A CRITICAL ANALYSIS OF A TEXTBOOK ACTIVITY: THE TEACHING AND LEARNING OF ADDITION AND SUBTRACTION OF INTEGERS IN THE SENIOR PHASE

Stanley A. Adendorff

adendorffs@cput.ac.za

Abstract

The teaching and learning of the addition and subtraction of integers in the first year of Senior Phase mathematics (Grade 7) remains problematic. Many learners, even up to grade 10 also experience problems with this. Finding appropriate ways to teach this meaningfully in order to enhance understanding, is not always easy. In this paper I critically look at a context used in an activity on the addition and subtraction of integers, found in a Dutch text that relates to the process of mathematizing. This involves the translation of real life mathematical content extracted from reality into mathematical problems. The nature of the activity is analysed in terms of four levels of abstraction, namely the formal, general, referential and situational. I also share experiences that relate to ways of co-ordinating the activity and how learners generally interact with the context, concrete equipment and underlying mathematical content.

Introduction

The teaching of addition and subtraction of integers in Grade 7 (the first year of Senior Phase mathematics) in Western Cape schools in South Africa, have always been problematic. Many learners, even up to grade 10 also experience problems with this. This is simply because most learners never actually grasped the concepts and procedures involved. In order to make the topic more practical and to enhance sense-making, teachers make use of the horizontal number line, they use contexts related to golf, temperature scales on weather charts and height above sea-level among others.

As is the case with some other mathematical content as well, it is sometimes not easy to find relevant or meaningful contexts or models that would facilitate understanding and learning of particular concepts. One of the contexts that I came across in a Dutch mathematics textbook (Van den Born et al, 1998) works extremely well. I have used this method with many learners at different schools in the Western Cape Province of South Africa (such as Willemsvallei Primary School (PS) in Porterville, Laurie Hugo PS in Moorreesburg, Uitkyk PS Vredendal-Noord PS, IDMkize High School (HS) in Gugulethu, Masiphumelela HS in Fish Hoek, etc), and learners almost always grasp the concepts involved and apply them successfully. Through small group

work and proper facilitation learners usually get it right, and manage to complete most of the related exercises on their own.

Discussion in a mathematics education context

Realistic Mathematics Education

This discussion is situated in the realm of Realistic Mathematics Education (RME) and deals with one of the key principles on which it is based, namely that of "emergent modelling" (Drijvers, 2003, p. 52) or also referred to as "self-developed models"(Gravemeijer, 1994, p. 91). Realistic Mathematics Education originated with Freudenthal's (1983) view of mathematics as a human enterprise which essentially deals with the process of mathematizing, According to Gravemeijer (1994, p. 91) this is the act of translating real life mathematical content extracted from reality into mathematical problems. The emphasis is thus on mathematizing mathematics subject matter taken from learners' own mathematical experiences and activities. There is a deliberate effort to refrain from presenting mathematics as ready-made or pre-packed facts and laws (Gravemeijer, 1994; Barnes, 2005) to learners to passively acquire and apply.

The other three additional principles on which RME are based are the following:

> (1) Guided reinvention and progressive mathematization, according to which learners should be afforded the chance to experience a process similar to that by which a given mathematical topic was invented (Freudenthal, 1973, 1983; Gravemeijer, 1994). Progressive mathematization refers is the process by means of which mathematical knowledge is developed by progressively formalizing informal but meaningful strategies (Drijvers, 2003, p. 52; Gravemeijer, 1994, p. 90).
>
> (2) The notion of didactical phenomenology was developed by Freudenthal (1983).Situations applicable to particular mathematical aspects are probed for two reasons, namely it addresses the "relation between thought object and phenomenon from the perspective of teaching and learning" as well as "the question [of] how, mathematical 'thought objects' can help organising and structuring phenomena in reality" (Drijvers, 2003, pp. 52-53); and
>
> (3) Horizontal and vertical mathematization is considered the fourth principle. According to Drijvers (2003, p. 53) the former "concerns organizing, translating, and transforming realistic problems into mathematical terms", whilst the latter deals with "reflection on the horizontal mathematization from a mathematical perspective", by "developing a framework of mathematical relations".

Modelling to facilitate understanding

A very common problem for teachers of mathematics is to find appropriate ways to teach abstract mathematical knowledge and concepts. According to Gravemeijer (1994, p. 77) "In the mainstream information processing approach, one usually presents concrete models" to assist learners in acquiring this abstract knowledge. However, he warns that concrete "in the sense of tangible does not necessarily mean 'concrete' in the sense of making sense", and substantiates his stance by saying that "… even if a certain mastery of procedures is attained, their application appears to be problematic. The manipulatives approach fails, probably because – although the models as such may be concrete – the mathematics *embedded in the models* is not concrete" for the learners.

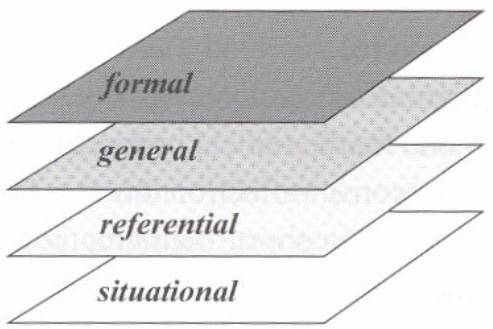
In this activity I make use of a concrete model (transparent plastic; blocks, cylindrical cardboard thermometer) where an "artificial environment is created" (Gravemeijer (1994, p. 78) and which compels learners to focus on the mathematical content at hand. To a large extent the activity, based on the teaching model put forward by Brown, Collins and Duguid (1989, p. 32) which holds that the knowledge encountered by learners is "… situated, being in part a product of the activity, the context, and culture in which it is developed and used". This activity thus aims to develop knowledge and skills in an applied situation (Gravemeijer (1994, p. 81). The successes experienced with this activity can be ascribed mainly to the use of the wooden blocks in combination with a 'modified' number line, since the concrete approach facilitates understanding and assist learners in making sense (Gravemeijer, 1994, p. 77). Furthermore, the mathematics contained in manipulating the blocks and number line is experienced as concrete.

It needs to be stressed that contexts involving negative numbers cannot actually be realistic, since negative numbers can be seen as a mathematical creation to serve a particular mathematical need. The context in which the activity occurs might not be experienced by some learners as being totally realistic. In this case it deals with a woman who brews potions which could either be medication, love potions or poison and who regulates the temperature inside the pot by adding or removing negative or positive blocks. Learners who have been exposed to traditional healers and herbal medication may identify closer to the context used in the activity, whilst others may experience it as fiction.

The nature of the activity would compel learners to grow more self-reliant and only turn to peers "for validation of their answers or for the directions for … solution procedures" (Gravemeijer (1994, p. 89). Upon scrutiny of the activity that appears below, it is possible to distinguish four (progressive) levels of abstraction as put forward by Gravemeijer (1994, pp. 101-102), namely:

- *the level of the situations* (situational)*, where … situational knowledge and strategies are used within the context of the situation …* [Compare Exercise 1 and 2 below]
- *a referential level, where a mathematical focus on strategies dominates the reference of the context* [Compare Exercise 3 below]
- *a general level, where a mathematical focus on strategies dominates the reference to the context* [Compare Exercise 4 below]
- *the level of formal [manipulations], where one [only] works with conventional procedures and notations* [Compare Exercise 5 below].

Diagrammatically the relationship between the four levels (reflecting vertical mathematization) referred to above is represented as follows (Drijvers, 2003, p. 54; Gravemeijer, 1994, p. 102):



The activity below is thus essentially based on a model which is "context-specific" since it refers to a "meaningful problem situation" (Drijvers, 2003, p. 54) that should be experientially real for the learner. Through the process of engaging and interacting with the model, it "gradually acquires a more generic character and develops into a model for mathematical reasoning that is possible because of the development of new mathematical relations that the model starts to refer to" (Drijvers, 2003, p. 54).

The nature of the activity itself

The whole activity is based on the following handout that learners receive at the outset of the lesson. It is taken (and somewhat adapted) from the text written by van den Born W. et al (1998). The idea was developed by Professor Hans Freudenthal of the Netherlands. It more or less reads as follows:

You visit a woman who uses a big black pot in which to brew a potion. She uses wonder cubes to regulate the temperature inside the pot. When there are the same amount of positive (hot) and negative (cold) wonder cubes inside the pot the temperature is 0 ^{0}C. When she adds 8 hot cubes,

the temperature increases to 8 ^{0}C. If she wants to lower the temperature by 5 ^{0}C, she adds 5 cold cubes.

1. One day she starts by placing 2 hot and 2 cold cubes inside the pot..

1.1 The temperature inside the pot is now ..... ^{o} C

1.2 The woman then adds another 4 cold cubes. The temperature changes to .... ^{o} C

2. On another day the woman makes a fresh brew and starts regulating the temperature by putting 12 hot and 7 cold cubes inside the pot.

2.1 The temperature inside the pot is now .... ^{o} C.

2.2 She then puts 8 hot cubes inside the pot. The temperature now becomes .... ^{o} C.

3. Complete the table below:

Starting temperature	Inside the pot is placed	Colder or hotter	The temperature now is
a. 2 ^{o} C	2 cold cubes		……..
b. -6 ^{o} C	3 cold cubes		……..
c. 7 ^{o} C	9 cold cubes		……..
d. -12 ^{o} C	5 hot cubes		……..
e			

4. Complete the following table by filling in the missing information (First complete one row before starting with the next):

Starting temperature	Inside the pot is placed	End temperature	Addition (summarised)
5 ^{o} C	6 hot cubes	11 ^{o} C	a. 5 + 6 = …
5 ^{o} C	7 cold cubes	…	b. 5+ -7 = …
…	9 hot cubes	…	c. -3 +9 = …
4 ^{o} C	9 cold cubes	…	d. 4 + …..

…	…	…	e. -6 + 8 = …
-2 ° C	4 cold cubes	…	f. …….

5. Add up the following.

5.1 -3 +4 = ….
5.2 –2 + (-13) = ….
5.3 –6 + 4 = ….
5.4 4 + (-9) = ….
5.5 -5 + (-15) = ….
5.6) 8 + 8 = ….
5.7) -8 + 8 = ….
5.8) 8 + (-8) = ….
5.9) -8 +(-8) = ….
5.10) -8 + 0 = …

6. It also happens that the woman has to change the temperature by taking blocks out of the pot when it is too full! Luckily she has a long tongs with which to take cubes out of the pot.

6.1 The woman takes two hot blocks out of the pot. Does the temperature drop or does it increase. ……..

6.2 What happens with the temperature if the woman takes 5 cold blocks out of the pot? ……

7. On a particular day the temperature in the pot is 10°C. She uses the tongs to remove 4 hot blocks from the pot.

7.1 What is the temperature now? ……. °C.

7.2 The temperature in the pot is changed to –5 °C. The woman removes 2 cold blocks. The temperature now becomes …… °C.

7.3.1 When the temperature inside the pot reaches –8 °C, she takes 5 hot blocks out. The temperature now changes to ……. °C.

8. Give 2 ways in which the woman can lower the temperature inside the pot by 4 degrees:
Method 1……..……..

Method 2:……..……..……

9. Complete the following table:

Beginning temperature	Out of the pot she takes	Colder or hotter?	End temperature
8 °C	4 hot blocks	Colder	……..
10 °C	3 cold blocks	……	……..
-2 °C	6 hot blocks	……	……
-7 °C	8 cold blocks	……	……
-10 °C	4 cold blocks	……	……

10. Complete the following table:

Beginning temperature	Out of the pot she takes	Colder or hotter?	End temperature	Subtraction (summarised)
-5 °C	3 hot cubes	Colder	……..	-5 – 3 = …
-6 °C	8 cold cubes	hotter	……..	-6 – (-8) = ..
……	……..	……	……	3 – 7 = …..
……	5 cold blocks	……	……	-2 - …
……	……..	hotter	……	-13 – (-6) = ..
……	……..	……	……	4 – (-9) = …

11. Subtract the following. Think about questions 9 and 10 above:

11.1 5 – (-8) = …….. 11.2 -4 – 8 = …….. 11.3 -1 –(-1) = …….

11.4 -15 – (-9)= …….. 11.5 -10 – (-9) = …….. 11.6 0 – (-8) = ……

11.7 0 – 5 = …….. 11.8 23 – (-23) = ……

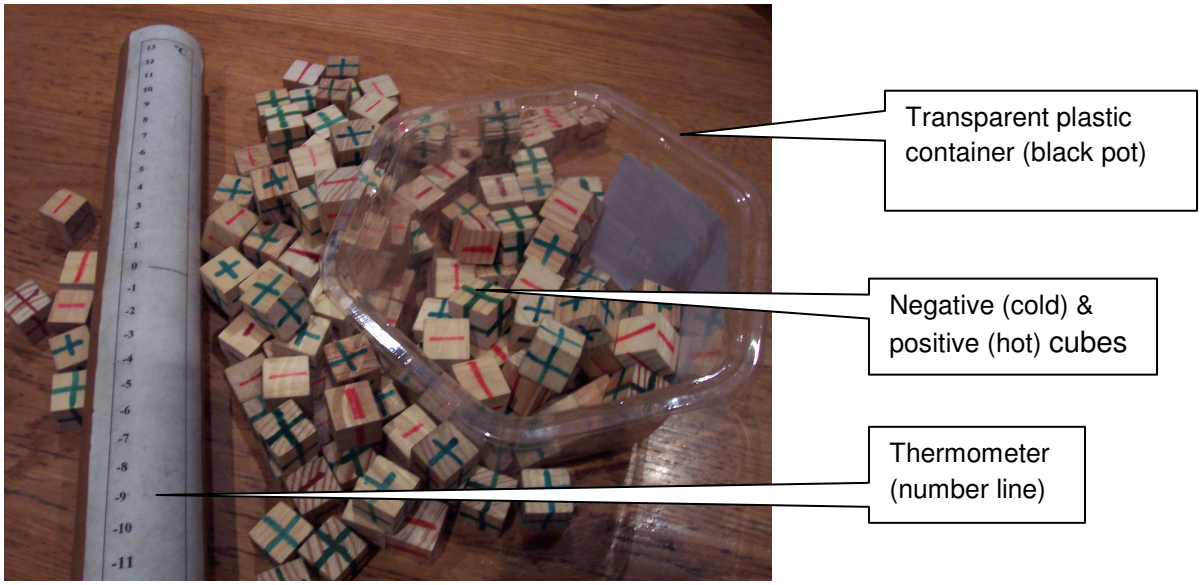
Coordinating the activity

The lesson is normally preceded by a baseline assessment including questions such as the following, written on the writing board to ascertain (through baseline assessment) whether learners do indeed have a problem with adding and subtracting integers.

+7 + (+4) = ……..; +7 + (-4) = ……..; -7 + (+4) = ……..; -7 + (-4) = ……..;

+7 - (+4) = ……..; +4 - (+7) = ……..; -7 - (-4) = ……..; -4 - (-7) = ……..;

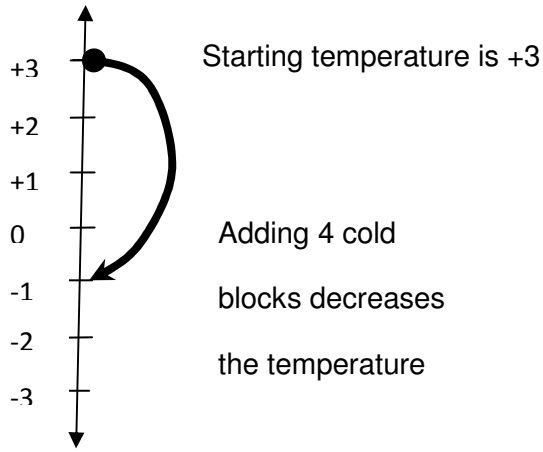
If learners struggle or cannot give the answers to most of the questions, the class is divided into smaller groups of 5 to 8 learners, depending on the class size. The activity works well in both small and large classes. Each group receives about 20 negative wooden cubes and 20 positive ones; a transparent plastic lid from a used cake container, and a thermometer made of cylindrical cardboard (the type used at material shops). Learners have to read the introductory paragraph in the activity first and the teacher must ensure that all the terms are understood by the learners by extracting all the relevant information from them. Make use of dictionaries to allow learners the opportunity to look up 'difficult' words such as potion; regulate themselves. This is especially of value to Second and Third Language speakers. Let one or two learners relate (in their own words) the information in the first paragraph to the whole class – in their own words. This helps them to focus on the task at hand.



Get learners to explain what the different apparatus would be used for. While busy doing the exercises learners need to put different blocks into the pot or take them out in terms of what the exercise requires. In most cases when the blocks inside the pot are grouped in pairs of positive and negative, some 'odd' ones will remain, say for instance one negative block remains, the answer to the sum would be -1, in the case of [+3 + (-4)].

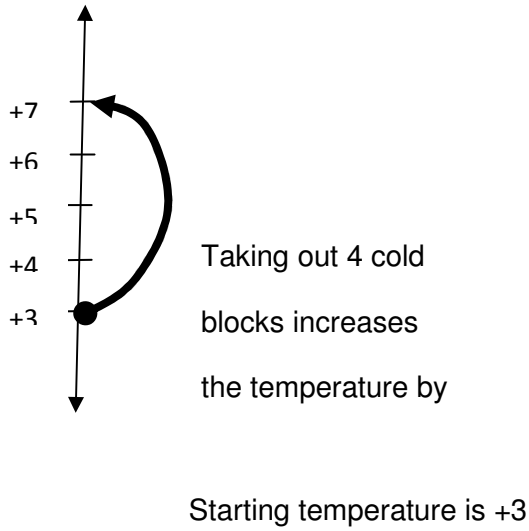
Learners should understand the different signs as meaning the following:

+3 (three hot / positive blocks) + [(added / put into the pot) - 4(four cold/negative blocks)] = -1.



In the case of [+3 -(-4) = +7]

+3 (three hot / positive blocks) - (subtracted / taken out of the pot) - 4(four cold/ negative blocks) = +7

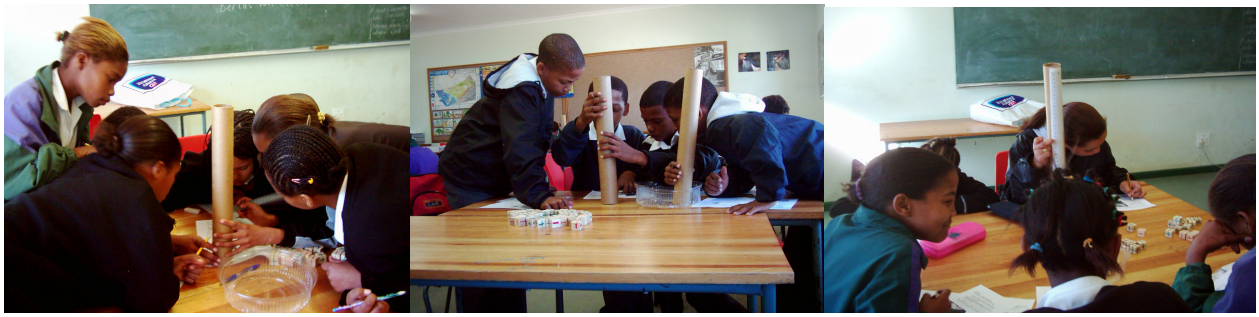


It is important that this is done in conjunction with the use of the cardboard thermometer, which is actually a 'modified' number line. Using the number line in a vertical position seems to enhance learners' understanding of increase and decrease significantly. One learner should be indicating to the group what the starting temperature is (+4), when negative or positive blocks are added or taken out, his or her finger should go up or down (in this case it should go down four units) to indicate the increase or decrease in temperature. In this case the finger should indicate -1 as final temperature, which corresponds to the temperature (1 negative block) inside the pot. The thermometer should circulate among learners, since it is crucial for all learners to actively take part. The teacher needs to make certain that tables are completed row by row. When learners get to the later exercises where bigger and smaller values are used, the blocks are not needed anymore. Those learners who still struggle must draw their own extended number line to use as reference. It is, thus, important that they make use of a vertical number line, since they seem to conceptualise better then when using a horizontal one.

The model applied here essentially forms the foundation of mathematical reasoning that is indirectly linked to the initial context or story, rather it develops into “a referential base for the level of formal mathematics (Drijvers, 2003, p. 55). As such, it succeeds in “bridging the gap between informal knowledge and formal mathematics” (Gravemeijer, 1994, p. 91).

Experiences with learners

Irrespective of who they are, where they come from, or where the school is situated, learners always become totally engrossed in this activity, and the enjoyment is evident. They seem to be oblivious of the presence of myself (facilitator), the teacher and the camera. As can be seen from the pictures taken at Lutzville PS and Masiphmulela HS the grades 7 and 8 learners (respectively) became totally absorbed into the activity. Disciplinary problems disappear, productive noise levels increase as learners try to make sense, and completing the questions on the activity sheet.



Grade 7’s at Uitkyk Primary in Lutzville in 2004

Feedback from learners is generally very positive. Throughout the activity the teacher does very little talking, and occasionally learners pose questions for clarification. The pace at which the different groups work, differ significantly and the teacher should have additional exercises ready for the fast workers. The lesson ends by revisiting the baseline assessment questions which would be able to answer effortlessly.

Being able to deal with addition and subtraction of integers meaningfully has obvious implications for the teaching and learning of algebra, and facilitates the solving of algebraic expressions and equations. The experiences of our South African teachers indicate that a substantial number of learners in the FET (Further Education and Training) phase, which include grades 10, 11 and 12, struggle with applying basic strategies, discussed in this paper, correctly

References

Barnes, H. (2005). The theory of Realistic Mathematics Education as a theoretical framework for teaching low attainers in mathematics. Pythagoras 61, pp. 42-57.

Brown, J.S. Collins, A. and Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18(1), 32-42.

Drijvers, P.H.M. (2003. *Learning algebra in a computer algebra environment.* Utrecht: Freudenthal Institute.

Freudenthal, H. (1973). *Mathematics as an Educational task.* Dordrecht: Reidel.

Freudenthal, H. (1983). *Didactical phenomenology of Mathematical* Structures. Dordrecht: Reidel.

Gravemeijer, K. (1994). *Developing realistic mathematics education*. Culemborg: Technipress.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics.*Mathematical Thinking and Learning*, 1(2), pp. 155-177.

van den Born, W.; Breugel, I.; Dijkstra, J; Gelderblom, G.; Goemans, H.; Hoekstra, B.; e.a. (1998). *Moderne wiskunde* – 7e editie, 1a mavo havo (vwo).

Groningen: Wolters_Noord