HOMOMORPHISMS, COUNTING AND THE CONSTITUTION OF ADDITION IN A GRADE 1 LESSON

Zain Davis

School of Education, University of Cape Town

The idea of a homomorphism is used to inform descriptions and analyses of some aspects of the teaching of addition in a Grade 1 lesson. That all but one student working on a set of addition tasks could not relate disjoint union over finite sets to addition over the natural numbers in a coherent way suggests that the students were taught in a manner that was dislocated from the grounding homomorphism enabling addition to be derived from disjoint union. It appears to be the case that students were sometimes taught number bonds without the mediating work done by the counting function, the latter being that which enables the shift from operations over collections of objects to operations over natural numbers. What this means is that the students' ideas of addition are likely to be underdeveloped despite the fact that many of them can use their knowledge of number bonds to perform certain addition computations.

INTRODUCTION

The primary motivation for writing this paper is to present for consideration and discussion an idea that suggests itself as productive for the description and analysis of the constitution of mathematics in pedagogic situations generally, but which is also productive for researching the teaching and learning of mathematics in the pedagogic situations of primary schooling. The idea is that of a *homomorphism*, considered in relation to objects constituted by taking an operation and the domains and codomains of objects that serve as arguments and values for the operation of interest.

Homomorphisms

If $*$ indicates an operation and *S* is the collection of objects that serve as a source of its arguments and values, then $(S,*)$ is an object of the type in relation to which I wish to consider the idea of a homomorphism. A familiar example of an object like $(S,*)$ is $(\mathbb{N},+)$, which is constituted by taking addition over the natural numbers. Now suppose that along with $(S,*)$, we have another such object, $(T,\circ)$, where *T* is a collection of arguments and values for the operation $\circ$. Is it possible for us to substitute mathematical work that uses $(S,*)$ by work that uses $(T,\circ)$? Expressing the question as I have done here may mask

the fact that a question of this sort is routinely asked by teachers when they think about how to teach a bit of mathematics. Foundation phase teachers, for example, regularly consider the use of teaching resources that enable their students to do computations involving the basic operations in an indirect way—like using separate collections of blocks that are brought together to form a single collection to assist their students to learn about addition. Or take the Grade 8 teacher who attempts to exploit their students' intuitive grasp of area and uses rectangular areas to teach them about the multiplication of polynomial expressions.

In each of these cases the substitution of one thing by another is productive only if the essential features that are of mathematical interest are preserved as we move from the one to the other. To make sure that there is at least a reasonable degree of the preservation of what's essential to the mathematical idea or process we wish to teach, we need to find an appropriate way of getting from the one to the other (and, sometimes, back again). In different terms, the problem is one of being able to ensure the *preservation of structure* as we move from an object like $(S,*)$ to one like $(T,\circ)$, where *structure* is understood as the essential computational features of a collection of things—like *S* and *T*—when acted on by some operation—like $*$ and $\circ$.

Structure preservation means that computations done using $(T,\circ)$ are compatible with those done using $(S,*)$, so that one can productively substitute $(T,\circ)$ for $(S,*)$ if one wishes to do so. How we achieve the preservation of structure as we move from $(S,*)$ to $(T,\circ)$ is by choosing an appropriate mapping from $(S,*)$ to $(T,\circ)$, if such a mapping exists. Suppose that *f* is, indeed, such a mapping, then we can write $f:(S,*)\rightarrow(T,\circ)$, which means that $\forall s_m,s_n \in S, f(s_i)\in T, f(s_m)\circ f(s_n) = f(s_m * s_n)$.

$$\begin{tikzcd}
s_m,s_n \arrow[r, "*"] \arrow[d, "f"'] & s_m*s_n \arrow[d, "f"] \\
f(s_m),f(s_n) \arrow[r, "\circ"] & f(s_m)\circ f(s_n) = f(s_m*s_n)
\end{tikzcd}$$

Figure 1: A commutative diagram showing $f(s_m)\circ f(s_n) = f(s_m * s_n)$.

The ideas will become a bit clearer and more tangible once an actual example is discussed. What the idea of homomorphism is useful for as a research resource is its enabling of the production of mathematically-attuned descriptions and analyses of the pedagogic transformations of mathematics contents routinely produced by teachers and their students as they go about doing school mathematics. The mathematics education research literature

on morphisms and homomorphisms is virtually non-existent, so the idea has not been explicitly explored in relation to pedagogic situations to any reasonable extent, and it is thus unfamiliar to most mathematics teachers, and especially to those who have not done advanced mathematics courses. See Krause (1969) and Baker, Bruckheimer & Flegg (1971) for discussions of the idea in the professional literature.

An empirical case

While watching a selection of video records of Grade 1 teaching in order to familiarise myself with current practices in the teaching of mathematics to foundation phase students, it occurred to me that something curious was happening in the teaching of counting in a number the lessons I reviewed. It also appeared to be the case that addition was often taught in a manner that dislocated it somewhat from counting. A more careful viewing of the video records, accompanied by closer readings of transcripts where they were available to me, got me to thinking that a study of the constitution of the basic arithmetic operations from counting and in other ways in Grade 1 classrooms was a worthwhile project to pursue because of the fundamental role of the counting algorithm and the counting function in the development of elementary arithmetic for the child (Carpenter, Moser & Romberg, 1982; Fuson, 1992; Gelman & Gallistel, 1978) on the one hand, and the pedagogic peculiarities of its teaching on the other.

Given the limitations on space, I've elected to focus on the constitution of addition in Grade 1 and, more specifically, on the elaboration of the idea of and process associated with addition in a particular Grade 1 class.

HOMOMORPHISMS AND ADDITION IN GRADE 1

In the video record of the Grade 1 lesson I'll discuss here, the teacher used a worksheet to deal with addition. The worksheet listed four tasks, each of which required students to fill in empty ovals and rectangles to produce a number sentence that realised a sum for 8. The ovals were to be populated with objects—the teacher and students used little ovals that resembled zeros or the letter 'O'—and the rectangles with numbers. Two of the four tasks required students to compose a sum consisting of two addends, while the remaining two required the composition of a sum consisting of three addends (see Figure 2). The general idea of the task is that students use their knowledge and experiences of counting to construct two or three sets of objects, the union of which has a cardinality of 8, and then write the cardinal values of those sets in the rectangles to produce an appropriate number sentence. To do the tasks students had to recognise that, given any whole number (finite natural number), like 8, they could always construct a set consisting of 8 elements by counting from 1 to 8 and associating each number with a new object as they drew the objects that populated the set. They also had to recognise that they should distribute the objects to two sets in two of the tasks and to three sets in the remaining two tasks. To complete a number sentence the student had to count the objects in the sets and then write the results in the rectangles.

In more formal terms, allowing ourselves a little flexibility here, what we have in use is a homomorphism relating a structure formed by taking disjoint union over the class of finite sets to a second structure formed by taking addition over the natural numbers. The transformation that makes the homomorphism possible is the counting function mapping finite sets to cardinal values.

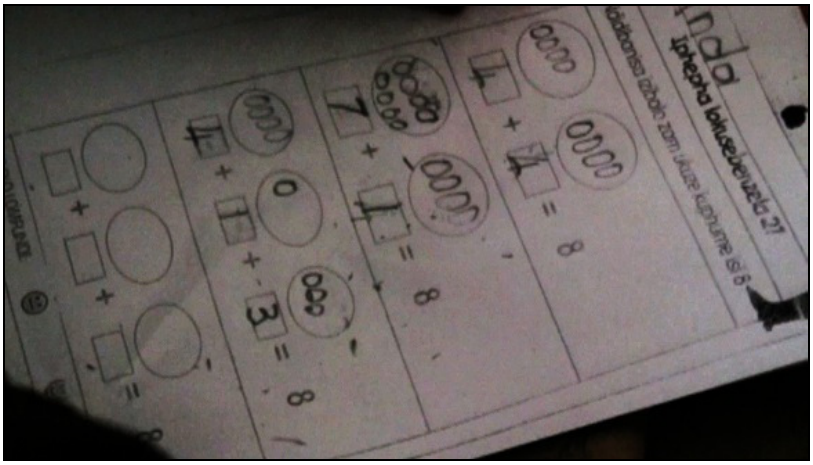

Figure 2: Worksheet on addition, showing a student's working

Consider the set $\mathbb{N}(n) = \left\{ m \in \mathbb{N} \middle| 1 \le m \le n \right\}$ with $\mathbb{N}(0) = \varnothing$ (the empty set). If *A* is some set, then the bijective function $f_c : \mathbb{N}(n) \to A$ defines a *count* on *A*, and *n* is the value that registers the cardinality of *A*. Let $(\mathrm{FINSET}, \cup)$ indicate disjoint union over finite sets, and let $(\mathbb{N}, +)$ indicate addition over the natural numbers. It seems that the intended pedagogic task is to use counting to enable the student to employ their knowledge and experience of working with collections of objects to learn about addition over the natural numbers. That is, the worksheet is grounded in and authorised by the existence of the mapping $f_c : (\mathrm{FINSET}, \cup) \to (\mathbb{N}, +)$. f_c preserves structure as we move from $(\mathrm{FINSET}, \cup)$ to $(\mathbb{N}, +)$. In less precise terms, we might say that counting enables the binary operations of taking the union of disjoint finite sets and the addition of natural numbers to be related metaphorically—metaphor being the linguistic trope effecting substitution—so that the one structure can be used to illuminate the other in ways that are productive for the teaching and learning of a bit of school mathematics.

Figure 3 shows a commutative diagram of the relations described here. The counting function, f_c, is used to count the elements in each of the sets to produce 3 and 5 as cardinal values. The union of the two sets produces the set with a cardinality of 8. What the diagram shows is that one can get from a pair of sets to 8 in two ways: one can either take the union of the sets and then count the objects in the resulting set, or one can count the objects in the initial sets and than add the results.

While the homomorphism described in Figure 3 indicates that it is counting that enables the preservation of structure from $(\mathrm{FINSET}, \cup)$ to $(\mathbb{N}, +)$, the task for the student starts with a consideration of how the number 8 can be used to generate an appropriate number sentence by way of the construction of collections of objects.

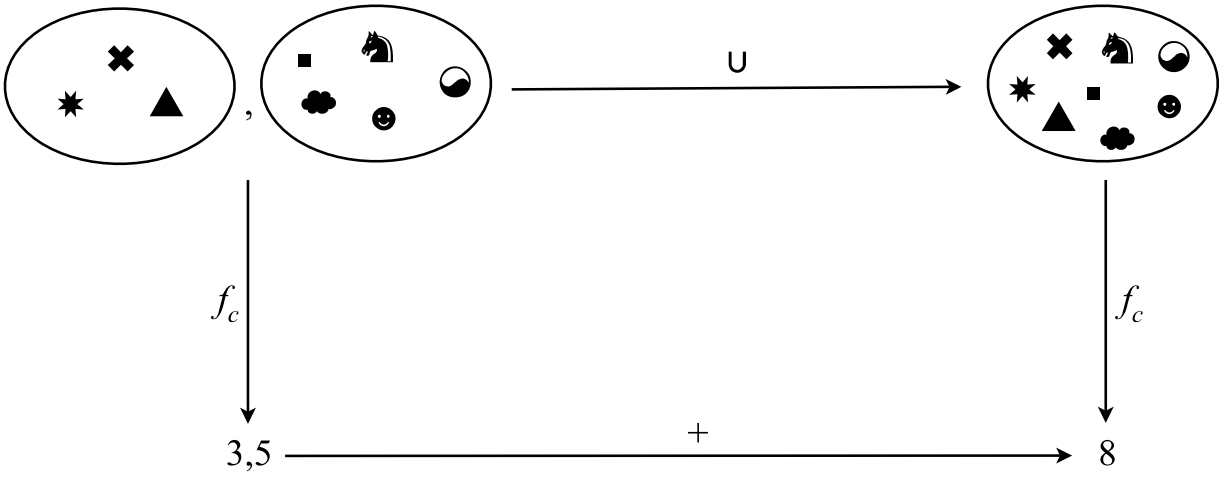

Figure 3: An example of a homomorphism mapping disjoint union over finite sets to addition over the natural numbers.

Using the natural number 8, the student is to select any set from the equivalence class of sets having cardinality 8—the counting of objects as they are drawn is an adequate procedure for the selection of an appropriate set. In Figure 4 that procedure is indicated by the symbol f_v (in honour of John von Neumann who thought of a natural number *n* as a set consisting of *n* elements in his work on set theory).

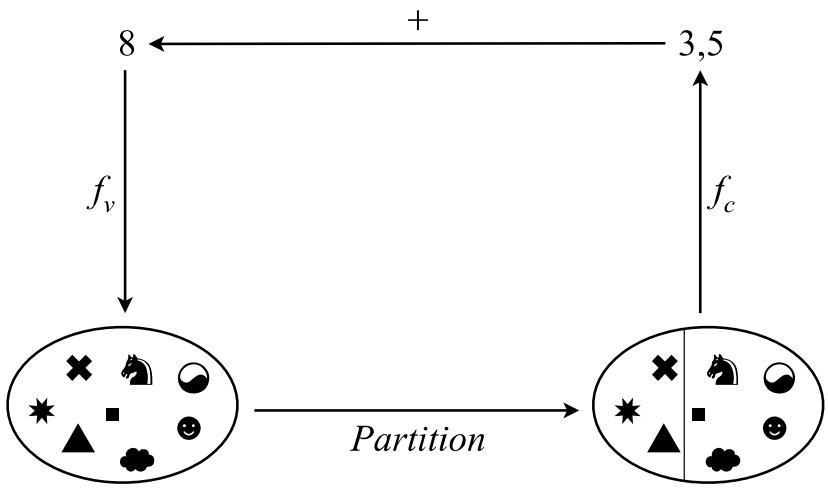

Figure 4: A diagram describing the computational processes to be used by a student to do a task, and the relations between the processes.

Clearly there are an infinite number of such sets for any given *n*, thus forming an equivalence class associated with that natural number, and any one of the sets will do. The student also has to produce two subsets of the set they associate with 8 by producing a partition of that set, thus ensuring that the subsets have no elements in common and that their union produces the original set once again. By counting the objects in each of the collections produced by the partition the student generates numbers that will sum to 8. From this process students are meant to learn that there are combinations of numbers that sum to 8, like $5+3=8$, as well as to acquire a general resource for computing sums of small numbers in situations where they don't yet know the required result. When young students

resort to counting on their fingers to compute a sum they are using just such a resource (see Figure 5).

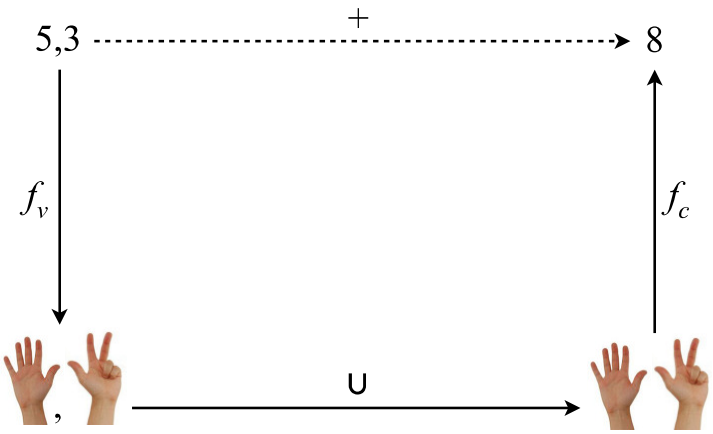

Figure 5: Adding 3 and 5 by using collections of fingers.

An interesting thing about the lesson in which the worksheet shown in Figure 1 was used was that only one of the students required to engage with the worksheet was able to complete the tasks fully (see Figure 6), but many of the students were, in fact, able to write down numbers producing true number sentences without being able to fill the ovals with the correct number of objects to match the numbers that they wrote down in the rectangles (see Figure 7).

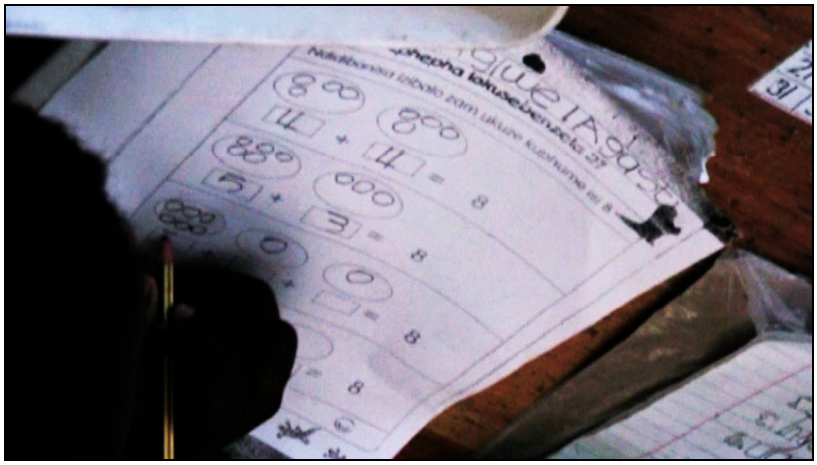

Figure 6: Work of the only student able to complete the tasks as required.

So, what is happening here? Is it perhaps it is the case that students do not need to use the ovals because they can perform the set computations and relate those computations to numbers without writing out the sets? Or, is it perhaps it is the case that the students are using number knowledge learned by some means that did not require them to consider collections of things? The students' workbooks showed that a full two months before the lesson discussed here, the teacher had been teaching them to use devices like the one shown in Figure 8 to learn number bonds. The information displayed by the device is to be read as: $8=5+3$, $5+3=8$, $8-5=3$ and $8-3=5$. The number bonds

enable the student to perform computations without needing to resort to the use of collections of objects to work out the result of a computation.

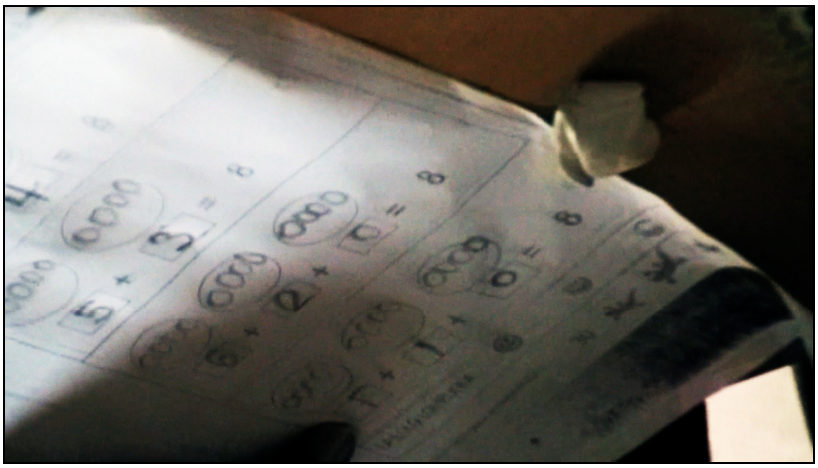

Figure 7: Work of the a student able to produce correct number sentences but unable to populate the ovals correctly.

Further, when one examines the teacher's responses to students' work where they produced correct number sentences alongside collections of objects showing cardinalities that did not match the numbers in the number sentences, she ignored the errors. Such evaluations of their work by their teacher would, arguably, have indicated to many of the students that the relation of the collections of things in the ovals to the numbers in the number sentences was not one of cardinality, or that it was unimportant and irrelevant.

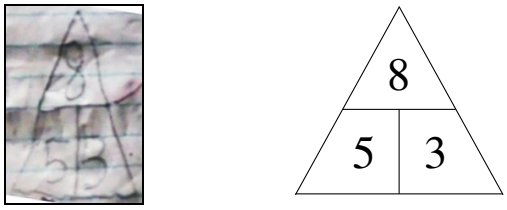

Figure 8: A device for learning number bonds, as it appears in a student's workbook (left), and as redrawn (right).

CONCLUDING REMARKS

The lesson presents a pedagogic situation in which all that Grade 1 students require are mental lists of a small collection of sums, differences, products and quotients to do most of the computations they'll be asked to do. Learning the number bonds certainly facilitates for such knowledge to be acquired. However, since counting is a central constituent of the *notions* of addition, multiplication, subtraction and division (Carpenter, Moser & Romberg, 1982; Ensor et al., 2009; Fuson, 1992; Gelman & Gallistel, 1978), student learning of those ideas is compromised if counting is dispensed with as a computational resource too soon.

The situation described in this paper is not unique to the lessons I reviewed. In fact, a lack of appreciation of the implications for the learning of elementary arithmetic of the relations between binary operations over collections of objects and the binary operations of elementary arithmetic is even evidenced in curriculum resources disseminated widely by the Department of Basic Education. Consider, for example, the extract from the Department of

Basic Education's Grade 1 Workbook prescribed for use by teachers and students in state schools, shown in Figure 9.

Figure 9: An attempt by the Department of Basic Education to use disjoint union to develop the notion of addition for Grade 1 students (DBE, 2010: 121).

We do not have to examine the representation too carefully to notice that the collection of sweets that represents the union of the two smaller collections of sweets is not, in fact, the union of those collections. While the cardinality of the larger collection shown is equal to the cardinality of the collection that can be formed from the two smaller collections, the depicted collection is not the same thing as the union. Many of the problems of this type in the Workbook are flawed in the same way: the expression of computations by arithmetic symbols is permitted to have an overdetermining effect on descriptions of operations over finite collections, so that the ground for the development of arithmetic shifts away from its intuitive base in collections of objects. It is little wonder that number bonds become the dominant computational ground for developing foundation phase arithmetic, and so it should not surprise us that students fail to grasp the basic arithmetic operations adequately.

REFERENCES

Baker, J.E., Bruckheimer, M & Flegg, H.G. (1971). A Pedagogic Approach to Morphisms, *Educational Studies in Mathematics*, 4(2), pp. 252-263.

Carpenter, T.P., Moser, J.M. & Romberg, T.A. (Eds.) (1982). *Addition and subtraction: a cognitive perspective*. New Jersey: Lawrence Erlbaum.

DBE (2010). *Numeracy Workbook Grade 1, Book 1*. Pretoria: Department of Basic Education.

Ensor, P., Hoadley, U., Jacklin, H., Kühne, C., Schmitt, E., Lombard, A. & van den Heuvel-Panhuizen, M. (2009). Specialising pedagogic text and time in Foundation Phase numeracy classrooms, *Journal of Education*, 47, pp. 5-29.

Fuson, K. 1992. Research on whole number addition and subtraction. In Gouws, D. (Ed.), *Handbook of research on mathematics teaching and learning*. New York: Macmillan, pp. 243–275.

Gallistel, C.R. & King, A.P. (2009). *Memory and the Computational Brain: Why Cognitive Science Will Transform Neuroscience*. Chichester: Wiley-Blackwell.

Gelman, R. & Gallistel, C.R. (1978). *The Child's Understanding of Number*. Cambridge, MA.: Harvard University Press.

Krause, E.F. (1969). Homomorphism: A Unifying Concept, *Mathematics Teacher*, 62(8), pp. 617-622.