APPLICATION OF TRANSFORMATION GEOMETRY: THE STUDY OF REFLECTIVE PROPERTIES CONIC SECTIONS

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The purpose of this paper is to provide some insight into the teachers' knowledge on the application of transformation geometry. Some illustrative examples of the application of transformational geometry to the study of conic sections are presented. Though conic sections are not in grade 10-12 curriculum they provide high school learners with their first introduction to class of curves beyond circles and straight lines; and their reflection properties have important applications in both science and industry. In this connection two conic section properties and their proofs will be discussed, namely, the reflection property of the parabola and hyperbola. It is hoped that teachers can get more instructional insight from reading this paper.

INTRODUCTION

The introduction of the new mathematics topics in the National Curriculum Statement (NCS) poses a challenge to both teachers and learners thus contribute to weak performance at classroom level. According to Wu(2003), for teachers, this manifests itself in (i) insufficient mathematics conceptual understanding and subject content knowledge, and (ii) limited understanding of matching teaching methodologies. Transformation geometry is one such new topic in the National Curriculum Statement. Most of the teachers teaching in the secondary school have not been formally trained in the teaching of transformation geometry, let alone its application to other section of mathematics or real world situations. According to Toh (2009), how a subject is being taught in schools is largely affected by the teachers' competency of the subject knowledge.

In 2007, the Limpopo Department of Education prioritised the upgrading of the skills of the mathematics educators in the province. This was particularly exercised by the urgent need to see improved performance in Mathematics. The intention was that, in due course, teachers in the province should benefit from structured mathematics continuing professional development. Usiskin (2001, p.86) concurs when stating that in order to teach mathematics well, a teacher should know a great deal of mathematics. He refers to the entire branch of mathematics that forms what he calls "teachers' mathematics" that includes explanation of new ideas, alternative ways of approaching problems, and how ideas studied in school relate to idea learners may encounter in later mathematical study(Usiskin; 2001). The primary goal of this paper is to address the above. In fact, one of the important purposes of Mathematics in the Further Education and Training (FET) band is the establishment of proper connections between Mathematics as a discipline and the application of Mathematics in real-world contexts.

WHY APPLICATION OF TRANSFORMATION GEOMETRY?

In South Africa Transformation geometry has been introduced in the National Curriculum Statement (NCS) in 2006 as learning outcome 3 (Department of Education, 2003). Internationally there is an overwhelming agreement among mathematics educators on the importance of introducing transformation geometry into the high school mathematics curriculum (NCTM, 1989). The grade 9-12 section of the NCTM's Curriculum and Evaluation Standards For School Mathematics recognizes the importance of applying transformation geometry to high school mathematics curriculum by listing it among the "topics to receive increase attention" (NCTM, 1989). According to Usiskin and Coxford(1975),the ideas and techniques of transformational geometry can be applied to provide theorems and solving problems that cannot be so easily considered using Euclidean geometry. In this regard, Hirschhorn and Viktora(1995) mention some of the qualities that makes school mathematics desirable as follows:

- the contents are widely applicable,
- they are rich in connections to other areas of mathematics,
- they are intuitive at the outset, they are elegant in presentation,
- they are consistent with the syllabus, and
- hopefully they are interesting to learn and teach.

Transformational geometry fits such a list (Chakerian, 1992).

WHY CONIC SECTIONS?

Usiskin and Coxford (1975) suggest that the entire geometry course could be taught from transformational point of view. Furthermore, Barbeau (1988), in proving that an angle inscribed in a semicircle is a right angle, uses a rotation through 180 degree. In their article "Integrating Transformational Geometry into Traditional High School Geometry" Okolica and Macrina (1992) shows that alternate interior angles formed by parallel lines are congruent by rotating the segment of the transversal connecting the two angles 180 degree about its midpoint. None of these articles, though agreeing that applications of transformation geometry make geometry interesting and meaningful, say anything about its application to the study of conic sections.

In South Africa the mathematics National curriculum statement Grades 10–12 indicate that transformation geometry should also be used to establish other geometric properties (Department of Education, 2003). This paper is an attempt to present to teachers of high school geometry on how transformational geometry can be applied to the study of conic sections. Conic sections were chosen for this article for the following reasons.

- They are not ordinarily studied in high school mathematics,
- They have a long and rich tradition in the history of mathematics;
- Their reflection properties have important applications in both science and industry; and
- More importantly, they provide high school learners with their first introduction to class of curves beyond circles and straight lines.

The main body of the paper is, however, devoted to the presentation of two specific examples of important properties of the conic sections that can be proven using transformation geometry. It begins with the definition of axial reflection and conic sections followed by presentation of the properties with comments.

DEFINATION OF AXIAL REFLECTION

In this paper the notation "$T: X \rightarrow X'$" will be used to mean that T maps X to X' where X' does not denote the complement of X but rather the image of X under the transformation T. The most basic transformation of all, the axial reflection, has a number of striking applications. We define axial reflection as follows (figure 1):

Suppose ℓ is a given line in a plane π. By the <u>axial reflection</u> of π in ℓ is meant the transformation

$[\ell]: \pi \rightarrow \pi$ given by:

(i) If X is a point on ℓ, then $[\ell](X) = X'$

(ii) If X is a point not on ℓ, then $[\ell](X)$ is that unique point X' in π such that ℓ is the perpendicular bisector of the segment XX'

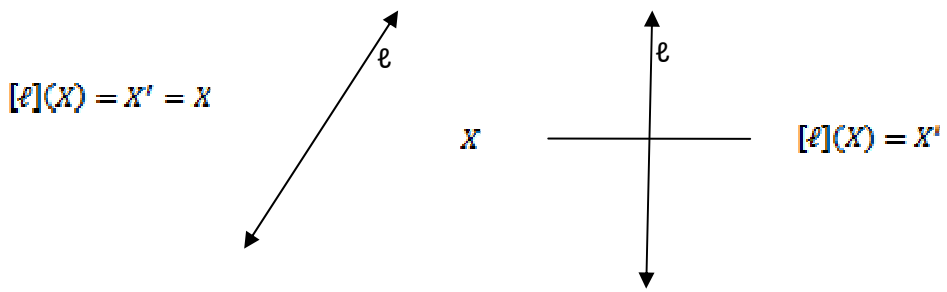

Figure 1

The above definition may be regarded as describing the <u>mirror image</u>. If for instance, the line ℓ is thought of as an ordinary plane mirror and X an arbitrary point, then X' will be the image

of X in the mirror. In the light of the discussions above, it is to be noted that every point X has a unique image X'.

THE CONIC SECTIONS- TWO ALTERNATIVE DEFINITIONS

Historically conic sections were first obtained as the intersections of a plane with a right circular cone. By changing the angle and location of intersection three general cases are obtained, namely, the ellipse, the parabola or the hyperbola. However, there is also a planar definition of conic sections. For planar definition, we suppose π as a given Euclidean plane, and that all points referred to are in π. Following are planar definitions of parabola and hyperbola.

(i) A **parabola** is defined as the set of all points P which are equidistant from a fixed line ℓ (the directrix of the parabola) and a fixed point F (the focus of the parabola). In other words, a parabola is the set of all points P satisfying: ***d(P, ℓ) = d(P, F).***

(ii) A **hyperbola** is the set of all points P for which the absolute value of the difference of the distances of P from two distinct fixed points F and F' is a constant. In other words, a hyperbola is the set of all points P satisfying: $\|(d(P,F) - d(P,F')\|$ **=2a** for some constant 2a. The points F and F' are called the foci. Note for the hyperbola
$d(P,F') < 2a$, Why? (The midpoint of the segment FF'.)

We assume that high school teachers are familiar with the concepts of distance, segments, midpoints of a segment, perpendicular bisector of a segment, angle, congruent angles, segments and triangles, etc. In fact, high school teachers are already exposed not only to Euclidean geometry but also to the concept of functions (i.e., domains, range, etc.). Then, using this knowledge, transformation such as axial (line) reflection can be defined and understood without much difficulty. Indeed, this is the transformation given primary consideration in this paper.

In all our discussions in this paper we shall assume the following properties of reflections:

- Distance is preserved by a reflection
- Lines go into lines under reflections
- Angle measures are preserved under reflection
- If X goes into X' by reflection $[\ell]$, then X' goes into X,
- All points on line ℓ are fixed under $[\ell]$.

All properties above can be proved easily on the basis of facts about congruence as activities in the class. Following are the examples of applications of transformation geometry.

EXAMPLE1: APPLICATIONS OF AXIAL REFLECTION TO THE MINIMUM PROBLEM

The idea of an axial reflection is significant in finding the point that gives the shortest path between two points on the same side of the line. However, the solution to this shortest path problem cannot be easily found by an ordinary Euclidean geometry argument, or even by an elementary application of calculus. We will see how this problem can be solved using transformational geometry.

Consider two points A and *B* on the same side of a line ℓ. The problem is to find a point *X* on ℓ such that the distance from *A* to *B* via *X* is a minimum, that is, $|AX + XB|$ is a minimum for all points *X* on ℓ. As shown in figure 2

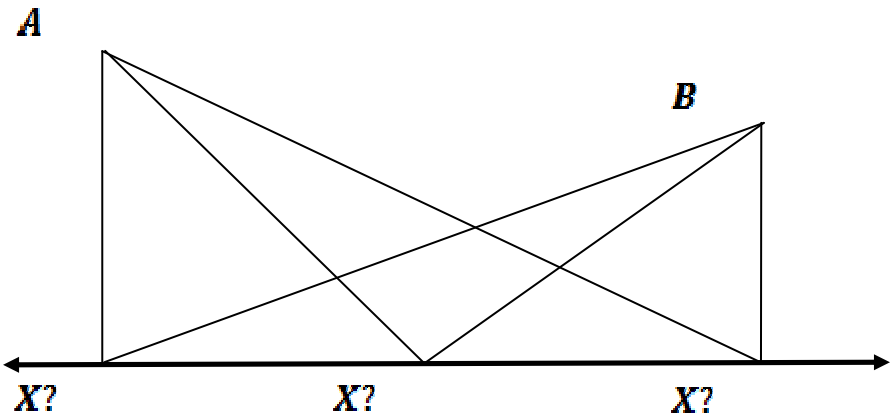

Figure 2

Solution: Let $B' = [\ell](B)$ be the reflection (Image) of B in the line ℓ , and construct AB. Let $M = \ell \cap AB'$, as shown in the figure 3 below. Then M is required minimum point.

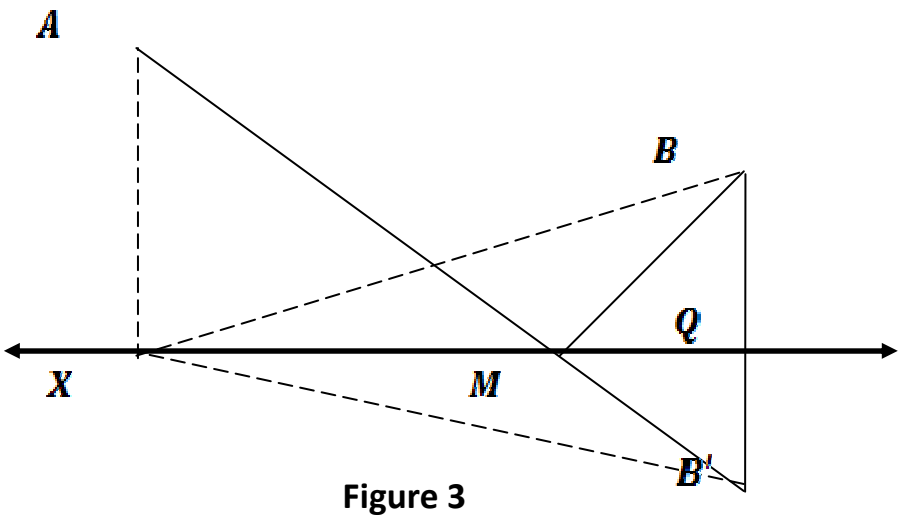

Figure 3

Proof: Suppose X is any other point on ℓ not equal to M. Consider $\Delta XB'A$. $AX + XB' > AB'$. But $AB' = AM + MB'$. Furthermore, $MB = MB'$

[$\Delta MBQ \cong \Delta MB'Q$] Hence, $AB' = AM + MB$. Also $XB' = XB$ [B' is a reflection of B in the line ℓ, so that $\Delta MBQ \cong \Delta MB'Q$]. Therefore, $AX + XB > AM + MB$. Hence M is a required minimum point on ℓ for A and B.

Comments:

Teachers may be asked to show the following as activities in the class:

(a) To prove the converse with the same setting as for the minimum problem

(b) The minimum point M is the unique point on ℓ where angle α (of incidence) is congruent to angle β (of reflection).

(c) To prove that if A and B are two points on opposite sides of a line ℓ such that the perpendicular distance between A and ℓ is not equal to the corresponding distance between B and ℓ then there is a point X on ℓ such that $|AX - XB|$ is a maximum for all points X on ℓ.

(d) The maximum point M is the unique point on ℓ where angle α (of incidence) is congruent to angle β (of reflection).

EXAMPLE 2: APPLICATION OF AXIAL REFLECTION TO THE REFLECTION PROPERTY OF THE PARABOLA

Let P be a point on the parabola ρ with focus F and directrix d, and let t_p be a line tangent to the parabola at P, that is t_p touches the parabola at the single point without crossing the parabola. Then the ray PG through P perpendicular to d (where G is in the interior of the parabola ρ) and the line segment FP will form congruent angles with the tangent line t_p to the parabola ρ at P.

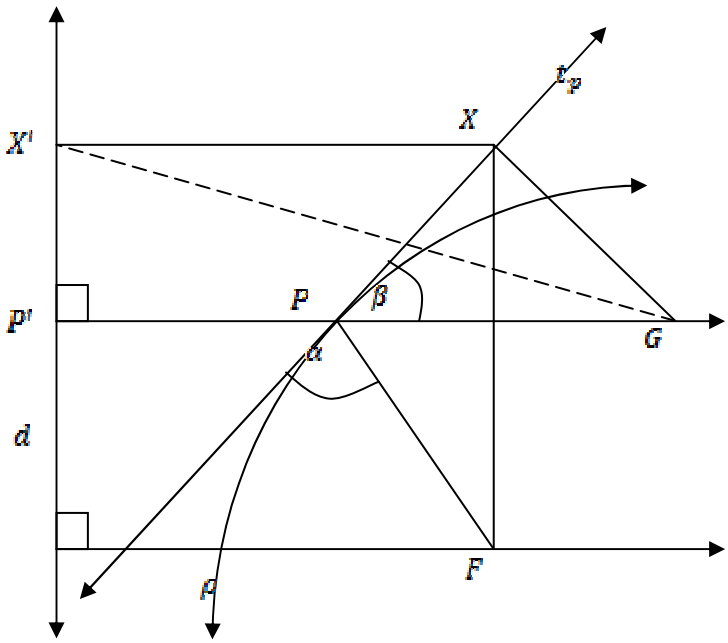

Figure 4

Proof: Suppose P' is the foot of P on d and that G is any point such that

$P' - P - G$. To show that angle α (of incidence) is congruent to angle β (of reflection). It suffices to show that P is the minimum point on the tangent with respect to the two points F and G lying on the same side of t_p .To do this let X be any other point on the tangent t_p not equal to P, and let X' be the foot of X on d. We want to show that $FX + XG > FP + PG$. Since X is in the interior of the parabola

$FX > XX'$. $FX + XG > XX' + XG$. $XX' + XG > X'G$ [Triangle inequality theorem on $\Delta X'XG$]. Therefore, $FX + XG > X'G$. $X'G > P'G$ [$X'P'G$ is a right angled triangle]. FX +XG > P′G. FX +XG > P′P +PG [Since P′-P-G, we have

P′G= P′P +PG]. , FX +XG > PF + PG [Since P is on the parabola, P′P = PF]. Then, P is the minimum point on t_p with respect to F and G. Thus angle α (of incidence) is congruent to angle β (of reflection).

EXAPLE 3: APPLICATION OF AXIAL REFLECTION TO THE REFLECTION PROPERTY OF THE HYPERBOLA

The reflection property of the hyperbola, emerge as the result of two cases, depending on its branches. However, only one case will be presented here. The other case will be treated as an exercise by teachers.

Property: Given a hyperbola **H** $(F, F'; 2a)$ with foci F and F′ and a constant difference 2a. Then for any point P on the hyperbola, the two focal radii FP and F′P form congruent angles with the tangent t_p to the hyperbola at P. Note that F and F′ are on the opposite sides of the tangent to the hyperbola as shown in figure 5. Let us consider a hyperbola **H** $(F, F'; 2a)$ and point P on the right hand branch of the hyperbola **H**. Let t_p be the tangent to the hyperbola at P, and suppose X is any point on the tangent line t_p not equal to P. We want to show that angle α (of incidence) is congruent to angle β (of reflection). We will do this by showing that P is the maximum point on t_p with respect to the foci F and F′. To this end, let X be any other point on the tangent t_p not equal to P. Let W = $F'X \cap t_p$ as shown in figure 5.

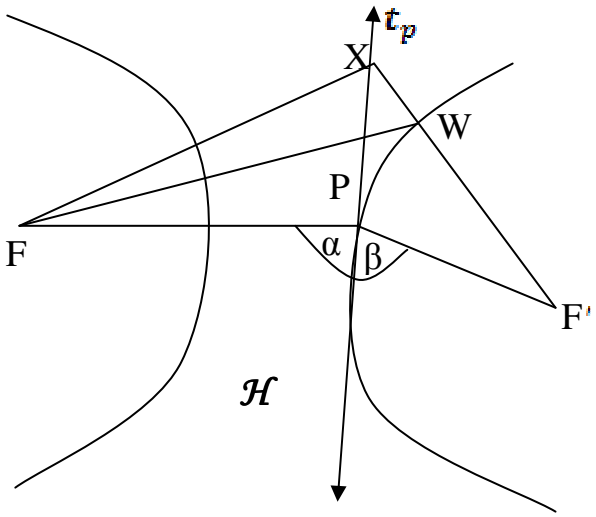

Figure 5

Since X is in the interior of the hyperbola then $F' - W - X$. Hence

FX-F'X=FX-XW-WF' since *F'X=XW+ WF',* as we have the relation

$F' - W - X$. FX-XW< WF, thus triangle inequality theorem on ΔFWX. Therefore, FX-XW-WF' < FW- WF'. Hence, FX-F'X< FW- WF'. However, FW- WF' $= 2a$. Since W is on H, by the planar definition of the hyperbola. Further, FP-F'P= $2a$, since P is also on H. Therefore, FW- WF'= FP-F'P. Hence, FX-F'X< FP-F'P. It follows from that since F and F' are on opposite side of t_p, P is the maximum point with respect to F, F' and t_p. Hence, as noted above the angle α (of incidence) is congruent to angle β (of reflection).

Comments

As an activity teachers may be asked to show the following:

If **H** $(F, F'; 2a)$ is hyperbola and point *P* is on the left hand branch of the hyperbola H, and suppose t_p is the tangent to the hyperbola at P with X on the tangent line t_p not equal to *P*, then angle α (of incidence) is congruent to angle β (of reflection).

CONCLUSION AND RECOMMENDATIONS

Since concepts of transformation geometry are familiar to learners in grade 10-12(Department of Education, 2003). The application of these concepts to other sections of

mathematics might provide an opportunity for effective teaching and learning. According to Adler (2002) it is critical for secondary mathematics teachers to have strong mathematical knowledge let alone its application. With respect to learners, they often see mathematical topics in isolation. Learners fail to see how a topic learned in their mathematics classroom can be applied to enrich understanding in another domain. Transformational geometry assists in eliminating some of this isolation as per examples illustrated in the sections above. Similar examples might have tremendous potential for making geometry an interesting and challenging subject for both teachers and learners.

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