A FURTHER EXPLORATION OF DIFFERENCES IN THE PERFORMANCE OF GRADE 12 PUPILS ON TWO TYPES OF GEOMETRY PROBLEM

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This paper reviews and reconsiders the author's earlier views on Grade 12 pupil performances in Euclidean geometry problems. Previous work indicated that pupils exhibited significantly lower competence in developing deductive proofs (proof-type problems) compared to the calculation of quantities (computational-type problems), even where problems drew on similar propositions but were of different types. For this study, pupils were tested on matched computation-type and proof-type problems to explore what they did more carefully. As before, pupils performed better on computational-type problems than on proof-type problems that required the use of the same propositions. From their performances it also emerged that pupils appear to focus on geometrical objects in a manner that de-emphasises their relations with other geometrical objects external to them, but that pupils are willing to construct arithmetic relations between quantities that refer to components of different geometrical objects. Why this should be the case is not entirely clear, but it does point us in a particular direction for future work.

INTRODUCTION

In this paper we discuss aspects of the examination performances of sixteen Grade 12 pupils, from five high schools, who participated in a teaching programme to prepare them for the 2011 Mathematics Paper 3 matriculation examination. The cohort comprised 26 pupils, but only 16 availed themselves for the two tests from which this paper derives. The content examined in Paper 3 was not offered at their schools. As was the case in 2010, reported on in Mackay (2011), our experience of pupils during teaching sessions suggested that: (a) they had very limited exposure to Euclidean geometry; (b) their mathematical activity was, generally, regulated in a manner that did not enable them to make coherent use of propositions appropriate to the problems they attempted to solve; and (c) the teaching of the reading and writing of mathematics proofs was not a common practice in their schools. Nevertheless, sixteen of the twenty-five pupils did pass the final 2011 Grade 12 Mathematics Paper 3 examination. To what extent their achievement can be attributed to successful completion of geometry problems, which is a part of the question paper, cannot be verified since access to their examination scripts is restricted.

In a pilot study we set out to construct a researchable problem to explore pupils' responses to two different types of geometry problems, described as *computational-type problems* and *proof-type problems* (MacKay, 2011). A computational-type problem explicitly asks pupils to calculate the sizes of angles or lengths of sides or areas of geometrical figures, given quantitative data on lengths or/and angles or/and area; a proof-type problem expects pupils to generate arguments consisting of strings of propositions that establish the mathematically necessary intermediate and final results that constitute a proof as a whole. The pilot study

indicated that even where both types of problems drew on the same propositions to regulate computational activity as well as to construct proofs, pupils exhibited competence in solving problems involving calculation of quantities but showed significantly less competence in arguing deductively and writing proofs.

A FEW REMARKS ON METHOD

To explore more carefully what might be happening with these students, we constructed two tests, Test A and Test B, consisting of three matched pairs of problems for which the solutions drew on the same sets of propositions. Test A was made up of computational-type problems and Test B of proof-type problems.

The two tests were administered one week apart and in both instances the same information was provided diagrammatically and discursively. A quantitative analysis of the pupils' answers is undertaken, specifically a measure of the correlation between their performances on the test, globally and by matched problem, was produced.

The pupils' aggregate performances across the two tests are represented by the graph in Figure 1, from which it can be seen that students, generally, performed better on Test A (computational-type) than they did on Test B (proof-type).

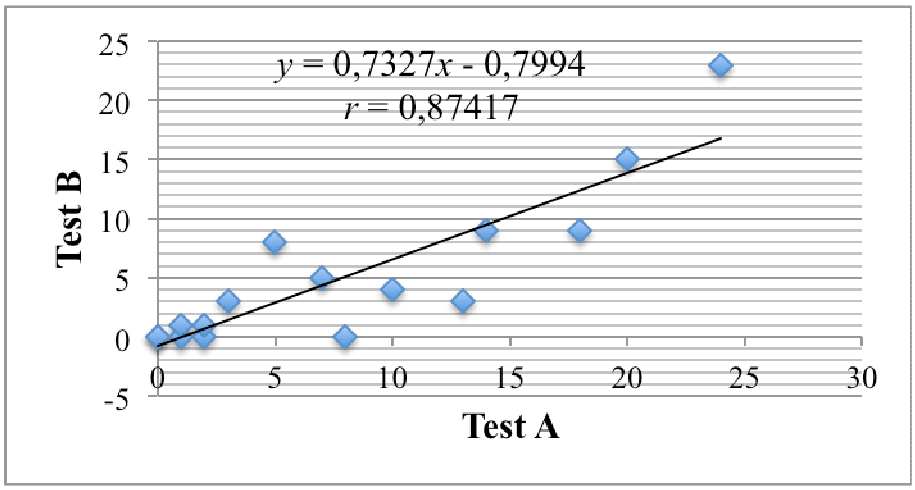


Figure 1: Pupils' aggregate performances across Tests A and B.

The value of the correlation coefficient ($r = 0{,}87417$) and the line-of-best-fit clearly points to a fairly strong linear positive correlation between the performances across the two tests, indicating that the hypothesis in MacKay (2011) that pupils tend to perform better on computational-type problems than they did on proof-type problems is not unreasonable. The problem for this paper is to begin to draw out ideas as to the possible reasons for this trend in the performances of Grade 12 students on geometry problems.

A quantitative analysis of the pupils' aggregate performances in Questions 1 to 3, across the two tests is represented in the graphs in Figure 2.

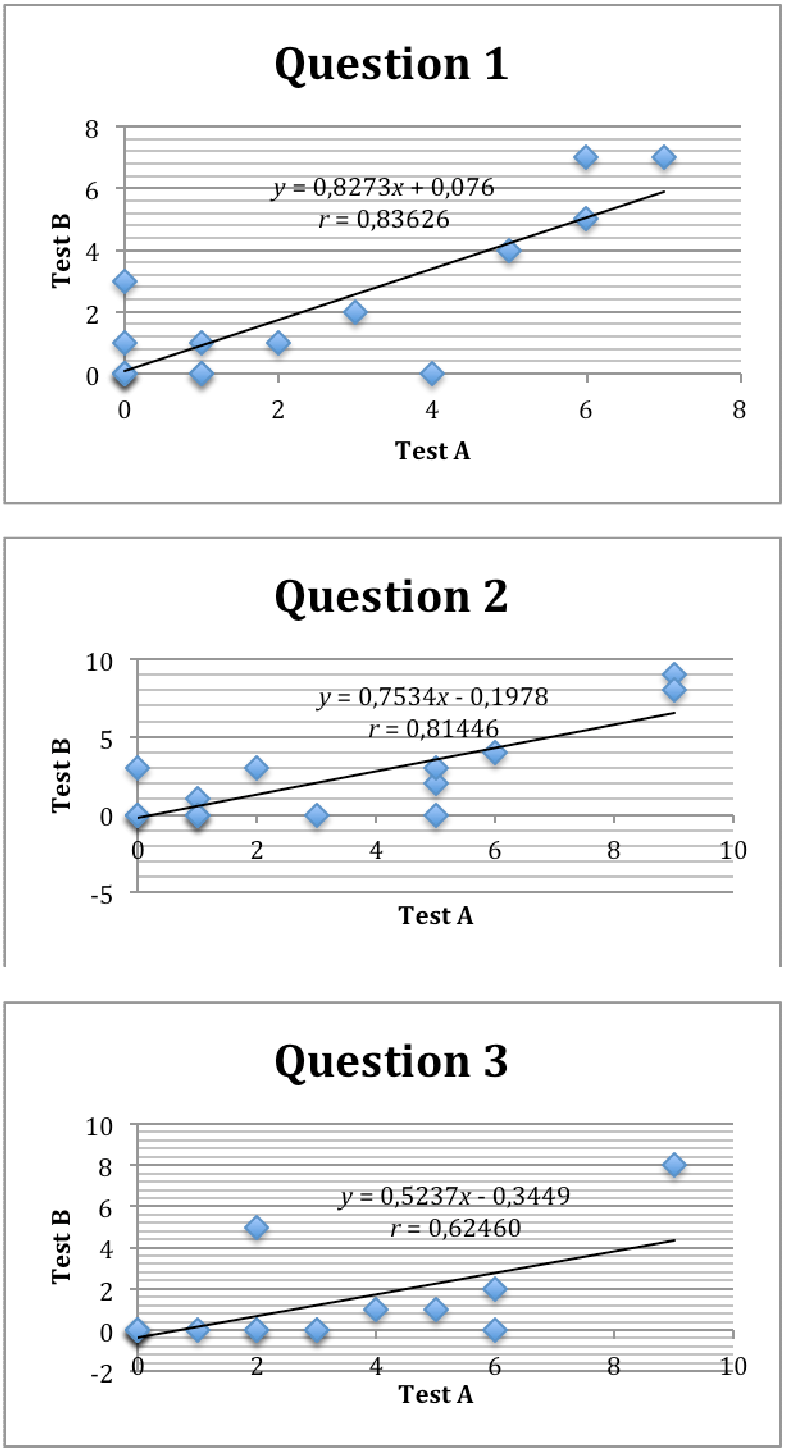


Figure 2: Pupils' aggregate performances across two tests, by Question.

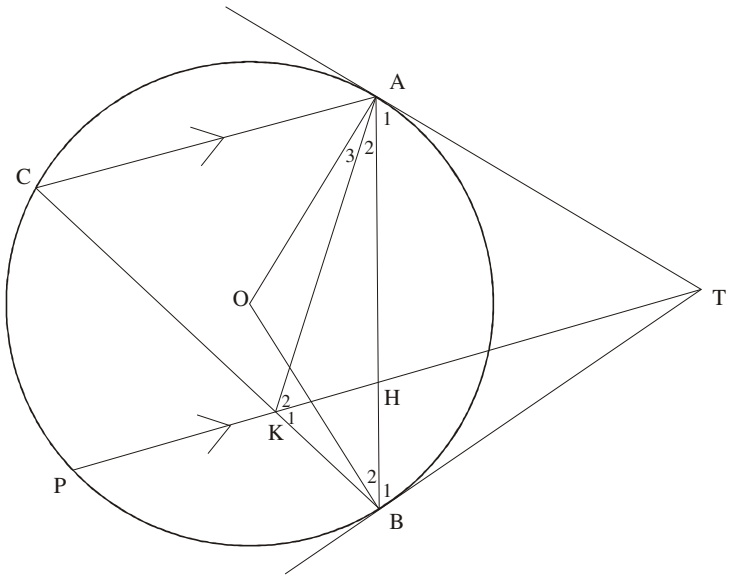
The correlation coefficients (r = 0,83626 and r = 0,81446) for the aggregate scores in Questions 1 and 2 are very similar and show a strong positive correlation between the two tests. In Question 3, r = 0,62460. This indicates a moderately strong correlation in performances across test A and B, with most scores falling below the least squares regression line. However, as can be seen from the graph, the performances of pupils on Question 3 give the clearest indication of the idea that they perform significantly better on computational-type problems than on proof-type problems. Since it is with Question 3 that we see a more pronounced difference in the performances of pupils on problems of the two

types, a detailed analysis of pupils' performances on Question 3 (Figure 3) might give us some additional insights into the possible reasons for the differences in performance on different problem types, so enabling us to generate one or more hypotheses for further investigation.

AN ANALYSIS OF SELECTED PUPIL RESPONSES TO QUESTION 3

We therefore focus on pupil responses to Question 3 (shown in Figure 3), and we examine the work of three pupils (given the limitations on space). Pupils are referred to with pseudonyms.

In circle with centre O, TA and TB are tangents at A and B, respectively. Chord AC is parallel to TP, with P on the circle. CB cuts PT at K, and AK, BO, OA and AB are drawn. AB cuts PT at H.



(Test A variation)	**(Test B variation)**
3.1 Calculate, with reasons, the size of $\angle OAT$.	<u>Prove that:</u>
3.2 By calculation, show that AOBT is a cyclic quadrilateral.	3.1 AOBT is a cyclic quadrilateral.
3.3 If $\angle A_1 = 70°$, calculate the sizes of	3.2 $\angle BKH = \angle BAT$.
(a) $\angle K_1$	3.3 $\angle ATB = 180° - 2.\angle C$.
(b) $\angle ATB$.	3.4 AKBT is a cyclic quadrilateral.
3.4 Give a reason why AKBT is a cyclic quadrilateral.	

Figure 3: Question 3 from Grade 12 Paper 3 tests, showing the variations in the problem statements.

It may be useful to briefly describe the propositions on which pupils must draw in the sub-questions of Question 3, to better understand the qualitative analysis of selected pupil answers later in this paper. The particular propositions are represented in Table 1.

Test A	Test B	Propositions
Q 3.1 & 3.2	Q 3.1	(a) The angle between a tangent and a radius, at the point of contact, is a right angle. (b) The opposite angles of a quadrilateral are supplementary
Q 3.3 (a)	Q 3.2	(a) The angle between a tangent and a chord is equal to the inscribed angle subtended by that chord in the alternate segment. (b) If two parallel lines are cut by a transversal, the pairs of alternate angle will be equal. (c) If two lines cut, the vertically opposite angles are equal.
Q 3.3(b)	Q 3.3	(a) The angle at the centre of a circle is double the inscribed angle subtended by the same arc or chord. (b) The opposite angles of a cyclic quadrilateral are supplementary.
Q 3.4	Q 3.4	If a side of a quadrilateral subtends a pair of equal angles, on the same side of it, then the quadrilateral is cyclic.

Table 1: Propositions on which pupils must draw to answer sub-questions in Question 3 of Tests A and B

Question 3.1 (Test B) requires pupils to prove that a quadrilateral is cyclic. Lionel offered the solution shown in Figure 4.

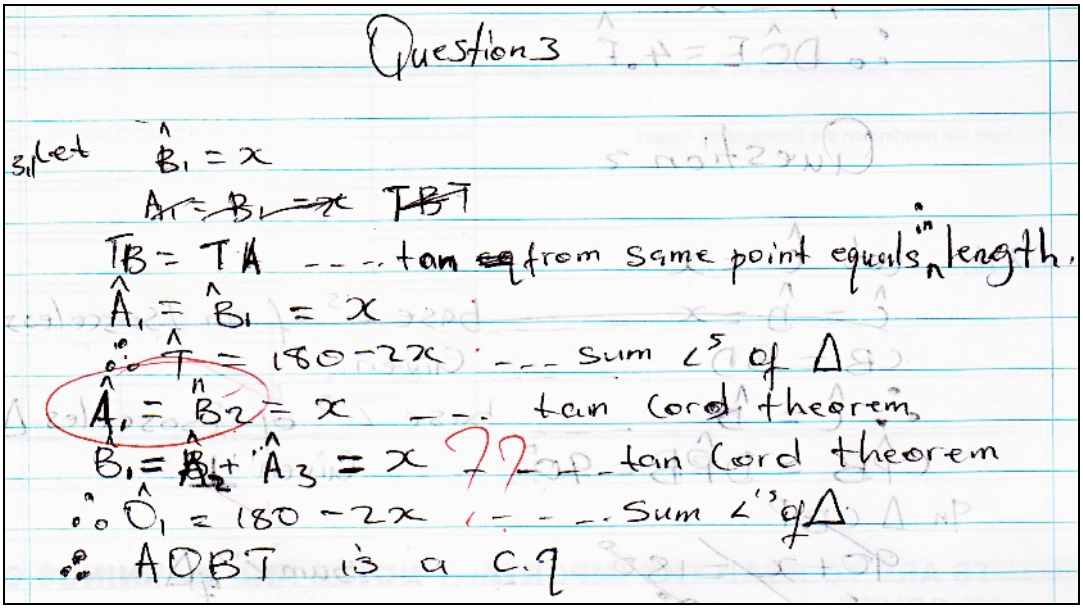


Figure 4: Lionel's answer to Question 3.1 (Test B)

To the point where Lionel proves that '$\hat{T} = 180 - 2x$', it is evident that he is able to string together a coherent sequence of mathematical statement, notwithstanding the grammatical errors in his statements of justification. However, the statements $\hat{A}_1 = \hat{B}_2 = x$ and

$\hat{B}_1 = \hat{A}_2 + \hat{A}_3 = x$ and his statements of justification 'tan chord theorem' point to a breakdown in his reasoning. From experience, he knows that 'the angle between a tangent and a chord' must equal another angle at the circumference of the circle. He fails to recognise that the chord must subtend the angle at the circumference. Consequently, he shows that $\hat{O}_1$ is also equal to $180 - 2x$. Lionel appears to recognise that he must describe the relationship between a pair of opposite angles of the quadrilateral, but erroneously accepts equality as the criterion for proving that the quadrilateral is cyclic. It is also possible that not having proved the known relation and being unable to recover, he makes the final required statement and proceeds to the next question.

In Figure 5, Sandra's answer to Question 3.1 (Test B) shows an almost complete disconnection between the mathematical statements and the intended outcome. The final statement of the attempted proof shows that Sandra appears to know what types of statements should be made at the conclusion to such a proof. In other words, a pair of opposite angles must be supplementary to conclude that the quadrilateral is cyclic. However, she is not successful in demonstrating that the opposite angles of the quadrilateral AOBT are supplementary. She contradicts this in the way that she sets about proving equality of pairs of angles and ultimately possibly that $O\hat{A}T = O\hat{B}T$. While sets of statements may be legitimate in their own right, she fails to string together a coherent argument in support of the outcome.

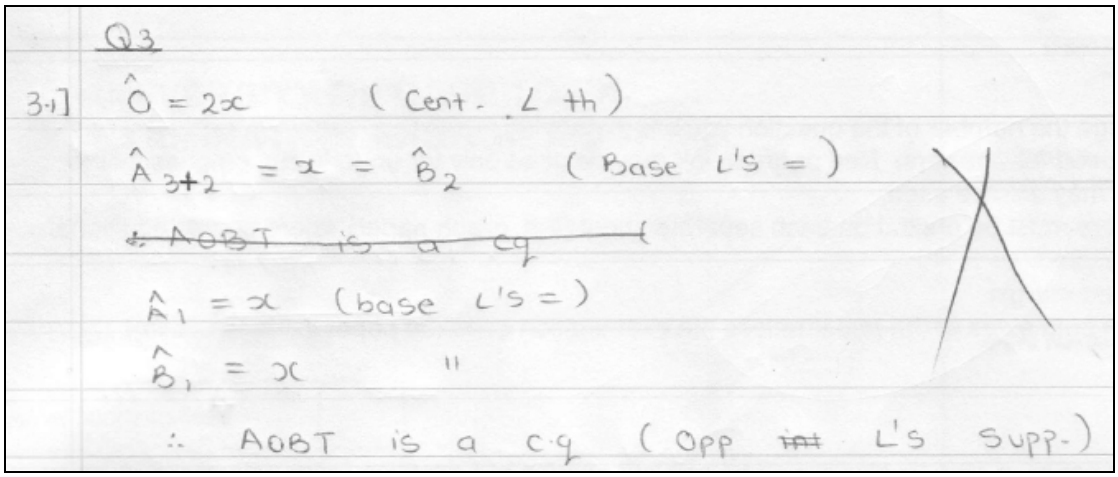


Figure 5: Sandra's answer to Question 3.1 (Test B)

We now consider and compare Simon's answers to Question 3.3 in both Test A and Test B (Figure 6a) and Test B (Figure 6b). In both instances, the pupil is required to determine the value of a particular angle, either in degrees or in terms of another angle.

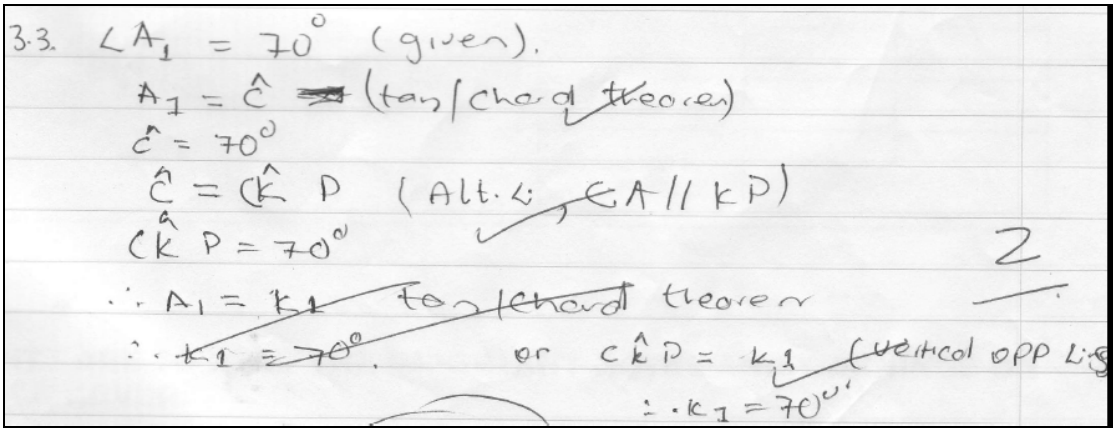
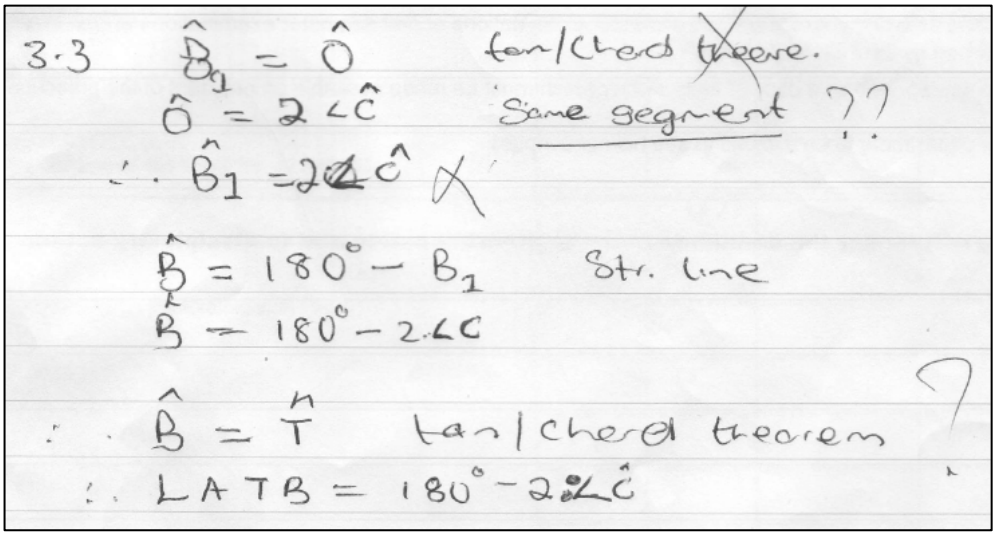


Figure 6a: Simon's solution to Question 3.3 on Test A



Figures 6b: Simon's solution to Question 3.3 on Test B

In Figure 6a, Simon is able to appropriately use the tangent-chord theorem, that is, $\hat{A}_1 = \hat{C}$. It is also apparent that he is capable of stringing together a series of mathematical statements, but like Lionel, does not use the 'tan-chord theorem' appropriately in his response to Question 3.3 of Test B where an incorrect proof attempt consequently emerges. In Figure 6b, a correct final statement emerges from a series of incoherent mathematical statements and justifications, presented in support of the statements. Thus, it appears that Simon is able to solve computational-type Question 3.3 (Test A) with ease but has great difficulty with comparable proof-type Question 3.3 (Test B).

One could attempt to address the discrepancy by arguing that the emergent difficulty in the answer to Question 3.3 (Test B) is a consequence of the absence of the actual value of $\hat{A}_1$. This idea was also considered in Mackay (2011).

CONCLUDING REMARKS

An interesting aspect of pupil work on geometry is suggested in the work of Lionel and Sarah on Question 3.1 of Test B. In this instance, the proposition that fixes quadrilateral AOBT as cyclic derives from relations between propositions that refer to geometrical objects (generally, things like points, lines, angles, polygons, planes and so forth) that are *internal* (a part/component of) to the quadrilateral, and objects that are *external* to it (not a part/component of). It may well be the case that Lionel and Sarah are attempting to prove that the quadrilateral is cyclic by operating only on elements that are internal to the quadrilateral; that is, that are part of it. For example, Question 3.3 (Test B) requires the pupil to express the value of one angle in terms of the value of another angle. In this instance, $\hat{C}$ is an inscribed angle of the circle, while $A\hat{T}B$ is an element of the cyclic quadrilateral. Although Simon correctly identifies the angle between the tangent and a chord, the 'inscribed' angle reference remains internal to the quadrilateral, precipitating the inappropriate relation $\hat{B}_1 = 2.\hat{C}$. Concatenations of arbitrary statements emerge from Simon's inability to recognise the appropriate relations between geometrical entities internal and external to the quadrilateral ($\hat{C} \rightarrow B\hat{O}A \rightarrow A\hat{T}B$) and easily solve the problem. Jones (2002) ascribes this dilemma to an inability to "produce a short sequence of statements to logically justify a conclusion and to understand that deduction is the method of establishing geometric truth" (p. 130).

On the basis of the analysis undertaken in the previous section, we now hypothesise that the inability of pupils to successfully realise a required output by appropriately drawing on the relations between propositions that are both internal and external to the announced object, may be causal to pupil failure. In other words, the appreciation and grasp of transitivity, which refers to the ability to recognise relationships among various things in a serial order, has not developed appropriately in relation to geometry, and possibly not in relation to other topics on the school curriculum (see Brainerd, 1973 for a discussion of transitivity as a precondition for seriation). The inclusion of values of angles in problems appears to enable the development of strings of statements constituting coherent arguments, possibly because pupils find numbers to be a useful resource. Hilbert & Bernays (1934), in a discussion about the consistency of theoretical systems, say

… one represents the object of a theory through numbers and systems of numbers and basic relations through equations and inequalities in such a way that on the basis of this translation the axioms of the theory become either arithmetic identities or provable assertions (as in the case of geometry) … (p. 4)

So, even though all solutions to Euclidean Geometry questions draw on propositions, pupils can begin to do computations without explicitly using the relevant propositions, but by acting on known formulae instead (see also Hilbert, 1927). $\hat{A} + \hat{B} + \hat{C} = 180°$ (the sum of the angles of a triangle) or $\hat{O} = 2.\hat{B}$ (the angle at the centre of the circle is double the inscribed angle subtended by the same arc) are examples of such formulae. The use of propositions as formulae is suggested by the pupils' responses particularly by their failure to solve proof-type problems.

In relation to the different forms of the questions, MacKay (2011) had postulated that many instances of pupil activity suggest that pupils are either not drawing on the expected

propositions, or they are employing some other resources in the treatment of different forms of the problems. We now, however, see that it may well be the case that the geometrical objects named in problems are focused on in a manner that deemphasises the relations between their immediate constituents and geometrical objects external to them, suggesting that Skemp's (1986; 1987) distinction between instrumental, relational and formal thinking might be productive for future investigation. In addition, the fact that pupils tend to perform better on computational-type problems suggests that they are able to construct arithmetic relations between quantities that refer to objects that are not necessarily only internal to a specific complex of geometrical objects (like a quadrilateral, for example).

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