employ the idea of *change sides-change signs* (Lima & Tall, 2008) use a resource that behaves somewhat like an operation that produces a spatial displacement, apparently of a number, and which changes its state from positive to negative, or vice versa. Whether or not one can do such things to a number is a question that will be addressed later.

For the moment, what we wish to focus on is the proposition that, in the absence of the explicit use of cancellation theorems, teachers and their students, either, (a) reproduce essential computational features of the proofs of such theorems as part of their solution procedures to simplify mathematical expressions; or, (b) they use operations or operation-like manipulations, like the change sides-change signs rule, to do the job. At times teachers and their students use both strategies to arrive at solutions to problems.

Linchevski & Sfard (1991), Herscovics & Linchevski (1992), Herscovics & Linchevski (1994), Linchevski & Herscovics (1996), Linchevski & Livneh (1999) and Gripper (2011a, 2011b) also refer to cancellation in the manner of the cancellation theorem discussed here when working with school students on solving equations. It is, however, rather rare to see such a direct and explicit use of cancellation theorems in schooling, because when school teachers and their students refer to cancellation they often intend something slightly different—something more akin to an operation rather than the result of an explicit appeal to a proposition.

I can now describe the specific focus of this paper as one that connects with a more general interest concerned with those elaborations of mathematics in schooling that could be taken as indexing the substitution of deductions—like cancellation theorems—by operations and operation-like manipulations that, apparently, do the same job. The use of what is termed cancellation in school mathematics lessons and in the work of school students present us with situations which make it is possible to study some aspects of the phenomenon.

AN INTERESTING CASE OF INSTANCES OF CANCELLATION

Given the severe restrictions on space for short papers, I'll briefly discuss only one case here and then generate a few propositions for further exploration. I'll also insert the specific methodological resources I need as I proceed rather than lay the methodology out in fuller form in a separate section. The case I wish to discuss is derived from the work of Doris, a student at a private school, who scored 75,3% on her 2010 mid-year Grade 10 examination. The work of Doris has been chosen because it presents us with uses of cancellation that crop up in a number of the students' work. In the discussion of her failed solution to a

problem that required her to simplify the expression $2+\frac{3}{a}+\frac{4}{a-1}$ (Figure 1), Doris used of a series of computational resources, amongst which were computations that she referred to as cancellation. Doris does not distinguish between equations and other kinds of expressions, but I'll not deal with that here (but see Jaffer & Davis, 2012).

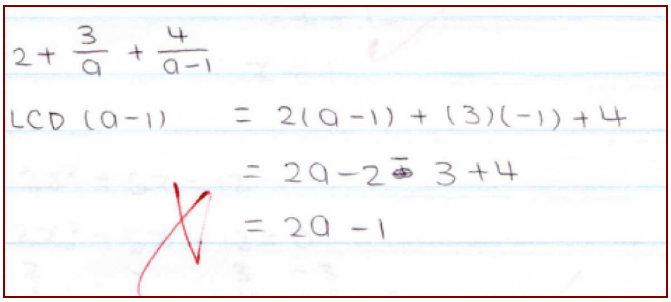

Figure 1: Doris's solution to the problem "Simplify: $2+\frac{3}{a}+\frac{4}{a-1}$".

Examine the solution presented in Figure 1 carefully and you will notice that the computation $\frac{3}{a}\times(a-1)$ gives an intermediate result of $(3)(-1)$, and also that the computation $\frac{4}{(a-1)}\times(a-1)$ gives the result 4. What is happening here? When Doris was interviewed she revealed that when she computed $\frac{3}{a}\times(a-1)$, she 'cancelled' the *a*s, so that the *a* in the expression $\frac{3}{a}$, as well as the *a* in the expression $(a-1)$, vanished, leaving her with the product $(3)(-1)$. In the case of $\frac{4}{(a-1)}\times(a-1)$, Doris argued that the expression $(a-1)$ was cancelled, causing it to vanish and leaving her with 4. When Doris was asked to compute $\frac{a}{a}$, she declared that the answer was 1. For Doris, then, $\frac{a}{a}=1$ but $\frac{a-1}{a}=-1$.

When the interviewer explored Doris's take on multiplication and division as she did computations described in terms of purely numerical expressions, Doris performed those operations as we might expect her to. For example, a look at the solutions that Doris had written out in her examination script indicated that she could use cancellation in ways consistent with multiplication and division over the reals, as well as with the rules for operating on expressions containing exponents (Figure 2).

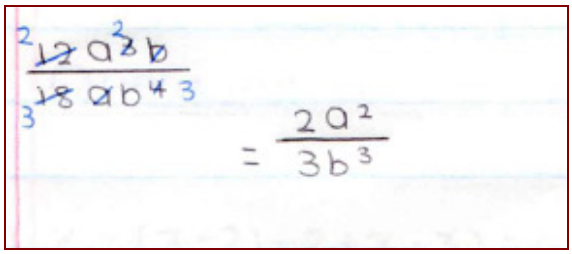

Figure 2: Doris's solution to the problem "Simplify: $\frac{12a^3b}{18ab^4}$".

In response to a request to calculate $\frac{3}{4}\times\frac{4}{1}$, Doris produced the solution 3, indicating thereby that she was able to use division and multiplication in ways perfectly consistent with mathematics, but there is some ambiguity regarding cancellation of the 4s. What are referred to as literal expressions could at times be treated differently from numerical expressions by Doris when performing arithmetic computations.

DISCUSSION

The phenomenon of so-called *anomalous cancellation* has been recognised for some time now—most famously in a paper by Ralph Boas (1979), which is an expanded version of his paper of 1972. There are many other references to the phenomenon (e.g., Schwartz & Trigg, 1961; Borasi, 1986), but I won't waste space listing all of those here. Boas (1972, 1979) and others have responded to students' 'computations' like $\frac{1\cancel{6}}{\cancel{6}4}=\frac{1}{4}$ by revealing the mathematical conditions needed to perform such cancellations legitimately. Boas derives theorems relating to a more general case—that of the conditions under which it is possible to cancel the *x* in $\frac{b^1x+b^0y}{b^1z+b^0x}$, where *b* is some base (like base 10, for example) and *x*, *y* and *z* are base *b* digits. This is all very interesting in its own right, but it does seem to register a missed encounter with something that is central to the teaching and learning of school mathematics, as we shall soon see. Where Boas and others are immediately concerned with the swift eradication of anomalies from the realisation of mathematics in schooling, and apparently intent on domesticating the activities of teachers and their students, I'd like to tarry with the negative a little while longer to see what I can learn about the constitution of school mathematics from students like Doris.

At this point I need to introduce aspects of my methodology. I describe the scriptural activity of the student, using their remarks on their work as a triangulating resource, in terms of the operations and operation-like manipulations that they appear to employ for computations, along with the collections of objects over which the operations and operation-like manipulations range. Here I am, broadly speaking, aligning myself with work in linguistics and cognitive science that subscribes to what is called the *representational*, or, *computational*, *theory of mind* (see, for example, Chomsky, 2006; Fodor, 2008; Gallistel & King, 2009), a central proposition of which is that thought and language are necessarily computational. This simply means that thought and language are taken to be combinatorial phenomena, entailing the deployment of genetically endowed generative rules for the production and transformation of thinking and its externalisation. How this proposition is played out in the description and analysis of, say, a student's mathematical work, is in the researcher's attempt to map out the operations and operation-like manipulations, along with the collections of objects that are their arguments, as apparently manifested in the written and verbal scriptural productions of that student. We have to kick off the exploration from somewhere, and the performance of the student and its products is an obvious place from which to start. In the case of Doris, I've already indicated that she employs different types of

cancellation that function somewhat like division but which are not the same thing as division.

Further, one cannot restrict the descriptive resources available to the collections of objects and the operations that range over them as formally prescribed by the topics of school mathematics. If one is interested in describing and analysing the work of teachers and students, one has to accept that there are universal features of mathematics that make it possible to realise desired mathematical outcomes in very many different ways—not only in terms of technique, but also in terms of the composition of operations and their arguments—and that teachers and students routinely exploit this characteristic of mathematics. Briefly: operations are functions; any rule that serves to realise a function can be replaced by a different rule that realises the same function; an operation can, therefore, be replaced by different rules that realise the same input-output relations as the original operation. Additionally, it may not even be the case that the operations/operation-like manipulations employed by teachers and students be those described in the axioms governing computations. For example, if we examine the work of teachers and their students in terms of the axioms for the field $(\mathbb{R},+,\times)$, which is the mathematical structure that serves to ground most of school mathematics, we find operations/manipulations and objects that are not described by the axioms, but which are routinely used in school mathematics. For example, when Doris cancelled the *a*s in her computation $\frac{3}{\cancel{a}}\times(\cancel{a}-1)=(3)(-1)=-3$, she introduced an operation-like manipulation that couldn't be aligned with multiplication and division over the reals, and which appeared to take the symbols themselves as arguments. In other words, we can argue that Doris implicitly employed a binary operation-like manipulation that takes the *a*s in $\frac{3}{a}\times(a-1)$ as arguments and which produces a void as value. Let's refer to this operation-like manipulation as CAN_V and note that it maps a pair of identical arguments to a void. So, $\text{CAN}_\text{V}:(x,x)\rightarrow void$. We also noted earlier that when Doris encountered expressions having identical numerators and denominators, like $\frac{a}{a}$, she cancelled the *a*s to produce 1 as the value. Here her cancellation functions in manner that is apparently consistent with multiplication and division over the reals. Clearly the type of cancellation used now is not the same as CAN_V. Let's refer to it as CAN_1, where $\text{CAN}_1:(x,x)\rightarrow 1$.

The objects that serve as arguments for CAN_V cannot be real numbers because one cannot do the sort of thing Doris is doing, to real numbers. She can, however, do what she's doing to numerical and algebraic *symbols*, and the operation-like manipulations she employs can exist and function alongside the operations and objects of $(\mathbb{R},+,\times)$ without causing problems in many situations. With respect to expressions like $\frac{a}{a}$, Doris could either be using multiplication/division or cancellation that takes literal symbols as arguments, but which produces 1 as value. We shall reserve CAN_1 for the latter operation and use either multiplication or division over the reals for the former. Using the same notation I used for CAN_V and CAN_1, I can represent multiplication and division as $\times:(x,x^{-1})\rightarrow 1,\ x\neq 0, x\in\mathbb{R}$

and $\div:(x,x)\to 1,\ x\neq 0, x\in\mathbb{R}$, respectively. It would appear that Doris has at least four different computations available to her whenever she believes that she's recognised an instance in which she can cancel numerical or literal terms of an expression.

CONCLUSION

We can describe the work of Doris as exhibiting at least two types of cancellation that require operational arguments that are not real numbers, and which are used alongside the usual operations described by the field axioms for $(\mathbb{R},+,\times)$. Doris's cancellations function like operations and operation-like manipulations and appear to have emerged as substitutes for the cancellation theorems in school mathematics, possibly because such theorems are not explicit objects of study in secondary schooling. This is merely one instance of the replacement of a particular class of deductions with operations and operation-like manipulations in school mathematics—there are many other examples.

We do, however, need to note that the notion of cancellation itself draws the unwary teacher and student to the symbols that make up expressions, surreptitiously suggesting that they be considered the direct objects of computation rather than merely as the markers of certain operations and arguments. In the context of secondary and primary schooling, the symbols used in mathematical expressions do appear to be used to generate a space of auxiliary operations/manipulations, all external to $(\mathbb{R},+,\times)$. Curiously, researchers in mathematics education tend to ignore the use of such auxiliary operations/manipulations and their particular arguments and values, despite the insistent productivity of such auxiliary resources in the elaboration of mathematics in school classrooms.

REFERENCES

Boas, R.P. (1972). Anomalous Cancellation, The Two-Year College Mathematics Journal, 3(2), pp. 21-24.

Boas, R.P. (1979). Anomalous Cancellation. In Ross Honsberger, (ed.), Mathematical Plums, Washington: MAA, pp. 113-129.

Borasi, R. (1986). Algebraic Explorations of the Error $\frac{1\cancel{6}}{\cancel{6}4}=\frac{1}{4}$, Mathematics Teacher, 79(4), pp. 246-248.

Chomsky, N. (2006). Language and Mind. New York: Cambridge University Press.

Fodor, J.A. (2008). LOT 2: The Language of Thought Revisited. New York: Oxford University Press.

Gallistel, C.R. & King, A.P. (2009). Memory and the Computational Brain: Why Cognitive Science Will Transform Neuroscience. Chichester: Wiley-Blackwell.

Gripper, D.B. (2011a). Describing and analysing the resources used to solve equations in a Grade 10 mathematics class in a Cape Town school. In Mamiala, T. & Kwayisi, F. (Eds.), Proceedings of the 19th Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education, North West University, Mafikeng Campus, 18 – 21 January 2011, pp. 136-151.

Gripper, D.B. (2011b). Describing and analysing Grade 10 learners' descriptions of the syntactic resources they use to transform expressions. In Venkat, H. & Essien, A.A. (Eds.), Proceedings of the Seventeenth National Congress of the Association for

Mathematics Education of South Africa (AMESA), 11-15 July 2011, University of the Witwatersrand, Johannesburg, pp. 81-92.

Grossman, A. (1924). An Analysis of the Teaching of Cancellation in Algebraic Fractions, Mathematics Teacher, 17(2), pp. 104-109.

Herscovics, N. & Linchevski, L. (1992). “Cancellation within-the-equation” as a solution procedure. In Geeslin, W. & Graham, K. (Eds.), Proceedings of the Conference of the International Group for the Psychology of Mathematics Education (PME) (16th, Durham, NH, August 6-11, 1992), Volume I, pp. 1-265 – 1-272.

Herscovics, N. & Linchevski, L. (1994). A Cognitive Gap between Arithmetic and Algebra, Educational Studies in Mathematics, 27(1), pp. 59-78.

Jaffer, S. & Davis, Z. (2012). Investigating the Relation Between top-performing Grade 10 Students’ Elaborations of School Mathematics and their Social Class Membership. In Nampota, D. & Kazima, M., Proceedings of the 20th Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education, University of Malawi, Crossroads Hotel Lilongwe Malawi, 16 – 19 January 2012, pp. 105-118.

Lima, R.N. de & Tall, D. (2008). Procedural embodiment and magic in linear equations. Educational Studies in Mathematics, 67, 3-18.

Linchevski, L. & Herscovics, N. (1996). Crossing the Cognitive Gap between Arithmetic and Algebra: Operating on the Unknown in the Context of Equations, Educational Studies in Mathematics, 30(1), pp. 39-65.

Linchevski, L. & Livneh, D. (1999). Structure Sense: The Relationship between Algebraic and Numerical Contexts, Educational Studies in Mathematics, 40(2), pp. 173-196.

Linchevski, L. & Sfard, A. (1991). Rules without reasons as processes without objects—the case of equations and inequalities. In Furinghetti, F. (ed.), Proceedings of the Conference of the International Group for the Psychology of Mathematics Education (PME) (15th, Assisi, 29 June to 4 July 1991), Volume II, pp. 317-324.

Schwartz, B.L. & Trigg, C.W. (1961). Problems and Questions: Illegal Cancellation, Mathematics Magazine, 34(6), pp. 367-368.