Deepening the quality of mathematics teaching and learning

PROCEEDINGS OF THE 21st ANNUAL NATIONAL CONGRESS
OF THE ASSOCIATION FOR MATHEMATICS EDUCATION OF SOUTH AFRICA

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Editors: Satsope Maoto, Benard Chigonga & Kwena Masha
Proceedings of the 21st Annual National Congress of the Association for Mathematics Education of South Africa

Volume 1

29 June - 3 July 2015

University of Limpopo
Polokwane

Editors: Satsope Maoto, Benard Chigonga & Kwena Masha
The 21st century challenge for South Africa as a Nation is to provide quality teaching and learning for all its young citizens. The teaching and learning of mathematics is at the core of this challenge. It thus makes sense that during our 21st Annual National Congress, we reflect on what AMESA can do to contribute to this national goal. The current theme, ‘deepening the quality of mathematics teaching and learning’ was conceptualised against this background.

South Africa has, at a system level, introduced Annual National Assessment whose purpose is, among others, to monitor the quality of teaching and learning at different points in the system. A number of presenters are engaging with these assessments picking up issues that arise from those and strategies that can be used to address them. Similarly, we have papers that address issues that arise out of our National Senior Certificate examinations. Our panellists and plenary speakers were also invited to address issues affecting quality mathematics teaching and learning from various perspectives that include higher mathematics learning, teacher education, international perspective, curricular issues and learners support material. The workshops that form part of these proceedings offer hands-on experiences through which facilitators share successful strategies and activities that they have successfully developed to improve the quality of mathematics teaching and learning in different phases and context.

This congress, like any other, will hopefully inspire you to take up the challenges of quality mathematics teaching and learning beyond the five days in which the activities unfold. You will continuously use the two volumes as a critical resource material in your classrooms, seminars, workshops, planning sessions, and so on.

Lastly, we remain thankful for the presenters and their reviewers for their immense contribution to the 21st Annual National Congress of the Association for Mathematics Education of South Africa. We are forever inspired! Enjoy!

The Academic Coordination Team

Satsope Maoto (Coordinator),
Benard Chigonga, and Kwena Masha
June 2015
REVIEW PROCESS

The papers accepted for publication for the 2015 AMESA Congress were subjected to blind peer review by at least two experienced mathematics education reviewers. The academic committee considered the reviews and made a final decision on the acceptance or rejection of each submission, as well as changing the status of submission. Authors of accepted submission were given the option of not to have their accepted long papers published in the AMESA 2015 Proceedings, to keep open the possibility to submit it to a journal. They were requested to submit an extended abstract rather than their full submission, and this extended abstract will be published in the Proceedings for publication.

Number of submissions: 112
Number of plenary paper submissions: 5
Number of long paper submissions: 44
Number of short paper submissions: 2
Number of workshop submissions: 27
Number of ‘How I teach’ paper submissions: 14
Number of poster submissions: 0
Number of Maths Market: 20
Number of submissions accepted: 103
Number of submissions rejected: 9
Number of submissions withdrawn by authors: 8

We thank the reviewers for giving their time and expertise to reviewing the submissions.

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iv
<table>
<thead>
<tr>
<th>Plenary papers</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sarah Bansilal</td>
<td>Using APOS theory to understand some of the demands of teaching and learning Mathematics</td>
<td>1</td>
</tr>
<tr>
<td>Kabelo Chuene</td>
<td>The quality of teaching and learning, reexamining the fluid concept</td>
<td>15</td>
</tr>
<tr>
<td>Sarah Jane Johnston</td>
<td>Why are special functions special?</td>
<td>16</td>
</tr>
<tr>
<td>Percy Sepeng</td>
<td>Discussion, argumentation and realistic considerations in mathematics word problem solving</td>
<td>18</td>
</tr>
<tr>
<td>Edward A. Silver</td>
<td>(How) can we teach mathematics so that all students have the opportunity to learn it?</td>
<td>32</td>
</tr>
<tr>
<td>Authors</td>
<td>Title</td>
<td>Pages</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>Stanley A. Adendorff &amp; Bertus van Etten</td>
<td>Observations and thoughts on workings with number in senior phase mathematics</td>
<td>45</td>
</tr>
<tr>
<td>Benadette Aineamani</td>
<td>Teaching algebra with understanding</td>
<td>57</td>
</tr>
<tr>
<td>Benadette Aineamani</td>
<td>Teaching mathematics to the 21st century mathematics learner: a focus on mobile learning</td>
<td>69</td>
</tr>
<tr>
<td>Sarah Bansilal</td>
<td>The grade 9 ANA: perceptions and performance</td>
<td>81</td>
</tr>
<tr>
<td>C.A. Akintade, U.I. Ogbonnaya &amp; L.D. Mogari</td>
<td>Deepening the quality of mathematics learning through computer assisted instruction (CAI) in Nigerian secondary schools</td>
<td>83</td>
</tr>
<tr>
<td>Clemence Chikiwa &amp; Marc Schafer</td>
<td>Teacher code switching in a multilingual mathematics classroom: a focus on precision, consistency and transparency</td>
<td>99</td>
</tr>
<tr>
<td>Hester du Plessis, Mdutshekelwa Ndlovu &amp; Magda Fourie-Malherbe</td>
<td>The role of advanced programme mathematics in bridging the gap between school and university mathematics</td>
<td>104</td>
</tr>
<tr>
<td>Roland Fray, Washiela Fish &amp; Allen Taylor</td>
<td>What can teachers learn from comparing different strategies to find the general term of quadratic sequences</td>
<td>107</td>
</tr>
<tr>
<td>A. Giannakopoulos</td>
<td>Application of a psycho-pragmatic approach to the teaching and learning of mathematics.</td>
<td>125</td>
</tr>
<tr>
<td>A. Giannakopoulos</td>
<td>Using critical thinking in the teaching of differentiation in a first year university vocational course</td>
<td>139</td>
</tr>
<tr>
<td>Faaiz Gierdien</td>
<td>On working with mathematics teachers from historically disadvantaged high schools through a continuous professional development initiative</td>
<td>153</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>-----------------------------------</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>VG Govender</td>
<td>The assessment of quadratics in grade 12 mathematics examinations in South Africa: implications for teaching and learning</td>
<td>166</td>
</tr>
<tr>
<td>VG Govender</td>
<td>What are the views of mathematics teachers regarding the implementation of problem solving in their classrooms? A study involving both primary and high school teachers</td>
<td>184</td>
</tr>
<tr>
<td>Rajendran Govender &amp; Monde Mbekwa</td>
<td>Connecting a constructed Sierpinski generalization to the concept of infinity</td>
<td>205</td>
</tr>
<tr>
<td>Gasenakeletso Hebe</td>
<td>Investigating teachers’ reflections on teaching through lesson study.</td>
<td>224</td>
</tr>
<tr>
<td>Ibeawuchi E, Ogbonnanya UI, &amp; Mogari D</td>
<td>Secondary school mathematics teachers’ subject matter knowledge: the case of quadratic functions.</td>
<td>234</td>
</tr>
<tr>
<td>Mark Jacobs &amp; Duncan Mhakure</td>
<td>Recorded classroom practice: unpicking the complexities of teaching</td>
<td>250</td>
</tr>
<tr>
<td>Zingiswa Mybert Monica Jojo</td>
<td>Using instructional design to enrich mental constructs of a mathematical concept: a case study</td>
<td>265</td>
</tr>
<tr>
<td>Cyril Julie</td>
<td>A systematic review of teaching approaches that lead to the development of mathematical modelling competencies in high schools</td>
<td>279</td>
</tr>
<tr>
<td>Nothile Kunene &amp; Percy Sepeng</td>
<td>An exploration of factors that affect academic achievements in mathematics story problems: a case of 6th grade learners in rural school</td>
<td>294</td>
</tr>
<tr>
<td>Rolene Liebenberg</td>
<td>Engaging pre-service teachers’ flawed solution strategies in number context to develop structure sense</td>
<td>295</td>
</tr>
<tr>
<td>Authors</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>----------------------------------</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>Caroline Long &amp; Erna Lampen</td>
<td>Professional identity and teacher agency; necessary and sufficient</td>
<td>307</td>
</tr>
<tr>
<td>France Machaba &amp; Willy Mwakapenda</td>
<td>Teachers’ views on mathematics and mathematical literacy tasks</td>
<td>324</td>
</tr>
<tr>
<td>P. T. Mahlabela &amp; S. Bansilal</td>
<td>Learner errors and misconceptions on ratio and proportion</td>
<td>328</td>
</tr>
<tr>
<td>Sello Makgakga</td>
<td>Rural secondary school learners’ perceptions of environmental variables influencing academic performance in Limpopo, South Africa.</td>
<td>344</td>
</tr>
<tr>
<td>Judah P. Makonye</td>
<td>Education policy and the role of teaching practicum in the preparation of quality mathematics teachers in South Africa</td>
<td>352</td>
</tr>
<tr>
<td>SJ Matlala; DCJ Wessels</td>
<td>The experiences of secondary mathematics teachers teaching mathematics through problem solving</td>
<td>360</td>
</tr>
<tr>
<td>Bruce May</td>
<td>Proof: do South African learners call a sheep a dog because of a lack of adequate knowledge?</td>
<td>375</td>
</tr>
<tr>
<td>Michael Kainose Mhlolo</td>
<td>The learning paradox: an analysis of high school learners’ experiences of the informal and formal worlds of reflective symmetry</td>
<td>383</td>
</tr>
<tr>
<td>Mogege Mosimege</td>
<td>Mathematical modelling: reflections on a specific aim which is central in the implementation of curriculum and policy statement in mathematics</td>
<td>393</td>
</tr>
</tbody>
</table>
INTRODUCTION

There is no shortage of theories to explain the learning of mathematics with some being more valuable than others. Understanding the mental mechanisms which are associated with conceptual development in mathematics lies at the centre of the Action, Process, Object, Schema (APOS) model of learning. The theory was introduced by Ed Dubinsky in the early 1991’s (Dubinsky, 1991) in a bid to investigate the ways in which students develop understanding in advanced mathematical topics studied at university level. The theory has been applied widely to most topics in mathematic. Dubinsky and McDonald explains that “a theory of learning mathematics can help us understand the learning process by providing explanations of phenomena that we can observe in students who are trying to construct their understandings of mathematical concepts and by suggesting directions for pedagogy that can help in this learning process.” (Dubinsky & McDonald, 2001). In this paper I look at how APOS theory can help to understand barriers to learning and teaching.

I firstly focus on expanding the view of learning as changes in conceptions before presenting a more detailed explanation of APOS theory. This is then followed by a discussion on the fragility of the learning process as exemplified by one learners’ struggles with gaining fluency in multiplication. I then focus on some demands of teaching mathematics and finally reflect on how we could contribute to improving the mathematics knowledge of teachers.

Learning as changes in conception

De Lima and Tall (2008) explain that learning occurs by making new connections in the brain, which can happen when a series of actions is repeated until it becomes automatic. The authors argue that this relegation of the routine to the subconscious allows conscious thought to focus on important issues. The process of making links leads to a compression of knowledge from complicated phenomena to rich concepts with useable properties and coherent links to other ideas (de Lima & Tall, 2008, p.4). This compression of thought into “thinkable concepts” is an effort to reduce the load of working
Progression to further concepts in mathematics can be made less painful if children were able to compress some of the processes into readily available “derived facts” (Gray & Tall, 1991, p.3). A ‘derived fact’ is not a rote learned fact but is actually an encapsulation of the process leading to the result and captures the rich inner structure of the process. The derived facts are available to the child, allowing him/her to use these facts in further operations, in ways that children who don’t have this conception, cannot do. This view is supported by Simon, Tzur, Heinz, & Kinzel (2004, p.310) who argue that the learners’ assimilated conceptions afford and constrain what the mental system can recognise and what it can carry out to accomplish a goal. Gray and Tall (1991, p.4) discuss how progress from counting all to counting on can be hampered when compression to derived facts does not take place leading to “intolerable difficulties and a high probability of failure”. Their discussion starts by considering a number, say “three” as representing the process of counting “one, two, three” which can also be seen as the outcome of that process. The child is engaging in the process of “counting all”, as in “one, two, three”. Usually the counting all conception is transformed or compressed into a conceptual entity when a child makes the transition to seeing the number three as an object. This development can facilitate the child’s progression to counting on, allowing the child to consider 3 as an object and to then count on from the 3.

In considering the sum of two numbers, say, 3+2, a child may have a process conception of the sum which involves “counting all” or an object conception of the sum as in “counting on”. A child whose conception of three is as a process, will only have available to her the “counting all” strategy. A child who is still working with a process conception of three, can still reach an answer for 3+2, but in so doing she will use a more cumbersome procedure which involves counting all the three, then continuing and counting all the two. By the time she gets to the end of the process, she may have forgotten the beginning and thus the sum 3+2 will not be available to her as an object which could be used in further processes, such as in subtraction, sums which depend on an understanding of place value, or in repeated addition leading to multiplication. As the demand in complexity increases, the strategies used by children who have not moved beyond counting all lead to increased difficulties associated with a high probability of failure, while the child who has encapsulated the operations is able to move on to using these derived facts or encapsulations which are available, leading to greater fluency.

Children whose conception of number have not transformed, must continually revert to the action of counting all each time and must coordinate it with the further actions that are required by the repeated addition or subtraction. They can only work at higher levels by performing sequential processes. This difficulty of keeping track in the midst
of juggling sequential processes is captured in the following excerpt taken from Bansilal (2013)

When asked to write out the eight-times table, Lizzy just wrote the answers 8, 16, 24, 32, … (as opposed to writing the fact 8×1=8; etc.) … However, when she got stuck, around the 50s or 60s, she found herself in a tight position. When adding the next 8, she needed to free up her fingers from keeping place or position of the number that was multiplying the 8 in order to use her fingers to add the 8 again. At this stage, her anxiety deepened while she was trying to juggle the different processes that need to be carried out. When that was done, it meant starting counting in 8s again, but one step further with the new sum she had worked out. With all these different operations going on, trying to keep track of the different processes became harder to manage as she moved on to higher multiples of 8 [which required her to draw upon the first two toes for the 11th and 12th multiples]. (Bansilal, 2013, p. 100)

Anghileri (1989) presented a similar description of a learner’s struggle to keep track of sequential processes involved in working with the three-times table:

Having counted the middle three fingers of her left hand, ‘One, two, three’ she raised one finger on her right hand. Now she focused again on her left hand to count, ‘Four, five, six’ and raised a second finger on her right hand. Using her left hand she counted, ‘seven, eight, nine’. When she raised a third finger on her right hand, her gaze passed from right hand to left hand and back again.

At this point, she abandoned her first attempt, apparently confused over the differing roles of the fingers on her right and left hands (Anghileri, 1989, p.373).

Many authors attribute the difficulty that children experience in progressing to further processes, to a non-encapsulation of the initial process. Gray and Tall (1991) wrote about the compression into derived facts, that eases access to further operations based on these facts; Later on Tall (2004; 2010; de Lima & Tall, 2008) wrote about the idea of a procept which is a process and a symbol representing the process. Sfard wrote about the reification of concepts which can then form the basis for a higher-level concept. Most of these ideas are related to Dubinsky’s (1991) original work on APOS theory (Action, Process, Object, Schema) which focuses on a series of transformations of a concept in a learner’s cognitive structure.
The APOS model

APOS theory asserts that:

an action conception is a transformation of a mathematical object by individuals according to an explicit algorithm which is conceived as externally driven. As individuals reflect on their actions, they can interiorise them into a process. Each step of a transformation may be described or reflected upon without actually performing it. An object conception is constructed when a person reflects on actions applied to a particular process and becomes aware of the process as a totality, or encapsulates it. A mathematical schema is considered as a collection of action, process and object conceptions, and other previously constructed schemas, which are synthesized to form mathematical structures utilized in problem situations (Trigueros & Martinez-Planell, 2010, p.5)

I will use an example of a Grade 3 learner who is just learning multiplication to explain these mental constructs from APOS theory.

Action level: Suppose that the learner understands $5 \times 3$ as 5 lots of 3. She can be seen as operating on an action level of multiplication because she works out the result by performing each step of the transformation explicitly. In order to work this out she needs to draw her five groups each consisting of three dots and then use repeated addition or counting on to work out the answer.

Process Level: By building on her action understanding, the adult (or teacher) would design numerous activities that allow her to interiorize the understanding of $5 \times 3$ to a process level, which takes place when she can find the answer without needing to physically draw the lots any more, and the same action can be performed wholly in her mind. However a child who does not have access to the repeated addition concept as an encapsulation will not be able to advance to this more efficient understanding of $5 \times 3$ – because she will always need the comfort of the physical counting of the dots to get to the answer.

Object Level: By providing activities that enable her to understand properties such as $5 \times 3$ equals $3 \times 5$, and $6 \times 5 = 2 \times (3 \times 5)$, that is, becoming aware of the process of $5 \times 3$ as a totality which can be acted upon, her access to “different ways of working on objects” (Valsiner, 1997, p.68) will be realized. Her understanding of $5 \times 3$ will move from seeing it as an action, to a process and then to an object and she would then be ready to understand the concept of division, or calculations such as $30 \times 5 = 10 \times (3 \times 5)$
which requires that a child understands the product $5 \times 3$ as an object which can be acted upon.

Hence within APOS theory a person’s understanding of a concept can be transformed from an externally driven entity into a totality upon which other transformations can act. As objects are operated on in further processes they contribute to layers of understanding. Sfard (1992, p.70) asserts that the phrase process before object, refers to one individual cycle in the development of mathematical ideas, which starts with a new idea which then gets encapsulated or reified and then forms a basis for a higher–level concept. This cycle adds another layer in the system of mathematical concepts.

In APOS theory the description of these layers or relationship between the concepts is referred to as a genetic decomposition (Meel, 2003) and is used to characterise the linkages and representations that are possible when developing an understanding of a concept. A genetic decomposition can provide a description of a possible path for a learner’s concept formation. Curricula are designed around particular genetic decompositions that are assumed by the curriculum designers. One possible genetic decomposition that can describe progression from single addition facts to multiplication appears in Figure 1.

![Figure 1. Layers of understanding in progressing from counting to multiplication tables](image-url)
The case of Lizzy

Meel (2003, p.154) cautions that a genetic decomposition “provides a possible path for a learner’s concept formation; however it may not be representative of the path taken by all students”. Sometimes when the curriculum is designed around the assumption of a particular genetic decomposition, certain learners’ own conceptual development may proceed differently. I present an example of how this path did not work particularly well for one child. This is recounted in more detail in the article: Lizzy’s struggles with attaining fluency in multiplication tables (Bansilal, 2013).

I first outline the details of the genetic decomposition represented in Figure 1.

**Initial Layer: Understanding single addition facts**

A first step in understanding single addition facts requires an action understanding, for example the addition that $5+6=11$. Learners can use their fingers, or dots, or other means to work such a sum where each single element (say $5+6=11$) is understood at an action level first. However by posing questions like “$5+\text{what}=11$” may help them to move to process level- seeing $11$ as a result of $5$ plus $6$. Questions such as “$11-5=\text{what?}$” focus attention on the relationship between addition and subtraction. Understanding these single addition facts and recognising the associated subtraction in each decomposition, implies that an object understanding of that addition fact has been achieved.

**Further Layer: Understanding number bonds associated with a certain number**

The next stage involves seeing the set of addition facts and its decompositions of one number, that is, the complete set of possible decompositions for $11$ say, such as $11+0=11; 10+1=11$ etc. This is commonly referred to as number bonds of $11$. An object understanding involves seeing this set of possible decompositions as a whole. It involves understanding each individual number bond set as an object. Being able to do the decompositions of each one separately is displaying an object understanding of each string of number bonds.

**Further Layer: seeing patterns of addition facts across different number bond strings**

When a concept has been encapsulated into an object, the learner can de-encapsulate an object, thereby returning to the process in a form prior to its encapsulations. De-encapsualtion enables the learner to use the properties inherent in the object to perform new manipulations upon it. (Meel, 2003, p.152).
Thus when the learner has developed an object understanding of various individual number bond strings, a learner should be able to pick out particular decompositions across different number bond sets. For example the learner will recognise patterns such as adding and subtracting 5 within the various number bond sets, for example, seeing that $6+5=11$ but $7+5=12$. When they are able to work out the difference between $5+6$ and $5+7$, as being 11 and 12 respectively, and when they can associate all bond pairs of 13 with 13, and simultaneously see the difference between groups of sums, they are developing an object understanding of the set of strings, because they are able to see the differences between the sets of decompositions belonging to individual numbers such as 13 and 14, say. Figure 2 below shows the decompositions for strings 12 and 13, by identifying patterns such as $5+7=12$ and $6+7=13$, across the different strings, a child should then be able to see a commonality in adding by sevens and subtracting by sevens. Similarly de-encapsulation will also allow them to see commonalities in adding and subtracting by nines.

![Figure 2: Recognising patterns across number bond strings](Bansilal, 2013, p.98)
In a study focused on one learner (Bansilal, 2013) Lizzy’s difficulties in reaching this stage were described. I realised that the bond patterns that she spent over two years practising in school so rigorously did not help her progress through the layers of concepts as envisaged by the curriculum. In fact for her, instead of focusing on the strings of number bond patterns, it worked better to look at patterns of adding a fixed number to various numbers such as 1+5; 2+5, 3+5, 4+5; 5+5 and so one, because this helped her see patterns in addition of 5 for example, which then eased the learning of multiplication facts. A colleague from Columbia suggested the use of two tables made up of ten cells each, as illustrated in Figure 3 with red and black counters representing different numbers. We practiced addition facts of 8 such as: 8+5 =? 8+7 =?; where we used black counters to represent the eight and red counters for the other numbers. The associated subtraction (13-8, say) could also be demonstrated by leaving the 8 black counters on the first table and then placing in red counters until there were ten in the first table, then three in the second table. We then blocked the black counters, and asked, “how many counters are left?” This helped her see addition and subtraction by 8 in a concrete manner, and also helped her concretise various decompositions of 10. If she got stuck, she placed the counters and physically counted them. After a while, she could imagine in her head what 8 +7 meant – two reds to make up the ten, leaving five reds for the next block, therefore 15.

![Figure 3: Using the “make ten” strategy to consolidate addition and subtraction facts](image-url)
Lizzy initial use of the counters represented an action level understanding of addition by 8. After some practice she became comfortable with reaching the answer without physically moving the counters around. By doing this repeatedly she could then see patterns in adding 8 to other one-digit numbers, leading her to an object understanding of the number facts associated with 8. Her progression to the multiplication facts of 8 was then eased and not as cumbersome as detailed earlier, where she confused herself with trying to keep count with her fingers.

This description was meant to highlight the usefulness of APOS theory in trying to diagnose barriers to learning experienced by Lizzy. This experience revealed the fragility of the process of concept development and how easily it can be disrupted. APOS theory emphasises that movement to higher level concepts will be limited if prerequisite concepts have not been encapsulated. This casts the role of the teacher into the spotlight, by highlighting how intricate the task of teaching mathematics can be.

The task of the mathematics teacher

Teachers often have to identify blockages in learners’ conceptual development and have to go back and help the learners resolve these gaps in knowledge before moving on to further concepts in mathematics. The task of teaching mathematics is not one that anybody can do. It requires much planning, and instruction that is tailored to the needs of the learners. As mathematics teachers we all agree that learners need much practice in order to improve. At different stages of the concept development they need different sorts of practice. As we gain experience we learn to recognise the signs which tell us which learners are ready for the next level of practice. However we often realise too, that sometimes we may move too soon and a child can lose their way.

As learners’ understanding of a concept develops, their engagement with concepts becomes deeper. At first they engage at an action level- by responding to external cues. As they get a better idea of what they are doing, they interiorise it into a process, and then with the right opportunities and support, they can encapsulate it into an object. At the different stages of development they require different scaffolding. A learner who is operating at an object level with the concept of quadratic equations requires different opportunities for practice as compared to when the learner is working on an action level. A learner who is still coming to terms with solving a quadratic equation can solve a quadratic equation similar to \(2x^2+5x+3=0\) by following the steps in the procedure.

However s/he cannot be expected to move to solving a problem such as \((2k+1)x^2-kx+1=0\) immediately. The first few examples that are used are usually in the standard form of the quadratic equation so that the learners can familiarise themselves with the
formula/ technique before moving to different factorisation techniques. As they develop confidence in knowing how the procedure works, then they are provided with different tasks.

Hence a key pedagogic skill of a mathematics teacher is the presentation of the tasks, or explanation or practice exercises in a way that allows the learner’s understanding of a concept to become increasingly more sophisticated. A teacher who tries to cover all the geometry theorems in Grade 10 in one week is being shortsighted because the time spent on that coverage will be wasted. Movement from an action to a process to an object level of understanding of a concept takes a long time and requires a lot of detailed attention to where the learners are and what needs to be done to move them on. Recently we have heard much about mathematics camps. For example when Grade 12 learners missed most of the year because of a problem with a road in a certain province, it was reported that the learners attended a camp to catch up on the work they missed. In another province last year, there was a well-publicised intervention just before the final examinations for matric learners. Many Grade 12 learners were bussed in from different districts to a few schools and at the camp they were taught most of the Grade 12 work. This constituted a quick tour through the syllabus, which politicians were very pleased about. However, when this did not result in improved scores for the learners who attended, there were no debates about the value of this kind of treatment of mathematics and under what conditions it could be used. The assumption seems to be that the teacher needs to “cover” the syllabus by providing explanations and that is all that a mathematics teacher does. Mathematics camps, like other interventions, can be very useful if well planned. At these workshops learners can be challenged with higher level questions or even just given the opportunity to work with basic mathematics tasks. But a single camp or workshop cannot replace three years of teaching – the Grade 10-12 curriculum cannot be compressed into a few weeks of teaching from 8 in the morning to 10 in the night. The only thing that could possibly happen is that the concepts remain as external entities for these learners. When so many topics are introduced in quick succession, the highest level of engagement will be at an action level. In order for interiorisation to occur one needs carefully sequenced tasks and time and object level engagement needs much more of the same.

Of course, even if teachers take their time to try to facilitate the knowledge building of the learners, it will not work if teachers do not have a sound knowledge of the content they need to teach. There has been much attention focused on the issue of teachers’ content knowledge in mathematics. In fact a recent study carried out with my colleagues (Bansilal, Brijlall and Mkhwanazi, 2014) attracted much attention because it reported
that the average mark obtained by a group of 253 mathematics teachers in a shortened form of a Grade 12 mathematics examination was 53%. The study generated many opinions about the state of mathematics teachers’ knowledge in general. I want to briefly consider Michael’s (a teacher in the study) response to a question based on the solution of a quadratic equation (Bansilal et al., 2013) presented in Figure 4.

![Figure 4: Michael’s written response to Q1.1](image)

Firstly note that Michael did not recall the quadratic formula correctly. (This is ironic because the formula sheet that accompanied the test contained the correct formula.) This incorrect formula served as a false stimulus triggering an action that led to a meaningless process. Thus for Michael movement to a process conception of solving quadratic equations was impeded by an ineffective action level, because the incorrect quadratic formula was used. This is a response of an FET teacher and this raises questions about the kind of instruction that Michael can offer to his learners. Wu (2004) comments that often a well-intentioned pedagogic decision in the classroom can be betrayed by faulty content knowledge. A better understanding of the content could lead to a different pedagogic strategy and a mathematics lesson that is easier to understand. In Michael’s case, he will not be able to recognise the correct and incorrect use of the formula. Furthermore, Michael will not be able to recognise learners who are working at more sophisticated levels and require more challenging work.

Hence if teachers are themselves working on action levels with concepts the quality of teaching can be severely compromised. In a further study (Bansilal, 2015), the Rasch analysis showed that the proficiencies of 66 teachers from the sample of 253 teachers were below the item difficulties of all level 3 and Level 4 items in the paper. This suggests that these teachers would only be able to deal with routine problems or questions based on recall of knowledge. This and other studies suggest that teachers need help in improving their knowledge of the mathematics they are teaching. However
this is a problem that has been identified and debated nationally, and attempts to solve this problem have yet to be successful.

**Supporting teachers**

The Department of Education has tried to initiate an intervention where teachers meet in groups to be taught the content before they teach their learners. This strategy seems to be a desperate but necessary step. But this and other interventions are not sufficient, and there needs to be more widespread agreement of the problem and a willingness to address it at all levels, from the individual teacher who needs to start taking responsibility for his/her own learning, to the subject leaders (HOD’s) who need to start providing support to teachers at their sites; to the subject advisors who need to ensure that they are providing support to teachers; and, to the Education departments who must commit to managing their departments.

What research does show is that schools which work well are those with strong support systems, where teachers meet regularly and plan together. In these settings teachers make up their teaching plans together; they plan and administer common assessments; they talk about how they teach; they talk about how they mark, and they talk about problems they are struggling to solve. In this environment it is the small things that make a big difference. If every teacher sits back and waits for the National Department of Education to sort out the problem of teachers’ mathematical knowledge, it will never be solved. As individual teachers, we can start chipping away at the problem individually and in school communities, in Amesa district forums, and so on. Every school must be a place where learners learn and it must also be a place where teachers learn from their teaching experiences and where they learn from one another.

I end this paper by presenting you with a challenge. The teacher Michael who did not know how to use the quadratic formula could be in your school. If he was, would Michael feel safe enough in your mathematics department to admit his difficulties? Or would he have to hide his problems for fear of being ridiculed? What is it that you can do to help identify teachers with these problems. Once identified, how will you help them to progress? How would you make your environment safe enough for people like Michael to share their problems so that they can be helped. My challenge to you is to take steps to make your department more supportive and a place where teachers feel safe to learn together.

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Performance in the last year of school mathematics has, in the last two decades become a principal measure of quality of education in South Africa. Performance in Annual National Assessment, National Benchmarking Test and Matriculation examination are scrutinized because they do not paint a good picture of the quality of teaching and learning mathematics. There is an outcry on how this impacts on the workforce in fields that relate with mathematics both in quantity and quality. All concerns raised necessitate measures through which the current quality of teaching and learning mathematics should be curbed and overhauled at both macro and micro systemic levels.

There is evidence that wide-ranging overt and covert attempts to deepen the quality of teaching and learning mathematics have been made. Minimum requirements for education have been revised to impact on the quality of graduating teachers. Curriculum and assessment statements have been revised to provide more on clearness of thought on what is expected of teachers and learners. Teachers who have, through the years, taught mathematics without the necessary training were given relevant training. There is improved monitoring of what happens in schools by stakeholders in education. Regardless of these noble practices, the quality of teaching and learning is not remarkable.

Quality teaching and learning as a principle is an erratic concept. Its interpretation is also relative and diversified. The quality of teaching and learning mathematics in elite urban schools is inarguably nuanced differently from schools in remote rural schools. Negotiated and none negotiated frameworks and ethos that inform practice in mathematics classrooms discriminates its quality of teaching and learning. Furthermore perceived identities on specialist in mathematics teaching sways perceptions the quality of teaching and learning. The picture painted goes to show how hard teaching and learning mathematics is. Also, it highlights how thought-provoking deepening the quality of teaching and learning mathematics is. Thus homogeneity in conceptualizing the quality of teaching and learning is necessary if it to be deepened.
WHY ARE SPECIAL FUNCTIONS SPECIAL?

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EXTENDED ABSTRACT

Special functions are mathematical functions which are classified in a certain way due to their importance in mathematical analysis, functional analysis, physics and other applications. Paul Turán once observed that special functions would be more appropriately labelled "useful functions". Because of their remarkable properties, special functions have been used for centuries. There is, in general, no formal definition of a special function, however the list of these so-called special functions contains functions which are commonly accepted as special and most of these functions were introduced to solve specific problems.

For example, trigonometric functions have been studied for over a thousand years since they have numerous applications in astronomy. The series expansions for sine and cosine (and most likely the arc tangent) were known in the fourteenth century and rediscovered by Newton and Leibniz in the seventeenth century. Since then, the subject of special functions has been continuously developed with contributions by many famous mathematicians, including Euler, Legendre, Laplace, Gauss, Kummer, Eisenstein, Riemann, and Ramanujan.

In the past thirty years, the discoveries of new special functions and of applications of special functions to new areas of mathematics have initiated a resurgence of interest in this field. These discoveries include work in combinatorics, orthogonal polynomials and hypergeometric functions. In particular, almost all of the elementary functions of mathematics are either hypergeometric functions or ratios of hypergeometric functions and many of the non-elementary functions that arise in mathematics and physics also have representations as hypergeometric series.

In this presentation, we will consider some of the elementary special functions taught at school and university and explore the properties that make them special. This list includes the trigonometric functions, logarithmic function, exponential function and more. We will look at their history, their eponyms, their applications, their usefulness in education and industry, and briefly, their link to the geometric and hypergeometric functions. We will also look at the operators and calculus governing each of them. We will discuss classic theory, contemporary theories, moonshine theories and ambiguities.
The issue of teaching Mathematics for understanding and the issues of language in Mathematics teaching and learning have been with us for a while. In fact, teacher educators and educational practitioners have argued for mother tongue instruction within South African multilingual classroom settings. However, language is just but one of many other reasons for learners’ poor academic achievement in mathematics amongst learners who speak language(s) other than that used in formal schooling and for assessments. In the work that I have done for almost a decade, poor problem-solving skills, failure to recognize reality and use of reasonable contexts in word problem solving, as well as inability for teachers to employ relevant and appropriate techniques in the teaching of problem-solving in school mathematics seemed to be a predictor of poor academic achievements (see for example Sepeng & Webb, 2012; Sepeng 2013a, 2013b, 2013c, 2014a, 2014b; Sepeng & Madzorera, 2014). In these studies, I reported factors that appeared to influence learners’ academic achievement in mathematics word problems and suggested possible remedies to this effect.

For the purposes of this talk, I will only focus on two issues: 1) the roles of discussion, argumentation and dialogue as a technique in the teaching and learning of mathematics, and 2) mathematics word problem solving.

I begin the talk by providing an overview of what constitute problem-solving in mathematics classrooms of South Africa, by looking at issues of connecting formal ‘classroom activities’ with those experienced by learners outside of the classroom, which I refer to as ‘informal out-of-school mathematics’. I do this in order to demonstrate the disconnect that exists between what is taught in school and the mathematics at play outside the school premises. In addition, I will use few examples and/or scenarios to illustrate the magnitude of a gap that contributes towards learners’ tendencies to relegate and ignore (willingly or otherwise) reality when the engage in formal classroom mathematics. In so doing, I will then create an intelligent space to provide an argument on how various strategies, such as discussion, argumentation, and dialogue, may be employed as a technique to address some of the concerns raised earlier on.

The second and last part of the talk addresses a notion of mathematics word problem solving within a multilingual classroom settings. I expose a common ground by demonstrating how senior primary school learners solve word problems using a case of
I then present and use examples in a form of word problem tasks as well as extracts obtained from learners’ focus groups. Issues of sense-making (or meaning-making) are discussed in relation to reality in problem-solving and learners’ personal interpretation of problem situation. I conclude and provide a resolution by discussing an overview of real context in word problem-solving regarding mathematical tasks given to learners.

**Connection between classroom mathematics and real-life knowledge**

The connection between classroom mathematics and learners’ everyday experiences is a complex issue because the two contexts differ significantly (Cooper & Harries, 2005). While there may be some inherent differences between the two contexts, these can be reduced by creating classroom situations that promote learning processes closer to those arising from out-of-school mathematics practices. I have demonstrated elsewhere, consistent with other reports in the world, that in normal teaching practice, establishing connections between classroom mathematics activities and everyday-life experiences still regards mainly word problems (De Corte, Verschaffel, & De Win, 1985; Sepeng, 2013b, 2014a; Sepeng & Webb, 2012). As such, word problems are often one of the means of providing learners with a basic sense experience in mathematisation, especially mathematical modelling (Reusser, 1995). In other words, there exists a need for effective pedagogies that empower mathematics teachers to teach mathematics word problems with understanding through connecting classroom mathematics with out-of-school mathematics.

I now present two examples of how learners connect activities in the classroom compared to their everyday-life experiences and informal knowledge about the mathematics. The two tasks were previously given to Grade 9 learners at township schools. Learners were encouraged to elaborate both in writing and verbally how and why they solved the problems the way they did. I start a discussion with a Bus problem and conclude with presentation on the Pencil problem:

**Bus Problem:** 100 children are being transported by minibuses to a summer camp at the sea-side. Each minibus can hold a maximum of 8 children. How many minibuses are needed?

When asked about how they solved this problem in the follow-up focus group discussions, one of the learners responded by saying:

“...when you divided children like that you will count that each minibus will get in 8 children, if you divided them in groups of 8, so
“that all 100 children can enter in all of the minibuses and you will know that you must have 12 minibuses.”

The following table shows learners written responses to this problem:

<table>
<thead>
<tr>
<th>Answer</th>
<th>Realistic reactions (RR)</th>
<th>Other reactions (OR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>i. 100÷8=1 2.5</td>
<td>therefore 13 minibuses are needed (situationally most appropriate answer)</td>
<td>so 12.5 minibuses are needed (mathematically accurate, but numeric answer is situationally inappropriate)</td>
</tr>
<tr>
<td>ii. 100÷8=1 2; remainder 5</td>
<td>so 12 minibuses are needed (answer that does not consider problem context and situation)</td>
<td>so 12 minibuses are needed and a car for remaining 5 children (mathematically incorrect, other answer that considered problem situation)</td>
</tr>
<tr>
<td>iii. 100÷8=8.5</td>
<td>so 9 minibuses are needed (computational error, but situationally appropriate)</td>
<td></td>
</tr>
<tr>
<td>iv. 100×8=180</td>
<td>so 180 minibuses are needed (incorrect operation used and calculation error, without situationally appropriate interpretation)</td>
<td></td>
</tr>
</tbody>
</table>

I therefore argue that everyday-life experience and classroom mathematics, despite their specific differences, should not be seen as two disjunctive and independent
entities. Instead, a process of gradual growth is aimed for, in which classroom mathematics comes to the fore as a natural extension of the learners’ experiential reality. The idea is not only to motivate students with everyday-life contexts but also, as Gravemeijer (1999, p.158) points out, “to look for contexts that are experientially real for the students and can be used as starting points for progressive mathematisation”. Studies conducted on word problem-solving (e.g., Säljö, Riesbeck, & Wyndham, 2009; Verschaffel, Greer, & De Corte, 2000; Verschaffel, Greer, & Van Dooren, 2009) reveal that as learners are acting in a complex situation, they have to consider what discursive practices are relevant and acceptable when solving problems and when arguing in a particular setting.

I now use a word problem task classified as a real-life mathematical word problems without real meaning to demonstrate learners’ tendencies to exclude realistic considerations when the engage with such problems in mathematics classrooms. To do this, I use an extract of a discussion that took place immediately after the learners solved the problem as an example.

_Real-life Mathematical Word Problem without Real Meaning (PWRM)_

You have 10 red pencils in your left pocket and 10 blue pencils in your right pocket. How old are you?

The situation in South African primary classrooms is similar to that in which many Western students have been challenged by the famous _shepherd’s age_ problem (e.g., _There are 125 sheep and 5 dogs in a flock. How old is the shepherd?_ (Greer, 1997; Nesher, 1980). Most of these students would answer —130, as they normally do in the classroom. Similar studies on realistic problem-solving have been conducted in China (see Liu & Chen, 2003; Xin, Lin, Zhang, & Yan, 2007; Xin & Zhang, 2009; Xu, 2007), where students are often confronted with word problems such as: _There were 5 birds on a tree. If one bird was shot down by a hunter, how many birds are left?_ A more realistic answer to the problem would be “None, because all of the other birds would be frightened away by the sound of the shot” (Xin, 2009), however, most of these students would give an answer as “4”.

**Extract**

**R(esearcher):** How did you approach this question?

**L(earner) 3:** My age is 20 years old, I added up 10 red pencils and 10 blue pencils and I got the answer 20.
R: OK, what about you L5? How did you approach the question?

L5: I added up 10 red pencils and 10 blue pencils then I got the answer 20.

R: Any other different approach?

L5: None.

R: Your solution may not work in real life because of real factors. Why did you answer that way?

L6: The question said you have 10 pencils on this side and another 10 pencils on the other side, so I thought because of the question did not ask anything on personal details and then I thought when you add up the pencils from both sides, it bring up the total of your age or something.

L3: It's because the question didn’t ask how old you are in real life.

The text in the Extract shows that learners failed to recognise everyday knowledge and their understanding of everyday practices described in the word problem. Cooper (1998) offers a different explanation for the reason behind the unrealistic solutions, suggesting that it originates from the socio-cultural norm of schooling that emphasises de-contextualised, calculation exercises. In actual fact, learner 3’s argument that “the question didn’t ask how old you are in real life” confirms Cooper’s view, that the learners’ unrealistic responses reflect the learners’ relationship to school mathematics and their willingness to employ the approaches stressed in school.

**Reality in problem-solving**

In solving this problem, learners readily responded “I am 20 years old”, as if their own age could be determined by the reasoning that ”I added up 10 red pencils and 10 blue pencils “. Schoenfeld (1991) characterised this type of problem-solving as suspension of sense-making, referring to disconnect between learners’ understanding of reality and problem-solving. As such, both the learners’ problem-solving was lessened and relegated to a procedural, mechanical task with little or no sense-making beyond the number procedures used in this problem.
Sense-making of problem statement

All the written responses that reflected the common-sense understanding of everyday practice were categorised as sense-making of problem statement. In extract presented above, learner 6 argued that —_So I thought because of the question did not ask anything on personal details and then I thought when you add up the pencils from both sides, it bring up the total of your age or something_”, symbolises Lave’s (1992) description of word problem-solving as stylised representation of hypothetical experiences separated from learners’ experiences. According to Inoue (2005), students’ minds could be torn between two types of knowledge systems that the word problem activates: one developed in the traditional mathematics classroom and the other developed in through real-world experiences. In this study, the learners who gave calculation answers appeared mindless and mechanical; however, pre-intervention observations into these classrooms revealed that what is really problematic seems to be the lack of the opportunity for the students to freely bridge the calculation answers and their everyday life knowledge.

Personal interpretation of problem situation

Most of the learners’ responses appeared to have been largely influenced by personal interpretation of the problem statement based on quantitative information that led to word problem-solving resulting in calculation exercises, and not a solution that makes sense in terms of their everyday knowledge and experiences. It is also interesting that majority of the learners in these classrooms interpret the problem situation the same way, with justifications pointing the same direction of reasoning. The extract used here showed that when the learners answered: —20 year’s old, it was not because they did not know their actual age, or they did not understand the relevant mathematical concept (Frankenstein, 2009). Rather, it was, as (Pulchaska & Semadini, 1987) suggest, because learners give illogical answers to problems with irrelevant questions or irrelevant data is that those learners believe mathematics does not make any sense.

Discussion and argumentation in mathematics classrooms

In mathematics education studies language has been conceived and examined in a number of ways including the nature of mathematical talk or discussion and argumentation in the classroom, the discourse practices entailed in the learning of mathematics, and the challenges and opportunities within linguistically and culturally diverse mathematics classrooms.

Argumentation in classroom contexts encompasses a process where learners make a claim, provide suitable evidence to justify it, and defend the claim logically until a
meaningful decision has been reached (Webb, Nemer, & Ing, 2006). These researchers reported that the use of discussion as a tool to increase reasoning has gained emphasis in classrooms worldwide, consistent with earlier reports (Yore, Bisanz & Hand, 2003). Discussion, however, requires scaffolding and structure in order to support learning (Norris & Phillips, 2003).

Wood (2006) found variation in students’ ways of seeing and reasoning, and these were assigned in the first place to the particular differences established in classrooms early in the year pertaining when and how to contribute to mathematical discussions and what to do as a listener, consistent with findings reported by a number of other researchers (e.g., Dekker & Elshout-Mohr, 2004; Ding, Li, Piccolo, & Kulm, 2007; Gillies & Boyle, 2006; Webb et al., 2006). Moreover, participation obligations put boundaries around the opportunities for students to share their ideas and to engage in mathematical practices (Ding et al., 2007; Webb et al., 2006).

Issues of interest to mathematics educators, such as, for example, knowing, can be examined from the perspective of participants in interaction, rather than as underlying cognitive processes which can be used to explain what people do and say (Edwards, 1997). As Edwards & Potter (1992) acknowledge, this is not to say that people explicitly talk about these things. As Sacks showed, these patterns of interaction arise through the social actions of the participants; actions which bring about the on-going organisation of their talk (see Sacks, 1987). For discursive psychology, the social action through which interaction is organised takes precedence over other aspects of interaction, so that the psychological structures and functions of language became shaped by language’s primary social functions (Edwards, 1997).

Talk is about more than its surface content. Every utterance, for example, also constructs the identity and reflects the interests of the speaker, who may present themselves as, loud or polite, knowledgeable or uncertain, biased or neutral. Each utterance, therefore, reflects the partiality or interest of the speaker (Antaki, 1994). Amongst empirical studies of foreign language attainment, a focus on recycling in local classroom communities can be seen in the work of Rampton (1999) on how foreign language teaching is recycled in peer group interactions and participation among adolescents as substantial resources in performance-based identity work.

For learners, discussion, debate and critique are all learned strategies. Sfard and Kieran (2001, p.70) emphasise that "the art of communicating has to be taught". As such, learners should be afforded appropriate time and intelligent space for exploring ideas and making connections (Stein, Grover, & Henningsen, 1996) between classroom mathematics and out-of-school mathematical knowledge and a sustained press for explanation, meaning, and understanding (Fraivillig, Murphy, & Fuson, 1999). Such a
move will support Carpenter et al.’s (2003) notion that the very nature of mathematics presupposes that students cannot learn mathematics with understanding without engaging in discussion and argumentation. It, mathematical discussions and thinking were greatly enhanced by the pedagogical practices that allowed learners to engage in argumentation (Empson, 2003; Goos, 2004). In doing so, learners were not only in is therefore important for mathematics teachers to encourage learners to not only discuss classroom activities and solve word problems, but train them how to get involved in taking and defending a particular position against the claims of others (O’Connor & Michaels, 1996). They pointed out that this teaching process depends on the skilful orchestration of classroom discussion by the teacher. In particular, teachers in the experimental group showed signs of improvement over time, and had begun to understand how to promote discussion in mathematics multilingual classrooms, via the use of the concept cartoon as a stimulus. However, as Ball (1993) pointed out, highly articulate students displayed a tendency to dominate classroom discussions and, as such, the management of classroom discussion appeared to be vital if one is to promote conceptual understanding via this technique (Steinberg, Empson, & Carpenter, 2004).

**Classroom Interactions and Dialogue**

The baseline observations revealed that classroom interactions in the experimental schools took the form of teacher initiated talk (Mercer, 1995), characterised by teachers’ regular use of inauthentic initiation turns. In cases where the teacher asked questions, learners responded in chorus (Mayaba, 2009). Moreover, these classrooms were embedded with social discourses that reflected learners’ socio-cultural backgrounds (Lemke, 1990). There were only few occasions that resulted in learners’ engagement in dialogue, which occurred between the teacher and a few individual learners. As such, there were no understanding and agreement of rules of engagement between the teacher and learners in these classrooms to actively engage with mathematical discourse in order to contribute positively in problem-solving initiatives. The tendency by learners to be passive may be attributed to the classroom linguistic structures that were restricted to English, characterised by teachers’ inability to attend to gestures, representations, and everyday descriptions that second language learners draw on to create and communicate meaning in mathematics classrooms (Nasir, Hand, & Taylor, 2008). In doing so, teachers inadvertently miss the multiple, rich resources that learners bring to the classroom. However, effective continuing teacher development initiatives should be planned in a way that allows teachers to demonstrate the abilities to allow learners to actively engage in mathematical discourses that paved way for the learners to effectively interact with the mathematics contexts in content via classroom discussion.
Contexts in word problem-solving

A report by Julie and Mbekwa (2005) raises concerns with the way in which the notion of what constitutes a relevant context might not be in the same for curriculum developers, teachers, and learners. I have argued elsewhere that mathematics word problems used in school curricula are not relevant to and do not address the socio-cultural situations faced by and known to the learners from poor socio-economic backgrounds. Sethole (2004) suggested that foregrounding of context may lead to a loss of focus on the development of conceptual mathematics knowledge and render the mathematics invisible or inaccessible. Contrary to this, I propose that with well-planned and effective teacher development interventions, the issue of context in mathematics teaching may play a pivotal role in the development of learners’ problem-solving abilities. If we don’t influence policy makers towards bridging a gap between what is done in the classroom and learners’ informal knowledge about the mathematics, we run a risk of remaining with current school instruction given for mathematical word problems which is likely to develop in students a tendency to exclude real-world knowledge and/or reality in their solution processes (Sepeng, 2014a; Cooper & Harries, 2005; Greer, 1997; Verschaffel, De Corte, Lasure, Vaerenbergh, Bogaerts, & Ratinckx, 1999).

According to a socio-cultural perspective, modelling implies engaging in inter-semiotic work, that is, one has to decide about the appropriate and productive manners of coordinating linguistic categories and mathematical expressions and operations in order to come to a solution of a problem (Säljö et al., 2009). In my observations of lessons conducted by mathematics teachers over the past decade or so, it seemed that more emphasis is put on syntax and mathematics rules rather than, what Xin (2009) refers to as, a description of some real-world situation to be modelled mathematically. As a result, it could be argued that a significant number of learners who produced nonsensical solutions to word problems presented earlier, might have made realistic considerations during the solution process, but finally have decided to neglect reality in their final answers. Learners may have simply anticipated that such ‘unusual answers’ would not be appreciated by the researcher and/or the mathematics teacher (Verschaffel et al., 2000). However, analysis of the effects of promoting discussion and use of out-of-school mathematics revealed that the ability to take into account real-world considerations properly when solving word problems increased, as expected, with sense making.

Word problems without real contexts

The data generated via word problems without real contexts (PWRC) in this study revealed that an inappropriate use of contexts, particularly when learners are invited to
engage in the real world, but then penalised for doing so, results in classroom inequalities (Boaler, 2009). As learners learn to answer nonsensical questions about *number of blue and red pencils in your pocket*, they come to believe that mathematics classrooms are strange places in which common sense cannot be used. In doing so, learners realise that when you enter *Mathslands*, you leave your common sense at the door (Verschaffel et al., 2000).

The finding that unrealistic solutions may not simply stem from mindless or procedural problem-solving, but could originate in students‘ diverse effort to make sense of the problem situation and the nature of the problem-solving activity in socio-cultural contexts, is consistent to the reports by Inoue (2009) and Verschaffel et al. (2009). For example, the learners‘ responses to the bus problem indicated that their everyday socio-economic life experiences and knowledge influenced the way they interpreted the problem situation (or context). In particular and as noted before, they produced different definitions of a week in their problem solutions. In addition to social and cultural class inequalities that result when certain contexts are used, one group of learners is prompted more than the other, to engage with real-world variables, thus compromising their performance (Boaler, 1994). In other words, learners who come from households of professionals, will interpret a week as having five days in cases of school teachers, or less than five days in instances where a parent is employed as a domestic worker, or any other job that requires them to report six or seven days in a calendar week. In fact, Zevenbergen’s (2000) studies have shown that learners of a particular linguistic or cultural background are similarly disadvantaged or advantaged.

**CONCLUSION**

The paper provided an overview of the roles of mathematics word problems as a way of bridging a gap between what is done in the classroom formally and the learners‘ informal ways of doing and applying the mathematics learned in school at home. Issues of using discussion and argumentation as techniques in the teaching and learning of mathematics formed part of strategies to use in assisting learners to comprehend mathematics word problems better. Such techniques may also be used to assist learners with a way to include, where possible, realistic considerations when they solve word problems. In doing so, the process of successfully making sense of the problem statement becomes much easier and as a result, may go a long way to improve learners‘ problem-solving skills. Use of reasonable and acceptable contexts may go a long way in improving learners‘ problem-solving abilities of primary school learners within South African contexts.
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(HOW) CAN WE TEACH MATHEMATICS SO THAT ALL STUDENTS HAVE THE OPPORTUNITY TO LEARN IT?

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The theme of mathematics for all students is not a new one. It has deep roots that stretch as far back as the compulsory education movements in many countries more than 100 years ago, influenced by the writings of John Dewey and others on the value of access to high quality education as an essential ingredient to ensure an informed citizenry in a democratic society. It has been a recurring theme in more recent times as educators and policy makers in many countries, especially those with evolving democracies, have sought to make access to educational opportunity more widely accessible to their citizens.

On the face of it, it might appear that “mathematics for all” would be a simple aim to accomplish. After all, mathematics is a key component of the curriculum of compulsory schooling across the globe, and some mathematics is typically taught every day to every child at every grade level in every school in every country in the world. Yet, even a cursory review of the history of mathematics education reveals both that many students have not had access to high quality mathematics learning opportunities and that most students have not found school mathematics to be a subject that captures their intellect and imagination, and too few have chosen to pursue mathematics vigorously beyond the boundaries of what is required by compulsory schooling. In fact, throughout the world, mathematics is the school subject most likely to be taught and learned poorly.

Mathematics is taught for long periods of time in formal schooling, yet it is too often presented as if the instructional goal were to teach students to dislike it and to fail at it rather than to grow in affection for and interest in it. As a consequence, many students do not learn school mathematics well. This has been a particular problem for us in the United States where the average performance of students on multiple domestic and international assessments of mathematics proficiency has long been viewed by most observers as unacceptably low, and where there have been disturbingly large and persistent performance gaps between groups of students differentiated by racial/ethnic identity or family income.

Similar problems—low overall performance and gaps between demographic subgroups—plague many other countries as well, especially as the heterogeneity of school populations has increased across the world. My reading of some research reports suggests that South Africa has not been immune to this problem, as is evident from the very low level of performance by South African students on the TIMSS assessments in

In response to concerns about low overall performance and performance disparities, education policy makers have sought remedies, but these have not been easy to identify. Much of the conversation about the issue has focused on the notions of “opportunity to learn” and “instructional quality” which are terms with intuitive appeal that are typically used in attempts to explain performance differences. Unfortunately, the indicators used to detect opportunity to learn and instructional quality are too often quite crude or largely impervious to change, such as the number of years of experience of teachers in schools or the number of lessons taught in relation to certain curriculum topics. As such, these policy conversations typically remain quite far from the core matters of mathematics teaching and learning in classrooms. In contrast to these policy-based discussions, mathematics education professionals in many countries have launched initiatives to enhance curricular goals, to develop teaching and learning materials that engage students’ intellect and interest, or to improve the quality of classroom mathematics teaching.

In the United States, the National Council of Teachers of Mathematics (NCTM) has sustained over more than 25 years a focus on improving mathematics teaching and learning, in an effort that many call “standards-based reform.” Beginning with the publication of the Curriculum and Evaluation Standards for School Mathematics in 1989, continuing with the publication of Principles and Standards for School Mathematics in 2000, and persisting with the recent publication of Principles to Actions: Ensuring Mathematical Success for All in 2014, the NCTM has promoted the goal of providing a high quality mathematics education to all students. This latter publication comes at a time when the United States, where curriculum decisions have long been the purview of states rather than the federal government, is in the midst of a revolutionary shift toward shared curriculum goals across the entire country.

The widespread adoption of so-called college- and career-readiness standards, largely in the form of the Common Core State Standards for Mathematics (CCSSM), by 45 of the 50 states in the country, has provided an opportunity to reenergize and refocus efforts to increase access to high quality mathematics learning opportunities for all children. Though the new standards provide guidance and direction, help focus and clarify common learning outcomes, and motivate the development of new instructional resources and assessments, they do not tell teachers what to do in the classroom to realize these ambitious goals. Moreover, they neither describe nor prescribe the conditions required to ensure mathematical success for all students. In response, NCTM developed Principles to Actions to fill the gap between the development and adoption
of CCSSM and the enactment of practices, policies, programs, and actions required for their widespread and successful implementation.

The overarching message of *Principles to Actions* is that effective classroom teaching is the nonnegotiable core required to ensure that all students learn mathematics at high levels. *Principles to Actions* identifies and discusses eight teaching practices that are claimed to be essential features of effective mathematics teaching. Each of the practices is summarized briefly in Figure 1.

| Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions. |
| Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies. |
| Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving. |
| Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments. |
| Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students’ reasoning and sense making about important mathematical ideas and relationships. |
| Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual |
understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.

**Support productive struggle in learning mathematics.** Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

**Elicit and use evidence of student thinking.** Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Figure 1. Eight Effective Teaching Practices (NCTM, 2014)

In the remainder of this paper, I focus on two of the eight effective teaching practices identified by *Principles to Actions*. In particular, I attend to implementing tasks that promote mathematics reasoning and problem solving and eliciting and using evidence of student thinking. For each, I discuss the practice itself and the research evidence that supports its designation as an important feature of effective mathematics teaching. Following this, I conclude the paper with a few thoughts about how these two distinct practices are linked to each other.

**Focusing on Cognitively Demanding Tasks and Formative Assessment**

As I elaborate below, I have chosen to focus on these two teaching practices because I think that research has consistently and convincingly identified them both as features of classroom instruction that are clearly linked to students’ mathematics achievement. In particular, research on mathematics instruction has established two distinct, robust findings. One is that students learn mathematics well in classrooms where they have regular opportunities to work on cognitively challenging tasks that promote mathematical problem solving, reasoning, and understanding, as long as their teachers support their work on the tasks in a manner that does not lower the cognitive demand as the lesson unfolds. A second robust research finding is that students learn mathematics well in classrooms where teachers employ formative assessment techniques to elicit, interpret, and use evidence about what students are thinking and what they have learned in order to inform their short-term and long-term instructional decisions.
These two features have been identified in other evidence-based characterizations of effective mathematics teaching (e.g., Anthony & Walshaw, 2007), but because they derive from different perspectives on classroom instruction and from distinct lines of empirical inquiry, they have been treated as separate rather than connected to each other in both the research literature and in practitioner-oriented outlets. The publication of Principles to Actions provides an opportunity to link these two powerful practices. I discuss each in turn below.

Implement tasks that promote reasoning and problem solving

Mathematics classroom instruction is generally organized around and delivered through mathematical tasks, activities, and problems. According to Doyle (1983, p. 161), “tasks influence learners by directing their attention to particular aspects of content and by specifying ways of processing information.” In fact, tasks with which students engage constitute, to a great extent, the domain of students’ opportunities to learn mathematics. Students in all seven countries analyzed in the TIMSS Video Study (NCES, 2003) spent over 80% of their time in mathematics class working on mathematical tasks.

Not all tasks provide the same opportunities for student thinking and learning. Tasks can vary not only with respect to their mathematics content but also with respect to the cognitive processes that they entail. Tasks that require students to analyze mathematics concepts or to solve complex problems offer opportunities for students to sharpen their thinking and reasoning in mathematics. In contrast, tasks that require little more than memorization and repetition offer less opportunity to develop proficiency with high-level cognitive processes.

Variations in the cognitive demand of tasks used in mathematics instruction affect students’ opportunities to learn. In the TIMSS 1999 video study, the ability to maintain the high-level demands of cognitively challenging tasks during instruction was the central feature that distinguished classroom teaching in countries where students exhibited high levels of mathematics performance when compared with countries like the United States, where performance was lower and teachers rarely maintained the cognitive demands of tasks during instruction (NCES, 2003; Stigler & Hiebert, 2004). Although 17 percent of the tasks used by teachers in the U.S. were coded as high level, none of these tasks was implemented as intended. Instead, most of the cognitively demanding problems were transformed into procedural exercises. The authors concluded that U.S. 8th grade students spend almost all of their time in mathematics classrooms practicing procedures regardless of the nature of the tasks they are given. This claim is consistent with an analysis of mathematics classroom instruction conducted by the Horizon Research Institute, in which only 15% of
observed mathematics lessons were classified as providing opportunities for complex thinking, or for mathematical reasoning or sense-making (Weiss & Pasley, 2004; Weiss, Pasley, Smith, Banilower, & Heck, 2003).

Research conducted in the U.S. context has found that textbooks are the main source of instructional tasks used in American mathematics classrooms (Grouws, Smith, & Sztajn, 2004; NCES, 2003). Textbooks and other curricular materials function similarly in many other countries as a primary source of mathematics tasks on which students work in and out of school. Because curricular materials vary widely with respect to the nature of the mathematical tasks they contain, teachers need to understand how different types of tasks can influence students’ opportunities for learning, how to identify and select cognitively demanding tasks for use in their classrooms, and how to use cognitively demanding tasks in a way that will realize their potential value for student learning. This latter point is especially important because, as I discuss next, several studies have found that selecting high-level tasks for use in mathematics classrooms does not guarantee that the tasks will be used in ways that maintain the demand characteristics that are essential to students’ opportunities to learn mathematical thinking and reasoning.

The cognitive demands of mathematical tasks can change as tasks are introduced to students and/or as tasks are enacted during instruction (Stein, Grover, & Henningsen, 1996). The Mathematical Tasks Framework (MTF) shown below models the progression of mathematical tasks from their original form to the tasks that teachers actually provide to students and then to the tasks as they are enacted by the teacher and students in classroom lessons. The tasks, especially as enacted, have consequences for student learning of mathematics. The first two arrows in the diagram identify critical phases in the instructional life of tasks at which cognitive demands are susceptible to being altered.
Researchers who have used the MTF, and related conceptualizations, as a lens through which to study mathematics classroom teaching have noted that implementing cognitively challenging tasks in ways that maintain students’ opportunities to engage in high-level cognitive processes is not a trivial endeavor (e.g., Henningsen & Stein, 1997; NCES, 2003). American teachers can (and often do) lower the cognitive demands of a task by breaking it down into sub-tasks (Smith, 2000), by focusing only on correct answers to the exclusion of reasoning and explaining (Henningsen & Stein, 1997; Romanagno, 1994), or by adapting the tasks or teaching suggestions to be consistent with their personal notions of effective teaching and learning (Arbaugh, et al. 2006; Clarke, 1997; Lloyd & Wilson, 1998; Remillard, 1999). Conversely, teachers’ practices in the classroom, such as effective use of questioning (Boaler & Staples, 2008) or cooperative learning groups (Schoen, Cebulla, Finn, & Fi, 2003), can maintain students’ opportunities to engage in high-level cognitive processes, such as mathematical problem solving and reasoning. Thus, it is critical to help teachers extend and improve their knowledge and their pedagogical repertoire in ways that will support their more effective use of cognitively demanding tasks in mathematics instruction.

With appropriate support (e.g., Stein et al, 2009), there is reason to believe that American teachers, and teachers in other countries where the use of cognitively demanding mathematics tasks is not the norm, can learn to incorporate this practice into their instructional repertoire with corresponding positive effects on student achievement. As Principles to Actions (2014) notes:

“To ensure that students have the opportunity to engage in high-level thinking, teachers must regularly select and implement tasks that promote reasoning and problem solving. These tasks encourage reasoning and access to the mathematics through multiple entry points, including the use of different representations and tools, and they foster the solving of problems through varied solution strategies.” (p. 17)

Engaging students with cognitively demanding tasks may require additional work on the part of the teacher to build students’ confidence and motivation to accept the challenges such tasks entail. Principles to Actions suggests some dimensions of the additional work that may be required of teachers and developers of instructional materials:

“Furthermore, effective teachers understand how contexts, culture, conditions, and language can be used to create mathematical tasks that draw on students’ prior knowledge and experiences (Cross et al. 2012; Kisker et. al. 2012; Moschkovich 1999, 2011) or that offer
students a common experience from which their work on mathematical tasks emerges (Boaler 1997; Dubinsky & Wilson 2013; Wager 2012). As a result of teachers’ efforts to incorporate these elements into mathematical tasks, students’ engagement in solving these tasks is more strongly connected with their sense of identity, leading to increased engagement and motivation in mathematics (Aguirre, Mayfield-Ingram & Martin, 2013; Boaler, 1997; Hogan, 2008; Middleton & Jansen 2011).” (2014, p. 17)

The feasibility and value of teaching school mathematics in ways that provide students with regular opportunities to engage in high-level processes, such as mathematical reasoning and problem solving, is evident from a number of sources. In particular, several studies have demonstrated that greater student learning occurs in classrooms where the high-level cognitive demands of mathematical tasks are consistently maintained throughout the instructional episode (e.g., Boaler & Staples, 2008; Hiebert & Wearne, 1993; Stigler & Hiebert, 2004; Stein & Lane, 1996; Tarr, Reys, Reys, Chavez, Shih, & Osterlind, 2008). For example, in a longitudinal comparison of three high schools over a five-year period, Boaler and Staples (2008) determined that the highest student achievement occurred at the school in which students regularly engaged in high-level thinking and reasoning. Boaler and Staples attributed students’ success to the teachers’ ability to maintain high-level cognitive demands during instruction; specifically, to the teachers’ use of pre-planned questions that elicited and supported students’ thinking. Studies by Tarr and colleagues (2008) and by Stein and Lane (1996) both found that classrooms in which teachers consistently encouraged students to use multiple strategies to solve problems and supported students to make conjectures and explain their reasoning were associated with higher student performance on measures of thinking, reasoning, and problem solving.

**Elicit and Use Evidence of Student Thinking**

Another body of research suggests that student achievement is amplified when teachers employ formative assessment techniques in classroom instruction. Formative assessment refers to a process of eliciting and interpreting evidence about what students are thinking and what they have learned and then using this information to make instructional decisions (Wiliam 2011, p. 50). In this sense, formative assessment is essentially identical to the practice of *eliciting and using evidence of student thinking* identified in Principles to Actions (NCTM 2014) as one of the eight non-negotiable teaching practices critical for successful implementation of ambitious standards. In contrast to summative assessment, which involves the evaluation of student learning, progress, or achievement in order to assign grades or appraise programs, formative assessment involves assessment *for* learning—gathering evidence within the stream of...
instruction about what students are doing, thinking, and learning and then using that evidence to inform decisions that affect teaching and learning.

Black and Wiliam (1998) synthesized the results of dozens of studies of formative assessment, and they found strong evidence of greater student achievement in classrooms where teachers used such techniques. Ehrenberg, Brewer, Gamoran, and Willms (2001) reported that the impact on student achievement of teachers using formative assessment as part of instruction was far greater than that obtained by reducing class size. Other empirical studies have demonstrated that teachers can learn to use formative assessment in the mathematics classroom with positive effects on students’ learning (e.g., Wiliam, Lee, Harrison, and Black 2004). Although some have pointed to weaknesses and gaps in the evidence base (e.g., Bennett, 2011), the preponderance of research evidence appears to support the positive influence on student learning of formative assessment in classroom instruction.

According to Leahy, Lyon, Thompson and Wiliam (2005, p.19), “in a classroom that uses assessment to support learning, the divide between instruction and assessment blurs. Everything students do—such as conversing in groups, completing seatwork, answering and asking questions, working on projects, handing in homework assignments, even sitting silently and looking confused—is a potential source of information about how much they understand.” Based on their analysis and synthesis of a number of studies of formative assessment in classroom instruction across a variety of school subjects, Leahy et al. (2005) identified several aspects of instruction that characterize effective formative assessment in classrooms, including engineering effective classroom discussions, questions, and learning tasks and promoting students’ ownership of their learning and encouraging students to be learning resources for one another.

*Engineering effective classroom discussions, questions, and learning tasks* involves at least three interrelated instructional practices: 1) engaging students in tasks and activities that provide insights into their thinking; 2) listening and analyzing student discussions and artifacts interpretatively, not just from an evaluative perspective; and 3) implementing instructional strategies designed to engage all students in tasks, activities, and discussions (Wiliam 2011). For this to work well, instructional tasks and activities should elicit thinking and reasoning, relate to key concepts and skills in the curriculum, and allow students to show what they understand and can do. Also, it is important that teachers and students engage in listening “interpretatively” (Davis, 1997); that is, not just listening for the right answers but for evidence about student thinking to inform the next instructional steps. In this way, a teacher can obtain evidence about how well students are learning important mathematical concepts and
skills and detect errors or misconceptions that are prevalent in student work, especially those that may interfere with learning new concepts or solving related problems.

Effective formative assessment also means promoting students’ ownership of their learning and encouraging students to be learning resources for one another. Providing students with challenging mathematical tasks and supporting them to develop persistence in solving such tasks helps students develop a sense of self-efficacy that also supports their motivation to tackle difficult mathematics topics. Also, teachers can engage students in self-assessment and peer-assessment, with an emphasis on listening interpretively as noted above rather than focusing only on right/wrong judgments. Classrooms in which students actively listen to their peers’ presentations and explanations can be communities in which each student supports the learning of other students in a mutually enabling manner.

**Coda**

In a recently published paper, Silver and Smith (2015) present an episode of mathematics teaching that exemplifies of how formative assessment and the use of cognitively demanding mathematics tasks in instruction can be seamlessly integrated. In order to elicit evidence of student thinking, a teacher uses a highly engaging, cognitively challenging task that stimulates students to engage in problem solving and reasoning. During the lesson, the teacher strives to maintain the cognitive challenge by asking purposeful questions that both support students’ persistence in solving the task and elicit evidence of student thinking as they progress.

The lesson that Silver and Smith describe illustrates how the teacher’s lesson planning and enactment afforded her a window into her students’ thinking and also the evidence and information she needed to make sound instructional decisions during and after the lesson. In this way they provide an existence proof that the two practices identified in *Principles to Actions* (NCTM, 2014) and discussed in detail in this paper can act in ways that are interactive and mutually supportive of effective instruction and powerful opportunities for all students to learn mathematics.

**REFERENCES**


In this paper we, the authors, share some observations of senior phase learners’ encounters with mathematics content that relate to different types of number, and rationalise possible reasons as to why some of these learners may encounter difficulties grasping particular concepts related to number, or why they may be using wrong mathematical procedures when solving problems. This observational study is theoretically informed by constructivism as learning theory. Interaction with learners in the senior phase mathematics classroom in Western Cape, South Africa had revealed that they often need to apply and use automatisms or procedures to accurately perform calculations in a variety of contexts, but procedures are often used incorrectly or are easily forgotten. It is found to be more beneficial to learn a particular number concept through understanding and comprehension, because this would make it easier to recall the rule, by for example remembering the original context in which it was initially introduced to the learner, or derived at through self-exploration.

INTRODUCTION

Workings with number in the Senior Phase (grades 7 to 9), especially in grade 7 in South African schools in the Western Cape Province, are characterised by the expanding of the number system which involves introducing ‘new’ numbers such as integers, irrational numbers as well as more complicated manipulations and operations with common fractions and decimal fractions (DBE, 2011). In this paper we share some observations of learners’ dealings with mathematics content that relate to different types of number, and look at possible reasons as to why some learners in the senior phase may encounter difficulties grasping particular concepts related to number, or using wrong mathematical procedures when solving problems.

What particularly interested us as researchers was the quality of communication that occurs between learners and teachers. Very often teachers talk about concepts or phenomena that fall outside of the learners’ frame of reference. Often communication occurs in a vacuum and the idea or concept formed by learners, radically differ from what was communicated or intended. This implies that in order to improve communication and understanding, teachers should be conscious of the learners’ frame of reference in order to facilitate the accommodation and internalization of new concepts or concept formation.
The terms “accommodation” and “assimilation” are terms used by Piaget (Copeland, 1974, p. 39) to describe the cognitive process involved with conceptualizing new ideas or concepts by linking them to existing ones in a meaningful way. Assimilation is described (Curzon, 1985) by Piaget as the process by which new experiences and information are placed into the cognitive structure of the learner, and accommodation is viewed as the product of any restructuring of the cognitive schema.

THEORETICAL FRAMEWORK

This study is theoretically underpinned by constructivism as learning theory. In essence constructivism is described as a learning theory that claims that “knowledge is not passively received, but is actively built up by the cognizing subject”, and “that the function of cognition is adaptive and serves the organisation of the experimental world” (Wheatley, 1991, p. 10; Matthews, 2000, p. 175). Carr, Jonassen, Litzinger and Marra, 1998, p 5) likewise state that constructivism emphasises learner activity and how they construct knowledge as a process of making sense and giving meaning (Adendorff, 2007).

The role of communication within constructivism is described by Atherton (2003, p. 1) as that which allows “conversational theories of learning [to] fit into the constructivist framework”. The “active mental involvement” (Adendorff, 2007, p. 66) of learners is reflected in the teacher’s deliberate effort “to enter into a dialogue with the learners, trying to understand the meaning of the material to be learned by that learner”. Carr et al. (1998, p. 5) similarly argue that constructivism emphasises the need for “learning [to] support collaboration”, allowing learners to “talk to one another about their learning”. What is considered to be an essential element of the process of collaboration and transfer of information is the fact that learners are obliged to “crystallize what may be internally fuzzy into concrete words, and encourages knowledge synthesis and meaning making” (Carr et al., 1998, p. 8).

In this paper constructivism is used to help to explain individual differences in mathematics learners’ abilities to communicate proficiently on mathematics related concepts and contexts. The intension was to get learners to use an array of different types of communication such as discussion, written notes and calculations, individual sharing of ideas, and small group interaction (Gruba & Sondergaard, 2001) to accommodate learner diversity.

DISCUSSIONS BASED ON OBSERVATIONS

The discussion essentially centres on aspects that relate to the Senior Phase mathematics curriculum (Grades 7 to 9) referred to as CAPS (Curriculum and
Assessment Policy Statement) and include the following discussion points: expanding the number system by introducing numbers with properties that are new to the learners, and the fact that many learners find it difficult, not only to cope with the new number terminology, but also to mentally absorb and comprehend all the related terms and manipulations. Specific attention is paid to manipulations involving common fractions as an area of concern; and calculations (operations) involving negative numbers using associations and contexts to enhance conceptualization and sense-making.

**Expanding of natural and whole numbers**

Learners in grade 7, as the first year of the three-year senior phase, are supposed to be familiar with the system (figure 1) of whole numbers (natural numbers plus nil), namely 0, 1, 2, 3 .... and calculations such as addition, subtraction, multiplication and division, involving these numbers. In the senior phase curriculum, rational numbers (fractions), negative numbers and irrational numbers (roots) are mentioned (DBE, 2011). Calculations involving positive numbers and decimal numbers are almost always used in everyday life situations.
On the other hand calculations dealing with proper fractions, negative numbers and roots function only within mathematics itself. Fractions, however, are commonly used to indicate quantities, such as: half a teaspoon of salt, three quarters of a cup of sugar, the last quarter of the soccer match. Furthermore, negative numbers generally appear in expressions such as: the temperature is -5 degrees Celsius; a negative bank balance is indicated as –R35000, whilst a positive balance is indicated as R455. The time difference between Cape Town and New York (to the west) is expressed as 6 hours, and between Cape Town and Jakarta (to the east) it is -5.

Only decimal strategies are applied to solve roots. For example a particular painter contractor measures the dimensions of a specific wall as 2,50 metres by 3,33 metres and then determines the area by keying in $2.5 \times 3.33$ on his calculator. One can for certain say that he definitely does not calculate the area in terms of: $\frac{21}{2} \times \frac{31}{3}$.

For a calculation or algorithm such as $(-3) \times (-2)$ no (realistic) context actually exists, but it is possible to create ‘artificial’ contexts to enhance learners’ understanding.

Fractions, negative numbers, square and other roots are considered to be an expansion of the number system that stems from mathematical necessity. Fractions and negative numbers (negative integers) are numbers (created) to solve equations such as $5 \times x = 3$, $5 + x = 3$ and $x^2 = 3$. Such equations created the need for expansion to include numbers other than 0, 1, 2, 3, …

The history of mathematics and numbers specifically reveals (Wolfram, 2002; Rogers, 2014) the process of expansion of the number system to include negative numbers and irrational numbers, as having been a difficult and drawn out struggle. As such negative numbers are not even 600 years old yet, and learners of 500 years or more ago had to use or apply ways of solving quadratic equations such as the following notation:

$$x^2 + 4x = 5$$
$$x^2 + 3 = 4x$$
$$x^2 = 4x + 5$$

The lack of negative numbers and nil as number then prevented the solution to equations such as the following:

$$x^2 + 4x + 5 = 0$$
For Pythagoras it was evident that no number existed to express the length of the diagonal of a square with one side equal to 1 cm! In our language we write that Pythagoras was aware of the fact that the equation \( x^2 = 2 \) had no solution within the number system that was known to him then. It would take about two thousand years before this impasse would be broken.

**The work of centuries squeezed into one year**

Currently learners encounter mathematics that humanity struggled with for many centuries! As such we should not be surprised that learners struggle with learning mathematics that humanity grappled with for centuries! Learners also need to get used to working with ‘new’ numbers they encounter, such as common fractions, decimal fractions, negative integers, etc. Experiences with these numbers are sometimes in contrast to what learners have encountered before. Consider the following examples:

- Learners in grade 7 have become used to the idea that when two numbers are multiplied the resulting product is a number or value, bigger than any of the two numbers that are multiplied. In the case of multiplying fractions, this is not the case:
  \[
  \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}
  \]

  Learners need to understand why the product of common fractions becomes smaller. So the appropriate question needs to asked, namely: what is half of one third? Or what is one third divided by two?

- In the following subtraction problem the result is a value bigger than the value subtracted from: \((+5) - (-3) = (+8)\).

- Multiplication such as \(3 \times 2\) can be understood to mean repeated addition, namely \(2 + 2 + 2\). However, this is not possible with fractions. One cannot add one half times or add 0,65 times in the same way as in \(3 \times 2\).

- Division with whole numbers can easily be understood as repeated deduction, but how does one divide four sweets among 0,4 friends?

The teaching of these ‘new’ numbers must take into account the unavoidable psychological resistance that learners may experience. It should be evident to the teachers that the known rules for calculation are generalised. A ‘new’ multiplication \(\frac{1}{2} \text{ cm} \times \frac{1}{3} \text{ cm}\) with the old numbers \((2 \text{ cm} \times 3 \text{ cm})\) reveal the same result as earlier. When multiplying fractions the generalisation can be found by means of calculating the areas of rectangles in figure 2:
2 cm × 3 cm = 6 cm²
\[ \frac{1}{2} cm \times \frac{1}{3} cm = \frac{1}{6} cm^2 \]

Figure 2

Calculations with fractions

In addition to introducing these new types of numbers (what are they; what is the correct notation is, the positions these numbers occupy on the number line (ordering)), learners also are expected to do calculations involving these numbers. Based on observations we suggest that up to grade 8 learners should to a large extent be dealing with fraction notation involving fractions such as the following:

\[ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{3}{4}, ..., \frac{1}{10}, \frac{1}{100}, ... \]

It would be crucial to pay special attention to decimal notation of fractions by emphasizing the integrated nature of concepts such as proper fractions, decimal fractions, ratio and percentages:

\[ \frac{1}{10} = 0,1 = 10\% \quad \frac{1}{2} = 0,5 = 50\% \]
\[ \frac{1}{100} = 0,01 = 11\% \quad \frac{1}{4} = 0,25 = 25\% \]

The multiplication of \( \frac{1}{2} cm \times \frac{1}{3} cm = \frac{1}{6} cm^2 \) should be directly connected to

\[ 0,5 cm \times 0,333 cm = 0,1665 cm^2 \]

Placing and positioning of decimal numbers on the number line serve to help learners to see these numbers in relation to other numbers. In the following illustrations (figure 3) learners are expected to write the appropriate value indicated by a particular dot on the number line. Learners are also expected to pronounce (say aloud) the number.
Conversely, learners need to write numbers such as 1,9 en 0,9 en 1,09 en 0,09 at the proper position on the number line:

![Number Line](image)

We are of the opinion that it is not really meaningful for grades 5, 6, 7 or 8 to be learning about fractions the way it is currently done. To become mathematical literate and to be functioning optimally in everyday life situations that require mathematical calculations the following type of calculations (algorithms or recipes) are never required anywhere outside of the classroom:

\[
\frac{2}{2} \times \frac{1}{3} = \frac{2 \times 1}{6} = \frac{2}{6} = \frac{1}{3}
\]

or

\[
\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6},
\]

yet we spend much time on it. Some may argue that this forms the foundation for working successfully with algebraic fractions. Even for developing mathematical insight or understanding, these calculations do not actually serve a purpose. So why do we burden learners with these? Calculations involving decimal numbers on the calculator seem to be an excellent alternative to calculations with proper fractions. Calculations with money in terms of rands and cents serve as an appropriate example, and strengthen this stance. Money received and money spent are then translated to the addition and subtraction of decimal numbers.

Confusion related to addition and multiplication of fractions: Consider the following two cases,

\[
\frac{1}{2} + \frac{1}{3} = \frac{2}{5} \quad \text{and} \quad \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}
\]

Case (1) points to a lack of understanding in respect of the meaning of the concept fraction. “1” and “2” in \( \frac{1}{2} \) should not be viewed as independent of each other but as jointly representing a particular value, the same with \( \frac{1}{3} \). Secondly, these two fractions can only be added if the denominators are the same, and then only the numerators are to be added. Learners need to be able to visualise what actually happens when these two fractions are added. A way to resolve this would be to go back to basics by illustrating, using physical models in figure 4.
Calculations with negative numbers via diverse mental (thinking) schemas

Calculations with negative numbers provide different experiences. To progress to more advanced mathematics, learners have to become familiar with calculations involving formulae, graphs, equations and inequalities using negative numbers.

In many school textbooks (Van den Born, 1998) authors present different calculations by means of miscellaneous mental schemas. In a similar fashion understanding of addition of integers can be facilitated by means of a temperature / thermometer story, for example the beginning temperature is negative two degrees, and the temperature increases by three degrees then the final temperature is one degree (-2 + 3 = 1).

Arrows on the number line (figure 5) is another mental schema to be used to enhance understanding. When adding the arrow points to the right, when subtracting the arrow points to the left, as indicated below:
at plus 2 the arrow points to the right
$2 + 3 = 5$

at minus 2 it points to the left
$2 - 3 = -1$

at minus 2 it points to the right
$-2 + 3 = 1$

at minus 2 it points to the left
$-2 - 3 = -5$

Figure 5
Consequently the ‘rule’ forthcoming from this:

*Numbers with the same sign: add.*

*Numbers with different signs: subtract.*

In this example the second number is still +3. The question arises: How does one add or subtract with a number such as -3? Does one stick to the arrow notation – it would become highly complicated and eventually lead to complication and confusion. It becomes even trickier when one wants to use the arrow method to explain subtraction in a way that facilitates understanding.

With respect to multiplication learners are compelled to know the following pattern by heart without actually understanding why:

\[
\begin{array}{c}
+ \times + = + \\
+ \times - = - \\
- \times + = - \\
- \times - = + \\
\end{array}
\]

This, however, remains a mere recipe.

The rules for multiplication can be grasped by focusing on the pattern or regularity that emerges in the multiplication table:

<table>
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All of these different mental schema employed to explain addition, subtraction, multiplication and division of negative numbers are often counter-productive as learners may experience confusion. Some mental schema may appropriately explain particular aspects (such as the use of the thermometer for explaining addition) only, but then cannot be used to meaningfully explain other operations or calculations. This situation is not ideal since learners are in the process of making sense psychologically and are grappling with assimilating new number related concepts into existing schemes.

When dealing with addition of integers, learners (and often teachers as well) do not distinguish between the plus-sign for the operation addition, and the plus-sign indicating that a particular number is positive, such as +(+3). Similarly the minus-sign also has two distinct meanings: subtraction and being negative. This creates confusion and may cause learners to make mistakes. Consequently, the learner needs to distinguish between the number (whether it is positive or negative and the operation that must be performed).

Operations with negative numbers using a particular thought schema

Professor Hans Freudenthal (van den Heuvel-Panhuizen, 2012) developed the idea to present all calculations of negative numbers within a single mental or thought schema. This approach to working with negative integers is used widely in textbooks in the Netherlands. This particular context is not a real context but a fictitious one, a fable:

- A witch has a big pot in which she brews potions. She uses hot and cold blocks to regulate the temperature inside the pot while brewing.
- When there are equal amounts of hot and cold blocks inside the pot, the temperature is said to be zero degrees.
- When there are for example three hot blocks more than cold blocks inside the pot, then the temperature is said to be three degrees above nil, thus +3.
- When there are six cold blocks more than hot ones, the temperature is said to be six degrees below zero, indicated as –6.
- Putting blocks into the pot is considered as as addition, notation used +
- Removing blocks from the pot is considered subtraction, notation used –

The addition sum: (-2) + (+8) = (+6) is understood as meaning:

- There are initially two more cold blocks in the pot than hot ones [that is before the hot blocks are added]: notation used (-2)
- 8 hot blocks are put in the pot, indicated by the notation + (+8)
- Now there are 6 hot blocks more than cold ones in the pot, indicated notation (+6)

The subtraction sum: (+30) - (-10) = (+40) means:
There are thirty more hot blocks than cold ones inside the pot: notation used: (+30)
Then 10 cold blocks are taken out of the pot, with notation: - (-10)
Now there are 40 more hot blocks than cold ones, indicated by notation (+40)

The sum: +4 x (-3) = (-12) means:
- Four times, three cold blocks are put into the pot.
- Now there are 12 more cold blocks inside the pot than hot ones, indicated as: (-12)

The sum: -3 x (-2) = (+6) means:
- Three times, two cold blocks are taken out of the pot
- Now there are 6 more hot blocks inside the pot than cold ones, indicated as (+6)

It needs to be said that when multiplying, the plus or minus sign of the first number undergoes a change in meaning: positive now means addition and negative becomes subtraction. This is the only flaw related to this thought schema.

Division always is experienced as an investigative process. It is also the case with negative numbers, for instance:

\[ (-6) \div (-3) = ? \] is the search for \[ ? \times (-3) = (-6) \].

CONCLUDING REMARKS

Obviously learners need to develop automatisms or procedures to accurately perform calculations involving negative numbers. As such they need to be weaned from the context that involves the witch. It is essential that they discover the rule that a negative number multiplied by a negative result in a positive product. This rule can be taught by using a particular recipe or procedure, but procedures are often used incorrectly or are easily forgotten.

It is more beneficial to learn a particular number concept through understanding and comprehension, because this would make it easier to recall and apply the rule with understanding, by for example remembering and relating to the story of the witch as discussed earlier.

REFERENCES


Over the years various researchers have conducted studies to understand how learners can be taught school mathematics with understanding. One of the fields of school mathematics that has been researched extensively is school algebra (Hart, Brown, Kuchemann, Kerslake, Ruddock & McCarthy 1981; Rojano, 2002; Herscovics & Linchevski, 1994). The studies conducted have uncovered some ideas that need to be addressed in order for school algebra to be successfully taught. Watson (2009) argues that there is much that can be done about attention from a mathematical perspective to help learners understand school mathematics, and she therefore advocates for deliberate use of variation in mathematical examples given to learners. In this paper, I will respond to Watson’s claim using the constructs: reification and the dual nature of mathematics, and other literature that has been written about the teaching and learning of algebra. Using Watson’s claim and the constructs that will be discussed in the paper, I will analyse some mathematics algebraic expressions in an attempt to explain how to teach algebra with understanding.

WHAT IS ALGEBRA?

Algebra has been defined by different authors. Debyshire (2007) defines algebra as a branch of mathematics that involves abstractions. Debyshire (2007) stresses that polynomials are at the heart of algebra. Marton, Runesson and Tsui (2004) define algebra as a component of mathematics that involves generality, multiple expressions and they also argue that algebra has a structure and it is generalised arithmetic. Watson (2009) agrees with Mason, Graham and Johnson-Wilder (2005) and she defines algebra as the way generalisations are expressed about numbers, quantities, relations and functions. Kilpatrick, Swafford and Findell (2001) argue that algebra is a branch of mathematics that emphasises on relationships between quantities. Kilpatrick et al. (2001) do not say that algebra is generalised arithmetic; instead, they argue that algebra builds further on the learners’ proficiency of arithmetic. The definitions of algebra I have drawn on in this paper will help me point out what algebra really means as I present the response to Watson’s claim, and as I construct an analytical framework that will be used to analyse the mathematics algebraic expressions.

What stands out for me in the definitions above is the fact that all the authors agree with Mason et al. (2005)’s argument that algebra is much more than simply using letters in place of numbers, algebra is about expressing generality. The language of algebra is used to express the idea of structure (Mason et al., 2005) and in order for the school children to be able to manipulate the generalities with ease, they have to be able to
understand the structure or else they are said to have pseudo structural approaches to manipulating the generalities (Sfard & Linchevski, 1994). Having defined what algebra is and what it is not, I now turn to the point of how learners can be helped to ‘see’ the structural meaning that algebraic activities carry. From the definitions discussed above, algebraic activities carry a lot of meaning which learners may not be able to ‘see’ easily. If the learners miss the point of algebra, they will not be able to manipulate the equations and the expressions with understanding. This paper will focus on algebraic expressions. An algebraic expression is made up of the signs and symbols of algebra. The value of the algebraic expression depends on the value assigned to the symbol within the expression. Algebraic expressions do not have closure because they are objects which learners have to ‘see’, yet learners struggle to accept this idea of lack of closure (Tirosh, Even & Robinson, 1998).

THEORETICAL PERSPECTIVE: VARIATION THEORY

Runesson (2005) describes variation theory as a theory of learning. Marton et al. (2004) agree with Runesson (2006) and they argue that within Variation theory, learning is defined as the “process of becoming capable of doing something as a result of having had certain experiences” (p5). According to Watson (2009), learning has an object, which is always the focus of attention within any learning process. Marton et al. (2004) refer to the object of learning as capabilities. The object of learning has critical features, which have to be discerned in order for learning to take place (Runesson, 2006). Variation theory therefore focuses on the differences in the dimension or values of a critical feature of the object of learning. According to variation theory, it is not possible to discern a certain way of thinking about something without the contrast of other ways of thinking about the same thing (Marton et al., 2004). Marton et al. (2004) argue that variation theory can be used to enable learners to experience the features that are critical for particular learning and also to help learners in developing certain capabilities. Marton et al. (2004) define capabilities as the object of learning. The object of learning is defined by its critical features and the critical features are those features that should be discerned in order for the meaning that is aimed for to be understood.

Mathematics as a subject has objects of learning, for example, within Mathematics, topics are objects of learning. This implies that algebra is an object of learning. The object of learning has the general and the specific component. The general component involves the nature of the capability, for example, the general component of algebra involves interpreting, discerning and grasping the various features within algebra. Variation theory can be used to direct attention to what is to be discerned in any algebraic expression. In order for learners to see certain features of the object of learning, they have to discern certain features of that object of learning. When learners are being taught algebraic expressions, the learners have to discern the critical features
of an algebraic expression such as the different terms of the given algebraic expression. Discernment of critical features of an aspect leads to powerful ways of seeing and hence powerful ways of acting (Marton et al., 2004). Marton et al. (2004) identify different patterns of variation. These are contrast, generalisation, separation and fusion.

**Contrast**

Contrast is a pattern of variation where learners are given the opportunity of experiencing something that is not a critical feature of the object of learning so that they compare the different features (Marton et al., 2004). This pattern of variation allows learners to see what something is, and what it is not.

**Generalisation**

Generalisation is a pattern of variation where learners are given an opportunity of experiencing the critical features of learning through various examples showing the critical features of the object of learning and the activities emphasise the dimension of the varying aspects of the features of the object of learning (Runesson, 2005). In algebraic expressions, different algebraic expressions such as $2+y$, $m+x$, $5x+9$, $-x+t-1$, and $5x+4y-2x-4x$ should be given to the learners so that they can see several examples of how algebraic expressions look like in order to discern critical aspects of an algebraic expression.

**Separation**

Separation is a pattern of variation where the aspect is separated from other aspects by varying that particular aspect and keeping the other aspects invariant (Marton et al., 2004), and this is referred to as controlled variation (Watson & Mason, 2006). Variation must be controlled because if everything is varying at the same time, nothing maybe discerned. If learners are given expressions such as $x+y$, $9x-t$, $n+2m+8$, learners may not discern the critical feature of learning because there are many things changing in these expressions such as the coefficients, the structure and the number of variables in the expressions. Therefore, it may be difficult for the learners to discern the critical feature within the activities. In other words, random variation does not help learners to focus their attention on the critical features of the object of learning (Watson & Mason, 2006).

**Fusion**

Fusion is a pattern of variation where learners are given an opportunity to discern different critical features of the object of learning at the same time (Runesson, 2005). Fusion is also known as synchronic simultaneity (Marton et al, 2004). Discerning simultaneously is experienced against the background of previous experiences (Marton...
et al, 2004). When teaching a concept, fusion may not be used before learners are capable of identifying and discerning critical features.

The Space of Learning

The space of learning is defined as the “pattern of variation inherent in a situation” (Marton et al, 2004: 20). The patterns of variation discussed above lead to what is known as the space of learning. The space of learning includes the aspects of a situation that can be discerned as a result of the variation that is present in a given object of learning. When the space of learning is created, a dimension of variation is opened. Once the space of learning is created, the taken for granted ways of seeing the object of learning is challenged and hence learning takes place (Marton et al, 2004).

Using Variation theory to focus learners’ attention on critical features of the object of learning

Having discussed variation theory above, I now move on to discuss how the theory can be used in the teaching and learning of algebraic expressions. The constructs: dual nature of mathematics, interiorisation, condensation, and reification will be used to explain the process of discerning the critical features of algebraic expressions.

The Dual Nature of Mathematics

The dual nature of mathematical objects is discussed by Sfard (1991). By the dual nature of mathematical objects, Sfard (1991) argues that a mathematical entity can be viewed as an object or a process. A mathematical entity is ‘seen as an object’ if it is seen as if it were “a real thing that exists in space and time” (Sfard, 1991: 4). The structural meaning should be the focus during the teaching and learning process by contrasting what is not structural, for example a learner can be shown expressions such as $2x+4$ and $7x$ to show him/her the features that make an algebraic expression an object.

A mathematical entity can be seen as a process. When an entity is seen as a process, this is known as operational. Seeing a mathematical entity as a process implies that computation has to be done on the entity (Sfard, 1991). In other words, something has to be done. For example, when a learner is given an expression such as $y=4x+2$, he can ‘see’ the equation as a process which requires him to carry out a computation of multiplying 4 by $x$ and then adding 2 to the product. There is a major difference in what is to be done when a mathematical entity is seen as a process or an object.
Three Degrees of Structuralisation

Sfard (1991)’s argument is that when a learner gets to know a mathematical entity, he begins by operating on it, and the learner then starts to turn this entity into an independent object, and finally the learner acquires the ability to see the new mathematical entity as an integrated, object-like whole. In other words, the learner goes through the ‘three’ degrees of structuralisation: interiorisation, condensation and reification.

Interiorisation

When the learner has acquired a skill of carrying out an operation mentally, he is said to have interiorised that entity (Sfard, 1991). An example of interiorisation is when a learner is given an expression such as ‘\(4x-10x\)’, and the learner, through performing the process of subtraction recognises the existence of a negative number, \(-6x\). With time, the learner becomes skilled at performing the operation of subtraction of numbers to an extent of carrying it out through mental representation, without actually performing the operation.

Condensation

When a learner is able to see that ‘\(4x-6x\)’ can also be written as ‘\(-6x+4x\)’, the learner is said to have started the condensation stage. At the condensation stage, the learner is also able to simplify an expression such as ‘\(2x+5y-7+6x+8-y\)’ into a shorter manageable expression. As the learner operates on this expression, he/she thinks about it as a whole without having to go into details of what he must do to operate on the expression. However, at the condensation stage, the learner still sees the mathematical entity as a process (Sfard, 1991). The condensation stage only ends when a learner stops connecting a mathematical entity to a certain operation or process, and once a learner is able to see the mathematical entity as an object, then the learner is in the reification stage (Sfard, 1991).

Reification

Reification refers to “our mind’s eyes’ ability to envision the results of processes as permanent entities in their own right” (Sfard & Linchevski, 1994: 194). Reification enables the learner to see a familiar mathematical expression say, \(7x+y\), in a totally new light, as an object and not a process which requires him to multiply 7 by \(x\) and add \(y\) to \(7x\). Seeing mathematical entities as objects is as a result of reification. In order for reification to take place, variation theory must be used. As learners experience various examples and tasks that require them to reify, they will focus their attention on what it is that they need to reify. However, reification is a very difficult process (Sfard, 1991).
Figure 1 illustrates the steps involved in the processes leading to reification, as a result of using variation theory, as discussed above.

![Flowchart](image)

**Figure 1: Summary of the process of reification**

The main idea of understanding algebraic expressions is being able to discern when to view the given expression as an object or a process. In the following discussion, I will discuss my analytical framework, and use the analytical framework to analyse some algebraic expressions. The algebraic expressions analysed in this paper are typical textbook activities. However, the paper does not draw on any textbook in particular to present the activities.

**Analytical framework**

From Watson’s claim about variation theory and the constructs discussed above, I developed an analytical framework for analysing various algebraic expressions, as shown in Table 1. The analysis was done specifically to find out what it is that is possible to learn in the activities. The activities were first analysed to find out the
patterns of variation, if any, that were used in the designing them. The activities were then analysed to find out why the patterns of variation were used. In other words, the activities were analysed to find out if the variation was used to focus learners’ attention on the dual nature of mathematics, to help learners interiorise, condense and reify.

Table 1: Analytical framework for analysing activities

<table>
<thead>
<tr>
<th>Construct</th>
<th>Definition of construct</th>
<th>Indicator of variation to focus attention on construct in the activity</th>
<th>Guiding questions</th>
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<tbody>
<tr>
<td>Deliberate variation</td>
<td>Carefully choosing an activity that will focus learners’ attention on the required feature of the object of learning</td>
<td>-examples and exercises that focus attention on one aspect at a time</td>
<td>-Does the activity have examples that are selected to deal with a specific aspect at a time?</td>
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<tr>
<td>Process</td>
<td>Seeing a mathematical entity as a process implies that computation has to be done on the entity</td>
<td>-including activities and examples that require computation</td>
<td>-Does the activity have activities that encourage learners to carry out computations?</td>
</tr>
<tr>
<td>Object</td>
<td>If an individual is able to see a mathematical entity as an object, he is said to have recognised a static structure</td>
<td>-including activities that have expressions such as 2x+4 as the final answer</td>
<td>-Does the activity have examples where learners get expressions as final answers after computations?</td>
</tr>
</tbody>
</table>
### Interiorisation
When the learner has acquired a skill of carrying out an operation mentally, he is said to have interiorised that entity.

- giving many examples on the same aspect
- Does the activity have carefully chosen examples and activities on the same aspect?

### Condensation
When the learner is able to see different sides to the same mathematical idea.

- giving examples and activities that require learners to simplify or to collect like terms
- Does the activity have activities that give learners an opportunity to simplify?

### Reification
Reification refers to our mind’s eyes’ ability to envision the results of processes as permanent entities in their own right.

- including activities and examples that have a final answer in the form where it is not a single value.
  - For example: $x + 4$
- Does the activity have activities that help the learners to identify the structural meaning of the expressions?

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**FINDING FROM THE ANALYSIS OF THE ALGEBRAIC EXPRESSIONS ACTIVITIES**

**Deliberate variation**

The activities given below are examples of using deliberate variation. The activity has different algebraic expressions with counter examples to show what an algebraic expression is and what is not.

Which of the following statements are algebraic expressions?

1. $x^2 + 4x = 3$
2. $x^2 + 4x - 3$
3. $x^2 = 4x - 3$
4. $x^2 + 4x + 3$
From the activity above, there is deliberate variation because the signs between the terms are being changed and also one of the statements is not an algebraic expression. So learners are given a counter example of the critical feature of the object of learning.

Another example of deliberate variation is given below, where learners are shown what polynomials are and what they are not.

The following are not polynomials:

i) \(x + 3 + \frac{5}{x}\)  
ii) \(\frac{7}{x}\)  
iii) \(\sqrt{4x}\)  
iv) \(x^4\)

The following are polynomials:

i) \(2x^2 + 3x + 5\)  
ii) \(7x\)  
iii) \(x^2 + y^2\)

The activity below was analysed to identify whether there was a pattern of variation.

Simplify

i) \((x^2 + 2x) + (3x^2 - 4x - 2)\)

ii) \((x^2 + 2x) - (3x^2 - 4x - 2)\)

In the activity above, the aspects of the expression: coefficients, terms, and degree of the expressions are all invariant and only the sign between the two brackets is changing. Therefore learners’ attention is focused on how the sign affects the expression. Some activities which have random variation were also analysed. In random variation, all the aspects of algebraic expressions are changing as shown below.

Multiply out and simplify:

i) \((x-3)^2\)  
ii) \((x+5)(x-7)\)

iii) \((x-5)(x-5)\)  
iv) \((2x-1)(2x-1)\)

In the activity above, the variation is not controlled because all the aspects of the expression are changing. From part (i) to (iv), the coefficients of \(x\) and the constant terms have all changed. This form of variation may be used after learners are able to discern different features of the object of learning at the same time.

Process- Object Focus

Within algebraic expressions, there is a section on substituting for the unknown and then solving the expression to get an answer. An example of an activity which requires substitution is shown below.

i) Determine the value of \(-x^2 + 2x\) if \(x = -2\)
ii) Determine the value of \(-b^2 + 2b\) if \(b = -2\)

iii) Determine the value of \(-m^2 + 2m + 3m\) if \(m = -2\)

Giving learners the activity above that requires them to substitute for the variable helps them to focus on the process, but it does not help learners to understand the structural meaning of algebraic expressions (Kieran, 2007). The activity simply emphasises that a variable can stand for any value (Tirosh et al., 1998). The learners are able to ‘see’ the expressions as processes. Letting learners experience activities that require them to substitute for the variable does not focus their attention on the idea of generalisation, but activities that do not require them to substitute for the variable will help them to understand the structural meaning of the variable and hence be able to see the final answer as an object because the value of \(x\) is not known. Therefore, it is very important to give learners activities that give them opportunities to discern the dual nature of mathematics.

**The Process of Reification**

As discussed earlier, the process of reification is a lengthy and painfully difficult one (Sfard, 1991). This is because various steps have to be passed before the reification takes place. Algebraic activities given to learners cannot enable reification to take place if the steps to be followed are skipped. The first step as discussed earlier is interiorisation. Interiorisation can only be said to have occurred if the learner is able to manipulate an algebraic expression mentally (Sfard, 1991). In mathematics, for such a process to happen, the learner should have calculated numerous activities about the same idea until he gets to figure out the pattern to follow. Learners should be provided with as many activities as possible on the same idea and this opens the space for the learners to interiorise. The second step is that of condensation where by the learner is now able to deal with longer algebraic expressions and also being able to tell the difference and the similarity between algebraic expressions.

After the two stages are passed, the process of reification begins and here, the learner is beginning to appreciate the idea of lack of closure in algebraic expressions activities and also understanding the structural meaning of having letters and numbers together in one expression. Algebraic expressions activities should be designed to help the learners to experience the reification journey. Algebraic activities should guide the learners on the journey of appreciating the meaning that is embedded within the expressions. However, Watson and Mason (2006) argue that there is need for instruction especially to guide learners on what is happening in some activities so that they do not misinterpret what is being communicated to them by the activities given.
CONCLUSION

In this paper, I have analysed some algebraic expressions that may be presented to learners. The analysis was done to find out how the given algebraic activities use variation theory to help learners focus on the dual nature of mathematics, in the processes that lead to reification. The activities show that patterns of variation are appropriate in helping to focus learners’ attention on the structural meaning that is carried by algebraic expressions. However, not just any variation is significant (Runesson, 2006) as was mentioned about the random variation which is seen in one of the activities, which might make learners fail to discern anything (Watson & Mason, 2006). Therefore, variation should be carefully controlled, until such a time when the learners are able to discern critical features of the object of learning simultaneously. Watson (2009) suggests a strategy of using ‘variation theory’ to guide the learners’ focus towards particular variables and differences and critical aspects of mathematical activities. I agree with Watson’s claim because algebraic expressions activities have critical aspects that need to be attended to.

REFERENCES


As we advance further into the 21st century, mathematics learners are being exposed to different forms of technology, and this has influenced learners’ behaviour (Sharples, 2009). The rapid and widespread use of technological tools such as tablets, smartphones and laptops has completely changed the way mathematics learners conduct their daily lives. The change in learners’ lives leaves the mathematics teacher with a question of how to use the different forms of technology to enhance teaching and learning mathematics concepts to the 21st century learner, without distractions that come with incorporation of technology into teaching and learning, especially in mathematics classrooms. This paper draws on Sharples, Taylor and Vavoula’s (2006) theoretical framework to discuss how mathematics teachers can draw on technology to teach mathematics to learners who are already exposed to the technology in their daily lives. This paper also offers mathematics teachers with a series of questions that may guide them in an effort to use technology to teach the 21st century mathematics learner.

INTRODUCTION

The 21st century has been characterised with different forms of technology innovations. Technology is defined as any tool which can be used to help improve human lives by simplifying the way things are done (Luppicini, 2005). Technology can therefore be used to simplify some teaching and learning processes, for example, technological tools such as calculators, smart boards, digital cameras, computers, mobile cell phones and many other tools, if used correctly can help improve the teaching and learning process (Sharples, 2009). Technology can have a profound impact on classroom teaching and learning, and above all, technology use in the classroom ensures that the 21st century learner is prepared for the world he/she will face going forward after school (Kukulska-Hulme, 2007).

Technology in Education

To tackle educational challenges, many attempts have been made to explore how the systemic integration of technology can help alleviate the effects of the crisis. A wide range of educational technology interventions initiated at institutional, provincial, national, regional and global levels focus on the enabling role of technology in improving the quality of teaching and learning, expanding access to learning opportunities, promoting social equity in education, and building inclusive ‘knowledge societies’ (UNESCO, 2012). Projects were rolled out with ‘end-to-end’ technology
solutions, which included personal computer laboratories (PC labs) equipped with curriculum content, teacher training modules and technical support, in six schools per country across sixteen countries in Africa. At the national level, SchoolNet Namibia, Egypt’s Smart School Network and the Jordan Education Initiative (JEI) were among the most prominent programmes. At the provincial level, notable initiatives included the Gauteng Online and Khanya projects in South Africa (Farrell and Isaacs, 2007; Farrell et al., 2007). Collectively, all of these initiatives involved significant financial, technological and human-capital investments, and worked to establish a global community of practice whose purpose was to catalyse a paradigm shift toward ‘twenty-first century learning’ and support the technology in education goals at various levels throughout the region’s education systems (Hungi et al., 2011). The main challenge of the projects was how teachers implemented the technology availed to them in the process of teaching and learning, especially in mathematics classrooms (Farrell and Isaacs, 2007; Farrell et al., 2007; Lin, 2008).

The 21st century mathematics learner

The 21st century mathematics learner is surrounded with many technology tools in his/her daily lives as discussed above. According to Vahey, Roy and Fueyo (2013), there are four characteristics that are associated with a 21st century mathematics learner: learning at his/her own pace, not being able to focus for very long periods on school work, looking for excitement in the process of learning, and wanting to prepare for the future using technology. Integration of technology may enable learners to engage with information at times that are most convenient for them and this may help to promote self-directed learning. Individualized instructional material may be accessed on a computer, tablet or cell phone. The learner is able to decide when to access specific information to enhance his/her understanding (Vahey, Roy & Fueyo, 2013). Technology gives learners access to information almost instantly, hence eliminating the time spent trying to look for information in a printed book. This information can be accessed via the internet. Technology also helps the learner to access information for research purposes more easily (Lin, 2008). Technology also provides the learner with the exciting aspects of learning, such as gamification, and being able to visualize some concepts (Lin, 2008).

Education is moving from just being about memorization to being able to collaborate with other, solve problems and develop ways of communicating ideas with people globally. 21st century mathematics learners would like to be engaged in the process of preparing to work in a technologically driven world, which they face after school (Vahey, Roy & Fueyo, 2013; Lin, 2008).
The 21st century mathematics teacher vs the traditional mathematics teacher

Having discussed the four needs of the 21st century learner, teaching mathematics to 21st century mathematics learners is not an activity to be taken for granted. Over the years, research in mathematics education has shown that there are different types of mathematics teachers (Koole, 2009; Kukulska-Hulme, 2007; Leadbetter, 2005; McKinsey & Company, 2012; Leont’ev, 1978; Lepi, 2013; Nardi, 1996; Nkosi, 2012; Noor, 2012; Sharples, 2005; Shiner, 2009; and Traxler, 2007; Lin, 2008). The table below shows the differences between a 21st century mathematics teacher and traditional mathematics teacher.

<table>
<thead>
<tr>
<th>21st Century mathematics teacher</th>
<th>Traditional mathematics teacher</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consider collaborative work and take it to be just as important as individual work</td>
<td>Stop collaborative work and ensure learners focus on their own work</td>
</tr>
<tr>
<td>Emphasise project based learning and not memorization of concepts</td>
<td>Teach to make sure that learners pass my test and examination</td>
</tr>
<tr>
<td>Acknowledge, edit, and share educational media with my students</td>
<td>Give learners notes and discourage them from checking from other resources for ideas and further explanations of concepts learnt in class</td>
</tr>
<tr>
<td>Learn from my learners</td>
<td>My learner cannot teach me anything, I know it all</td>
</tr>
<tr>
<td>Expect change</td>
<td>Prevent change</td>
</tr>
<tr>
<td>Create positive learning environments</td>
<td>Create an environment where learners are not free to present their findings during my lessons</td>
</tr>
<tr>
<td>Embrace technology and not fear it in any way</td>
<td>Avoid technology as much as possible and only use the computer to record learners’ marks</td>
</tr>
</tbody>
</table>
Embrace and encourage appropriate educational online interaction
Discourage educational online interaction, and argue that it is time wasting
Create own personal learning network or website
Ask questions why anyone needs a learning network, on top of a physical classroom

Table 1: 21st century mathematics teacher vs traditional mathematics teacher

| Technology is becoming more embedded, ubiquitous and networked, with enhanced capabilities for rich social interactions, context awareness and internet connectivity. Such technologies can have a great impact on learning. Learning will move more and more outside of the classroom and into the learner’s environments, both real and virtual, thus becoming more situated, personal, collaborative and lifelong. The challenge for the mathematics teacher is to discover how to use mobile technologies to transform learning into a seamless part of daily life to the point where it is not recognised as learning at all (Naismith et al., 2004). This paper focuses on mobile learning and how it can be used in the mathematics classroom. Mobile learning refers to any technology-enabled learning solution that allows learners to access educational content through any portable device such as a mobile phone, laptop or tablet (UNESCO, 2012; Brandt, 2012; Mckensey & Company, 2012). I will therefore draw on Sharples et al’s (2006) framework for analysing mobile learning to discuss how mobile learning can be used in the process of teaching 21st century mathematics learners. |

THEORETICAL PERSPECTIVE

In this paper, I draw on Sharples, Taylor and Vavoula’s (2006) framework for analysing mobile learning. The theoretical framework by Sharples et al. (2006) has been developed through a series of projects to design mobile learning technology. Sharples et al. (2006) argue that there is a need to reconceptualise learning for the mobile age, to recognise the essential role of mobility and communication in the process of learning. They also argue that there is need to indicate the importance of context in establishing meaning, and the transformative effect of digital networks in supporting virtual communities that transcend barriers of age and culture. Sharples et al. (2006) therefore offer a framework for theorising about mobile learning, to complement theories of infant, classroom, workplace and informal learning. Sharples et al’s theory draws on activity theory of learning. Activity Theory describes learning as a cultural-historical activity system, mediated by tools that both constrain and support the learners in their goals of transforming their knowledge and skills. Some researchers recognise activity theory as a powerful framework for designing constructivist learning environments and
student-centered learning environments (Jonassen, 2000; Jonassen & Rohrer-Murphy, 1999). However, certain limitations and unsolved problems in activity theory have been raised. Barab, Evans, and Baek (1996) pointed out that “life tends not to compartmentalize itself. Sharples et al. (2006) took this into consideration in the process of developing their framework. Sharples et al’s framework separates two perspectives, or layers, of tool-mediated activity: the semiotic layer and the technological layer. The semiotic layer describes learning as a semiotic system in which the learner’s object oriented actions are mediated by cultural tools and signs. The technological layer represents learning as an engagement with technology, in which tools such as computers and mobile phones function as interactive agents in the process of coming to know (Nardi, 1996; Bodker, 1991; Kaptelinin, Kuutti, & Bannon, 1995). Sharples et al. (2006) argue that a central concern must be to understand how people artfully engage with their surroundings to create impromptu sites of learning. The table below shows the convergence between learning and technology.

<table>
<thead>
<tr>
<th>New learning</th>
<th>New technology</th>
</tr>
</thead>
<tbody>
<tr>
<td>Personalised</td>
<td>Personal</td>
</tr>
<tr>
<td>Learner centered</td>
<td>User centered</td>
</tr>
<tr>
<td>Situated</td>
<td>Mobile</td>
</tr>
<tr>
<td>Collaborative</td>
<td>Networked</td>
</tr>
<tr>
<td>Ubiquitous</td>
<td>Ubiquitous</td>
</tr>
<tr>
<td>Lifelong</td>
<td>Durable</td>
</tr>
</tbody>
</table>

Table 2: Convergence between learning and technology

Learning is now regarded as a situated and collaborative activity (Brown, Collins, & Duguid, 1989), occurring wherever people, individually or collectively, have problems to solve or knowledge to share, so mobile networked technology enables people to communicate regardless of their location (Sharples et al., 2006).

Sharples et al. (2006) therefore argue that any theory of mobile learning must consider the following questions:
Is it significantly different from current theories of classroom, workplace or lifelong learning?

Does it account for the mobility of learners?

Does it cover both formal and informal learning?

Does it theorise learning as a constructive and social process?

Does it analyse learning as a personal and situated activity mediated by technology?

From the questions above, Sharples et al. (2006) came up with the following arguments below.

Sharples et al. (2006) argue that it is the learner that is mobile, rather than the technology; learning is interwoven with other activities as part of everyday life, learning can generate as well as satisfy goals; learning can be initiated by external goals (such as a curriculum or study plan), or by a learner’s needs and problems, or it can arise out of curiosity or serendipity, prompting the learner to form new goals which may then be explored through formal or informal study. Sharples et al. (2006) also argue that with mobile learning, the control and management of learning can be distributed. In a classroom, the locus of control over learning remains firmly with the teacher, but for mobile learning it may be distributed across learners, guides, teachers, technologies and resources in the world such as books, buildings, plants and animals (Sharples et al., 2006). Context is constructed by learners through interaction: To explore the complexity of mobile learning, it is necessary to understand the contexts in which it occurs. Context should be seen not as a shell that surrounds the learner at a given time and location, but as a dynamic entity, constructed by the interactions between learners and their environment (Sharples et al., 2006). The figure below shows the different aspects which are used in analysing mobile learning presented by Sharples et al. (2006).
Learning occurs as a socio-cultural system, within which many learners interact to create a collective activity framed by cultural constraints and historical practices (Sharples et al., 2006). Engeström analyses the collective activity through an expanded framework that shows the interactions between tool-mediated activity and the cultural Rules, Community and Division of Labour. Sharples et al adapted Engeström’s framework to show the dialectical relationship between technology and semiotics, so they renamed the cultural factors with terms: Control, Context and Communication. Sharples et al. (2006) attempt to clarify their meaning of Control, Context and Communication, as discussed below.

**Control**

The control of learning may rest primarily with one person, usually the teacher, or it may be distributed among the learners. Control may also pass between learners and technology. The technological benefit derives from the way in which learning is delivered: whether the learners can access materials when convenient, and whether they can control the pace and style of interaction. These are issues of human-computer interaction design (Sharples et al., 2006). However, technology use occurs within a social system of other people and technologies. Social rules and conventions govern what is acceptable, for example who is allowed to email whom, what kinds of document format should be used, and when to use your mobile phone. A person’s attitudes to technology can be influenced by what others around them think about it, for example, whether they are resentful at having to use the technology or are keen and eager to try.
it out. And individuals and groups can also express informal rules about the way they like to work and learn (Sharples et al., 2006).

**Context**

The context of learning is an important construct, but the term has many connotations for different theorists. From a technological perspective there has been debate about whether context can be isolated and modelled in a computational system, or whether it is an emergent and integral property of interaction (Sharples et al., 2006). Context includes all the people involved and how and when they interact with the technologies.

**Communication**

The dialectical relationship between the technological and semiotic layers is perhaps the easiest to see in relation to Communication. If a technological system enables certain forms of communication (such as email or texting), learners begin to adapt their communication and learning activities accordingly. For example children are increasingly ‘going online’ at home, creating networks of interaction through phone conversation, texting, email and instant messaging that merge leisure and homework activities into a seamless flow of conversation (Sharples et al., 2006). As they become familiar with the technology they invent new ways of interacting such as ‘smilies’, and text message short forms, which are new forms of communication (De Lanerolle, 2012). This appropriation of technology not only leads to new ways of learning and working, it also sets up a tension with existing technologies and practices. For example, children can subvert the carefully managed interactions of a school classroom by sending text messages hidden from the teacher. On a broader scale, technology companies see markets for new mobile technology to support interactions such as file sharing and instant messaging. Thus, there is a continual co-evolution of technology and human communication, with each new development creating pressures that drive the next innovation (Sharples et al., 2006).

**How to use mobile learning to teach mathematics to 21st century mathematics learners**

Mobile learning can both complement and conflict with formal education. Learners can extend their classroom learning to homework, field trips, and museum visits by, for example, reviewing teaching material on mobile devices or collecting and analysing information using technology. Learners could also disrupt the carefully managed environment of the classroom by bringing into it their own multimedia phones and wireless games machines, to hold private conversations within and outside the
classroom (Sharples, 2002). In order to use mobile learning, the mathematics must ask him/herself the following questions:

**Does it cover both formal and informal learning?**

Due to the nature of 21st century learning, learners need control over their technologies. If the informal part of learning is not considered, learners may abandon the tool used for formal learning and focus on the tool that offers them informal learning. Therefore, mathematics teachers should ensure that the tool selected for mobile learning should have opportunities for formal and informal learning.

**Does the mobile learning tool account for the mobility of learners?**

Learners are constantly on the move (Sharples, 2002). A mobile learning tool which is not accessible on the go may not be conducive to mobile learners (Sharples et al., 2006). Naismith et al. (2004) also argues that mobile technology should be incorporated in learners lives in such a way that learners do not get the feeling of having to differentiate between learning and living their social lives, while on the move.

**Does the mobile learning tool analyse learning as a personal and situated activity mediated by technology?**

The learners should be able to view the tool as personal. Once the learners view the tool as personal, they will be able to view the learning it offers as personalised and hence feel that they have to control their own learning (Sharples et al., 2006). Learners should feel like they are able to use the tool at any time without anyone asking them to stop or keep it away. However, the control issue maybe challenging when the learner wants to use the tool in the middle of mathematics lessons when the teacher is trying to explain a concept. The learner may misuse the control aspect and use the tool to access only non-educational material because no one is monitoring what he/she is doing with the tool. Therefore, any mobile learning project should consider the following issues below.

**Context**

The process of gathering and utilizing contextual information may clash with the learner’s wish for anonymity and privacy. Learners may not be in a position of wanting their ‘personal’ device to be accessed by their teacher or parent.
Mobility

The ability to link to activities in the outside world provides learners with the capability to ‘escape’ the classroom and engage in activities that do not correspond with either the teacher’s agenda or the curriculum.

Learning over time

Effective tools are needed for the recording, organization and retrieval of mobile learning experiences. If not monitored constantly and appropriately, it may be impossible to find out if learners are actually using the tool for learning formal content or not.

Informality

Learners may abandon their use of certain technologies if they perceive their social networks to be under attack. In an informal setting, learners would want to have as much privacy as possible.

Ownership

Learners want to own and control their personal technology, but this presents a challenge when they bring it into the classroom. In the following section, I will discuss some limitations of mobile learning, which mathematics teachers need to consider in the process of answering the question of how to use mobile learning to teach mathematics to the 21st century mathematics learner.

Limitations of mobile learning technology

Every technology has some limitations and weaknesses, and mobile devices are no exception. Kukulska-Hulme (2007) summarized these problems as follows: 1) physical attributes of mobile devices, such as small screen size, heavy weight, inadequate memory, and short battery life; (2) content and software application limitations, including a lack of built-in functions, the difficulty of adding applications, challenges in learning how to work with a mobile device, and differences between applications and circumstances of use; (3) network speed and reliability; and (4) physical environment issues such as problems with using the device outdoors, excessive screen brightness, concerns about personal security, possible radiation exposure from devices using radio frequencies, the need for rain covers in rainy or humid conditions, and so on. It is important to consider these issues when using mobile devices and designing the learning environment.
However, looking at how rapidly new mobile products are improving, with advanced functions and numerous applications and accessories available these days, the technical limitations of mobile devices may be a temporary concern. Also, the use of mobile technologies in education is moving from small-scale and short-term trials or pilots into sustained and blended development projects, in an attempt to address the educational needs of the 21\(^{st}\) century learner (Traxler, 2007).

CONCLUSION

South Africa is a country where mobiles cell phones are affordable by majority of our learners. The way mathematics learners live their lives has been influenced by the technology surrounding them, especially the mobile devices. Therefore, mobile learning can be used to influence and enable learning more and more (Vosloo, 2012). However, mobile learning can also be seen as a challenge to formal schooling, to the autonomy of the classroom and to the curriculum as the means to impart the knowledge and skills needed for adulthood. But it can also be an opportunity to bridge the gulf between formal and experiential learning, opening new possibilities for personal fulfillment and lifelong learning, which is needed for the 21\(^{st}\) century (Sharples et al., 2006).

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EXTENDED ABSTRACT

The Annual National Assessments (ANA) now plays a central role in the teaching and learning of mathematics - it is an intervention of the South African education authorities that is aimed at improving the quality of education. The ANA is administered to learners in Grades 1 to 7 and Grade 9 each year. One of the purposes of the ANA is to serve as a diagnostic tool identifying areas of strengths and weaknesses in teaching and learning of mathematics and literacy. The study reported in this paper, was carried out in KZN with seven schools from two districts. Data was generated from interviews with teachers from three of the seven schools, and the written responses of 1100 learners to the grade 9 mathematics ANA.

The purpose of this study was to 1) elicit teachers’ perceptions of the usefulness of the ANA and, 2) examine the responses of learners to the assessment by means of a Rasch analysis.

The ANA is perceived by many as being a first step in improving the education system because it aims to provide information nationally about learners’ understanding of mathematics which can be used to diagnose strengths in the system as well as threats to the education system. Information from the ANA can provide fine grained information that can range from specific details about an individual learner’s understanding details about trends in learner performance at a national level. However anecdotal evidence suggests that some teachers are not positive about the programme, and experience the ANA as yet another load that they need to take on. It is therefore important to find out about the benefits or challenges associated with the ANA as perceived by the teachers. Hence teachers from three schools were interviewed in order to find out more about the value they attached to ANA in their practice. The teachers were probed about how the ANA results were used as part of their learners’ Grade 9 results as well as the ways in which the administration of the ANA impacted on their delivery of the curriculum.

In addition, the responses of 1100 learners from the seven schools were analysed to search for broad trends in the results by looking at the actual performance of learners in the mathematics Grade 9 ANA. A Rasch analysis was carried out on the responses of these 1100 learners in order to investigate the functioning of the grade 9 mathematics ANA as an assessment of mathematics proficiency of grade 9 learners. When carrying out a Rasch analysis, an assumption is that the data must fit the model. The Rasch model
is based on measurement requirements and not on assumptions about data and prior to making inferences about assessment data, it is necessary to check the fit of the data to the model. When a test adheres to the requirements for measurement- like interpretations, then they allow for inferences related to comparisons of item and person proficiency locations. When some of the measurement criteria are not satisfied, the identification of the anomalies can contribute to a deeper understanding of the construct under scrutiny, and may help us understand some of the threats to the development of sound assessment instruments. This paper will report on these issues, in an effort to interrogate the instrument as well as the take-up of the ANA at the school level.
DEEPENING THE QUALITY OF MATHEMATICS LEARNING THROUGH COMPUTER ASSISTED INSTRUCTION (CAI) IN NIGERIAN SECONDARY SCHOOLS

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For the past few years, the West African Examination Council (WAEC) Chief Examiners’ reports on students’ performance in mathematics have indicated that some topics (e.g. the concepts of latitude and longitude) have posed a major problem for students at the senior secondary school level. The study investigates if Computer Assisted Instruction (CAI) can deepen students’ learning of mathematics, and in particular, the concepts of latitude and longitude in Ogun State, Nigeria. The study employs a pre-test post-test non-equivalent quasi-experimental design involving two group: experimental group (157) and control group (158). The principal instrument for data collection was Achievement Test on Latitude and Longitude (ATLL). Data were analyzed by using descriptive and inferential statistics. The mean scores of pre-test and post-test scores of the two groups were compared using t-test. Results showed non-significant difference in the pretest means scores of the experimental and the control groups but, a significant difference in the post-test mean scores of the two groups. The study concluded that CAI students attained higher achievement level than their counterparts who were exposed to the traditional method of teaching. Hence, a conclusion that the strategy can be used to deepen students’ learning of mathematics.

BACKGROUND TO THE STUDY

The continued decline of Secondary School students’ performance at both internal and external examinations has remained a source of concern to all stakeholders in education, including mathematics educators (Abakpa, & Iji, 2011). Whilst the students’ performance in mathematics has been generally poor, the West African Examination Council Chief Examiners’ reports in particular, indicate that the topic, Latitude and Longitude is a major problem area for their Mathematics examination candidates (West Africa Examination Council [WAEC] 2007; 2010). According to these reports, the candidates usually failed to understand the questions and hence did not address them adequately.

Mathematics educators have put up noble and spirited efforts aimed at identifying the major problems associated with the teaching and learning of mathematics and latitude and longitude as a topic in the nation’s schools (Maoah, Indoshi & Othuon, 2011; Yara, 2011). The same authors pointed out that the teaching methods used by teachers when teaching the topic could be the primary reason why students find it difficult. Some other researchers, such as Salman, Muhamed, Ogunlade and Ayinla, (2012); Adolphus,
(2011) reported on students’ negative interest in mathematics, teachers’ attitude toward mathematics, poor primary school background in mathematics, unqualified teachers in the system, learners’ interest in mathematics, wrong perception and psychological fear about the subject as other possible factors responsible for students’ poor performance in Geometry inclusive of Latitude and Longitude.

Education reforms in recent years, have advocated a shift from a teacher-centered to a student-centered method of teaching. Many reforms have also encouraged the use of different forms of ICT such Computer-Assisted Instruction (CAI) in the teaching of mathematics to improve students’ learning (Panaoura, 2006; Panaoura & Philippou, 2007). Various researchers (Kulik, 2002; Panaoura and Philippou, 2007) consider computer-assisted instruction (CAI) as a learner-centered method that allows students to progress at their own pace as they work individually or in groups. CAI is an interactive strategy that can illustrate a concept through attractive animations, sound and demonstrations in the mathematics classrooms. The strategy provides immediate feedback, letting students know whether their answers are correct or not. If the answers are not correct, it gives prompts/tips to enable them to attempt the question again. In addition, most CAI software has links to an array of related websites that can help students to deepen their learning of mathematics and in particular topics such as Latitude and Longitude. As a learner-center method, CAI is flexible and convenient with self-paced instruction that provides immediate and frequent feedback without the embarrassment that a mistake in a traditional classroom might cause (Chen, Lee, Hung, & Wei, 2011).

The use of ICT in mathematics classrooms has been found by some studies, not only to have enhanced students’ achievement in various mathematical concepts, but has also motivated them to learn, aid the development of their social skills and sustained their understanding (Devi, Chinnaiyan & Dhevakrishnan 2012; Chen, Lee, Hung, & Wei, 2010). According to Azuka and Awogbemi, (2012), the use of CAI is now gaining popularity among mathematics teachers in the teaching of mathematics around the globe. Many of the teachers according to their reports have begun to use mathematical software such as, Mat Lab and Mathcad during their mathematics lessons. This innovation to the teaching and learning of mathematics will enhance students’ skills, encourages them to learn the subject better and more effectively with deeper understanding.

It is against this background and our belief that the introduction of CAI in the teaching and learning of Latitude and Longitude will deepen and foster students’ achievement on the topic that we were motivated to carry out this study. Specifically, the study investigates how Computer Assisted Instruction can deepen the quality of students’
learning of mathematics and in particular, the concepts of Latitude and Longitude in Ogun State, Nigeria.

Latitude and Longitude is one of the topics in mathematics in the WAEC syllabus. **Latitude** is a geographic coordinate that specifies the north-south position of a point on the Earth’s surface. If the Earth is cut perpendicular to the polar axis, the circles formed on its surface are called lines of latitude (Macrae, Kalejaiye, Chima, Garba, Ademosu, Cannon, McLeish, Smith, & Head, 2009). It is an angle which ranges from 0° at the equator to 90° north or south at the North and South Poles. Harwood, (2012) describes **Longitude** as angular distance measured east and west of the prime meridian (Greenwich meridian). The prime meridian is 0° longitude. As one travels east from the prime meridian, the longitude increases to 180°, while, as one travels west from the prime meridian, longitude increases to 180°.

Specific objectives of the concept of Latitude and Longitude as contained in the Mathematics Curriculum for Senior Secondary (WAEC, 2008) are:

- To distinguish between Great Circle and Small Circle on the Surface of the Earth.
- Define the lines of Longitude (including the Meridian) and Latitude (including the Equator) on the Surface of the Earth.
- Determine and sketch the position of a point on the Earth Surface in term of its’ Latitude and Longitude (e.g. 14° N, 26°E and 37°S, 106°W).
- Calculate the distance between two points on the Great Circle (Meridian) or the Equator.
- Calculate the distance between two points on a parallel of Latitude.
- Calculate the Speed, the Shortest Distance between two points,
- Compare Great Circle and Small Circle Route on the surface of the Earth (Macrae, et.al. 2009 p.53).

Related questions on the concept of latitude and longitude include the following:

1. A boy walks 6Km from a point P, to a point Q, on a bearing of 065°. He then walks to a point R, a distance of 13Km, on a bearing of 146°. (i) Sketch the diagram of the coordinate of his movement, (ii) Calculate, correct to the nearest kilometer, the distance PR.

2. A Plane flies due East from A (53°N, long.25°E) to a point (Lat.53°N, Long.85°E) at an average Speed of 400Km/hr. The plane flies south from B to a point C 2000 km away. Calculate, correct to the nearest whole number. (Take π=22/7; R=6400km) (i) The distance between A. and B ;(ii) The time the plane takes to reach point B (iii) The latitude of C.
3. An aircraft flies due South from an airfield on latitude 36°N, longitude 138°E to an airfield on latitude 36°S, longitude 138°E (i) Calculate the distance travelled, correct to three significant figures. (ii) If the speed of the aircraft is 800 km per hour, calculate the time taken, correct to the nearest hour. (Take π=22/7; R=6400 km)

CAI and students’ learning

Findings on the positive impact of CAI in students’ learning of mathematics and other science subjects have been inconsistent (Awofala, 2012). While some studies claim that CAI has positively influenced students’ mathematics learning, others assert that CAI does not add any favorable advantage to students’ mathematics learning. Researchers such as Devi, Chinnaiyan and Dhevakrishnan, (2012) who have studied the effectiveness of CAI in the teaching of mensuration in mathematics at secondary school level reported that students who were taught using CAI, achieved better than their counterparts who were taught using the traditional method of teaching. Hsu’s (2003) study on the effectiveness of CAI in teaching introductory statistics has revealed a moderate, positive effect of CAI on students’ learning. Furthermore, Kumar, (2010) tested the effectiveness of CAI in teaching general science at secondary school level and found positive results in favor of CAI as compared with the conventional or traditional method. Study by Singh, (2010) revealed that the simulation mode is more effective than the tutorial and drill and practice modes of CAI for teaching science to 9th grade students. Mahmood, (2006) has examined students’ performance in developmental mathematics when CAI was combined with traditional strategies. His findings have revealed that CAI greatly enhances the mathematical skills of poorly-performing students. CAI benefits extend beyond mathematics as other researchers such as Yusuf and Afolabi (2010) found CAI to be an effective mode of instruction for teaching biology to secondary school students both in individualized and cooperative settings. They reported that CAI is an effective mode for knowledge, comprehension and application domains of learning, as well as for learning all content areas of general science, i.e. biology, chemistry and physics.

Critics of CAI

Despite the fact that some research findings agreed that higher test scores and acquisition of higher level thinking skills can be attributed to the use CAI in the classrooms, some studies indicate that computer assisted instruction does not positively affect the learning of mathematics.

For instance, a study by Hamtini, (2000) that evaluated students’ achievement by means of computerized instruction in comparison with traditional instruction, in a developmental algebra course at a university based on a pre-test and post-test, established that the traditional lecture-based group showed a significantly higher level of achievement than the computer-based group did. In addition, research conducted by Baker, Gersten and Lee (2002) on the influence of CAI on the mathematical achievement of low-achieving
students revealed that low-achievers did not perform significantly better. Similarly, Yusuf (2005) examined the effect of cooperative and competitive instructional strategies on junior secondary school students’ performance in social studies, and found out that achievement levels have no influence on academic performance of the low achieving learners/students. He reported further that there was no significant difference between the high and medium achievement level, and between students of medium and low achievement levels when taught social studies using videotape instruction. The apparent lack of literature on the use of CAI strategy in teaching the concept of Latitude and Longitude in mathematics, and the diverse findings of studies on the effectiveness of CAI in the teaching and learning of mathematics make a case for this study.

RESEARCH QUESTIONS

The research questions addressed in the study were:
- Can CAI method of teaching deepen students’ learning of mathematics with reference to the concept of latitude and longitude in Nigeria secondary schools? And
- If yes, in what way is CAI effectively deepening students learning on the topic?

RESEARCH HYPOTHESIS

The following null hypotheses were tested at, 0.05 level of significance, in order to provide answers to problems that were raised in the study.

H1: There is no significant difference between the pre-test scores of students exposed to the computer assisted instructional strategy and those not exposed to CAI.

H2: There is no significant difference between the pre-test and the post-test scores of students exposed to the computer assisted instructional strategy?

H3: There is no significant difference between the post-test achievement scores of students exposed to computer assisted instructional strategy and those not exposed to CAI.

THEORETICAL FRAMEWORK.

The research work was premised on social constructivism learning theory associated with Vygotksy (1978). The theory posits that Learning is an active, constructive process. This theory is considered relevant for the study because it emphasizes cognitive development through social interaction and the use of CAI technologies provides a platform for such social interactions. A reaction to didactic approaches such as
behaviorism and programmed instructions, constructivists such as Lev Vygotsky, Jean Piaget and Jerome Brunner stated that learning is an active, contextualized process of constructing knowledge rather than acquiring it. They emphasized that knowledge is constructed based on personal experience and hypotheses of the environment and that learners continuously test these hypotheses through social negotiations.

Vygotsky felt that social learning preceded development. He stated that every function in child’s cultural development appear twice, Firstly, on the social level i.e. between people (inter-psychological) and secondly, on the individual level i.e. inside the child intra-psychological). He believed strongly about developmental link with society and surroundings. Constructivism theory of learning opined that teacher as one of the more knowledgeable others (MKO) has the role to make the classroom as rich as interactive learning community as he or she can and through language, lead the children into new zone of proximal development (ZPD). The zone of proximal development according him is the difference between an individual’s current level of development and his or her potential level of development. He stressed further that learners’ peer, his teacher, a coach, an old adult or even computer as a more knowledgeable others can guide the learner to the zone of proximal development which is considered the distance between learner’s ability to perform a task and the ability to solve problems independently.

This is one of the views of CAI whereby students interact with the objects in the environment through variety of techniques such as quizzes, simulations and explore the world around them with the aid of their Computers.

**Application of the theory to CAI**

According to constructivism learning theory paradigm, appropriate teaching and learning may include number of components such as genuine discussion, cooperative group work, project work and problem solving for engagement, mastery autonomous project, exploration and investigative work (Sprandlin, 2009). The advent of Internet age and Websites facilities conveniently fixed into constructivist paradigm and actively promoted the learning theory. As society and the world are rapidly changing, a close look at the learning theory showed its relevance and appropriateness as a foundation for certain technologies like e-learning and computer assisted instructions. The integration of educational technology as suggested by the learning theory into a class situation will enhance learning and positively change the orientatiotns of both the students and the teachers towards the perceived difficult topics like the concept of Latitude and Longitude in Mathematics. Social constructivists opined that reality is constructed through human activity. Mathematics students can interact with each other and the objects in the environment through a variety of techniques such as quizzes, simulations, explorations, and tests with the aid of their Computers. One of the beauties of Computer Assisted Instruction is the ability to guide students to progress through a unit of study at their
own rate, checking their own answers and advance after answering correctly. The teacher in this case acts as a coordinator after he had plants a powerful mathematical idea in a personally meaningful context for students to investigate.

The More Knowledgeable Others (MKO) advocated by the constructivism learning theory refers to anyone who has a better understanding or a higher ability level than the learner, with respect to a particular task process or concept. In the CAI classroom, the peer tutor and personal computer can be the MKO; the peer tutor helps the struggling students work through problems by providing hints and instruction. After few periods, the struggling students will stop relying on MKO as they work through ZPD levels. Thus, the struggling students have reached the ZPD levels and no longer struggling.

The Computer Assisted Instruction package (CAIP) developed by the researchers for the study follows this trend of guiding the students in solving related questions on Latitude and longitude in mathematics.

**METHODOLOGY**

**Research Design**

The study adopted a quantitative research method described as a systematic empirical investigation of social phenomena via statistical, mathematical or computational techniques within the blueprint of pre-test, post-test non-equivalent control group quasi-experimental design (Creswell, 2013). The research design is symbolically presented below:

N 1: O₁ X₁ O₁ Experimental Group I (CAI)
N 2: O₂ O₂ Control Group II (Traditional Method)

The first row represents the experimental group. The second is the control group. O₁, O₂, represent pre-test and post-tests; X₁ represents the treatment and O₁ and O₂, were tested for Statistical significance using the Analysis of Covariance (ANCOVA).

This design is appropriate because it reduces the interactive effect of treatment and increases the external validity of the findings (Creswell & Plano 2011). Furthermore, quasi-experimental design using pre-test, post-test non-equivalent control groups is typically easier to set up than true experimental designs but lacks randomization of subject to treatment conditions (Shadish, Cook & Campbell, 2002). In addition, the choice of a quasi-experimental design for the study allows investigation of intact groups in real-life classroom settings since it was not necessary to randomly assemble students for any intervention during the school hour so as not to create artificial conditions (Emin, 2005). The quasi-independent variable-instructional strategies was manipulated
at two levels (CAI and TM) so as to provide answers to research questions raised for the study, and also assess the extent to which the CAI and TM have influenced students’ achievement about the concept of Latitude and longitude.

Population

The target population for the study consisted of all the 2nd year mathematics students in public Senior Secondary schools (SSS) in Ogun State, Nigeria while the accessible population is 2nd year mathematics students in public Senior Secondary students in Odeda Local government area of Ogun state, Nigeria.

Sample and Sampling procedure

Three hundred and fifteen (315) Senior Secondary School 2nd year mathematics students from two schools in Odeda local government participated in the study. The participants were purposively selected based on the following criteria:

(i) There is no impending and immediate external examination that could distract them from full participation in the study. (ii) The students are mature and have gained the confidence required to participate in the study having been taught some aspect of Plane and Solid Geometry at their first year at Senior Secondary School (iii) The topic used as an intervention in the study (Latitude and Longitude) is in line with WASCE mathematics curriculum (WAEC, 2008). (iv) The 2nd year mathematics students in the public Senior Secondary schools (SSS) have not received instruction on the topics before.

Furthermore, selected schools for the study was based on the under listed criteria, (i) The schools have computer laboratory (ii) the mathematics teachers are willing to participate in the study (iii) The school is a public co-educational Senior Secondary school. From the four Senior Secondary schools contacted and met the criteria stated above, two schools were purposively selected to participate in the study. The researchers randomly assigned a school to the experimental group and the other school to the control group.

INSTRUMENTATION

The instruments for the research were the treatment instrument “Computer Assisted Instruction Package (CAIP) and the test instrument, Achievement Test in Latitude and Longitude (ATLL). CAIP is a self-instructional interactive package that lasted for 3 hours for an average student to complete. It consist of five lessons structured into three modules namely; the introduction of Latitude and Longitude, identification of latitude
and longitude on the globe, and guided steps in solving mathematical problems on latitude longitude. The package was developed by the researchers, with the assistance of a professional program developer using both Mat lab and Java languages. The software was validated by mathematics and computer experts for the appearance, operation, spellings and many other things. The software was pilot studied by the researchers and the results obtained in the usability experience were used for improvement of the package.

Similarly, Achievement test on Latitude and Longitude (ATLL) developed by the researchers was used in the study. The questions were essay-type cognitive test that required students higher-order cognitive skills of Bloom’s taxonomy (Analysis, Synthesis and Evaluation) and objectives of latitude and longitude, as contained in the Mathematics Curriculum for Senior Secondary (WAEC, 2008). The instrument was used for pre-test and post-test in both the control and experimental groups. The instrument was validated by three mathematics experts to ascertain the coverage of the selected area of the topic for the study. The reliability coefficient was found to be 0.78, using the Kuder-Richardson method (formula 21). The test items yielded a discrimination power of more than 0.40 and a difficulty index of 0.40–0.60. This follows the rule of thumb which specifies that above 0.75 is easy and that 0.25–0.74 is difficult (Creswell, 2011).

**DATA COLLECTION AND ANALYSIS**

Quantitative data were collected in this study. Data collection which lasted four weeks was by means of the Achievement test on latitude and longitude (ATLL). Before instruction, all the groups (experimental and control) were subjected to ATLL pre-test. Thereafter, the experimental group was exposed to CAIP which was installed in their computer. Each class was taught a 3-weeks unit of instruction, concerned with latitude and longitude, by different teachers. After the intervention, ATLL items were rearranged in order to prevent the cognitive bias before the test was administered as a post-test to both experimental and control classes.

Data collected from the pre-test and post-test were used to answer the research questions and test the stated hypotheses. The research questions were answered using descriptive and inferential statistics while the null hypotheses were tested at the 0.05 confidence level. The mean results of the two classes on the pre-test, post-test level of achievement were compared using independent t-test for a possible significant difference.
Results

A pre-test conducted before the treatment was to establish the baseline in the two groups on the concept of Latitude and Longitude as indicated in Table 1 below.

**Hypothesis One**: There is no significant difference between the pre-test scores of students exposed to the computer assisted instructional strategy and those not exposed to CAI.

<table>
<thead>
<tr>
<th>Group</th>
<th>N</th>
<th>Mean</th>
<th>S.D</th>
<th>S.E</th>
<th>t-value</th>
<th>P-Value</th>
<th>Sig. Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experimental</td>
<td>157</td>
<td>30.24</td>
<td>12.76</td>
<td>2.20</td>
<td>1.56</td>
<td>P&gt;0.05</td>
<td>Not Sig.</td>
</tr>
<tr>
<td>Control</td>
<td>156</td>
<td>23.22</td>
<td>8.62</td>
<td>1.62</td>
<td>1.56</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Analysis of ATLL Pre-test scores (Experimental and Control)

The result shows that the mean achievement scores are 30.24 and 23.22 for experimental and control group respectively. The t-value of 1.56 is obtained, which is not significant at 0.05 levels. It could then be concluded that there is no significant difference between the experimental and the control group in the pre-test. This confirms the equivalence of the two groups. Therefore it was evident that before the treatment, the two groups were comparable in term of their knowledge on the topic. The null hypothesis stated above was not rejected but was upheld.

**Hypothesis Two**: There is no significant difference between the pre-test and the post-test scores of students exposed to the computer assisted instruction strategy?

After the treatment, the pre-test and post-test scores for the experimental group were compared and the result is presented in Table 2 below.

<table>
<thead>
<tr>
<th>Group</th>
<th>N</th>
<th>Mean</th>
<th>S.D</th>
<th>S.E</th>
<th>t-value</th>
<th>P-Value</th>
<th>Sig. Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-Test</td>
<td>157</td>
<td>30.24</td>
<td>12.76</td>
<td>2.20</td>
<td>3.54</td>
<td>P&lt;0.05</td>
<td>Significant.</td>
</tr>
<tr>
<td>Post-Test</td>
<td>157</td>
<td>55.85</td>
<td>17.04</td>
<td>1.86</td>
<td></td>
<td></td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 2: Comparison of pre-test and post-test scores for the experimental group

Table 2 reveals the mean and the standard deviations of the pre-test and the post-test scores of students in the experimental group after the treatment. Comparison of difference between the mean pre-test and post-test showed significant difference \((t=3.54, p<0.05)\)
in favor of post-test class. This is evidently the positive impact of CAI on the students’ achievement in the topic. The null hypothesis is therefore rejected.

**Hypothesis Three:** There is no significant difference between the post-test achievement scores of students exposed to computer assisted instructional strategy and those not exposed to CAI.

The post-test scores for the two groups (experimental and control group) were analyzed and the result is also presented in Table 3 below.

<table>
<thead>
<tr>
<th>Group</th>
<th>N</th>
<th>Mean</th>
<th>S.D</th>
<th>S.E</th>
<th>t-value</th>
<th>P-Value</th>
<th>Sig. Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experimental</td>
<td>157</td>
<td>55.85</td>
<td>17.04</td>
<td>1.86</td>
<td>7.44</td>
<td>P&lt;0.05</td>
<td>Significant.</td>
</tr>
<tr>
<td>Control</td>
<td>156</td>
<td>34.24</td>
<td>13.68</td>
<td>1.54</td>
<td>7.44</td>
<td>P&lt;0.05</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**Table 3. Post-test ATLL scores (Experimental and control group)**

The result above indicates that the mean scores of the experimental and control group are 55.85 and 34.24 respectively in the post-test. The t-value is 7.44 which are significant at 0.05 level of confidence. It is interpreted that the experimental and control group differ significantly in the post test (21.61) and this favored the experimental group. It shows that students who are taught by CAI learned more and show higher achievement than the traditional method of teaching. Hence, the stated null hypothesis is rejected because of remarkable significant difference that exists between the two groups.

**Discussion of Findings**

From the results of the study, a significant difference exists between the post-test scores of the control and of the experimental group. This indicates that the CAI teaching strategy is effective in deepening students’ achievements in mathematics and the mathematical concept of latitude and longitude. The teaching strategy promotes students’ achievements in subject content and gave them opportunities to become more confident in following instructions (Akinsola & Igwe 2002).

This finding is consistent with the studies of Kumar (2010), Singh (2010), and Hsu (2003), who report that CAI is a useful tool in enriching, supporting, and mediating the learning of mathematical concepts. The findings also support those of Yusuf and Afolabi (2010), that the use of CAI mathematics software to teach students for a better
output is an effective mode of instruction for teaching in both individualized and cooperative settings.

CONCLUSIONS

Based on the findings of the study, we conclude that CAI strategy is an effective medium of instruction for teaching the mathematical concepts of latitude and longitude in Nigerian secondary schools. Evidence has shown from the results, that the strategy has the potential of not only improving students’ achievement, but also deepens the quality of learning the topic and mathematics generally.

LIMITATIONS OF THE STUDY

Even though the study is limited to Ogun State, in the South West geo-political zone, and one of the educationally advantaged states in Nigeria, it is an indication for the existence of a potential for future researches to be undertaken by expanding its scope that cover many other states. The research’s investigation is limited to the problems encountered by students specifically on the concept of latitude and longitude in mathematics. Hence, the study has the limitation that the results of this study will be applicable to Ogun State.

Although, the researcher has done to minimize the basic and administrative research bias, the inherent flaws of the methodology adopted in this research may have its effect on the analyses of the study. Furthermore, the applications of the recommendations will be appropriate for the improvement of teaching methodology of mathematics and other science related subjects in Nigeria secondary schools.

RECOMMENDATIONS

Based on our findings from the study, we recommend the following:

1. That, the teaching strategy, if implemented, will improve students’ performance and consequently arouse their curiosity and interest to learn new concept in mathematics, and perceived difficult topics such as the concept of latitude and longitude in particular.

2. The use of CAI strategy as method of instruction should not be limited to mathematics alone, but rather be more encouraged in other subjects in Nigerian schools.

3. Future researches are undertaken on the use of CAI by adopting other research method so as to triangulate the findings.
4. A CAI strategy should be implemented in the teaching and learning of mathematical concept latitude and longitude, and mathematics generally in Nigerian secondary schools in order to deepen students’ knowledge.

If the recommendations are considered for implementation by the Federal Ministry of Education, students’ leaning quality in mathematics will not only be deepened, but will also influence their attitude towards the subject.

References


TEACHER CODE SWITCHING IN A MULTILINGUAL MATHEMATICS CLASSROOM: A FOCUS ON PRECISION, CONSISTENCY AND TRANSPARENCY

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EXTENDED ABSTRACT

The presence of eleven official languages available for a school to choose and use as its language of learning and teaching (LOLT) has resulted in new opportunities and challenges for the South African Education system since 1994. The language in education policy in South Africa which had to accommodate competing social, political, economic, historical and educational factors (Setati, 2008) has led to continued debates as to which LOLT is best in the teaching and learning of Mathematics and other subjects at school level. Preference by both parents and learners of English as the LOLT, as driven by various factors from within and outside school settings (Probyn, 2009) points to the conclusion that English is likely to remain enjoying official status as the LOLT at upper primary school, secondary schools and in teacher education institutions. As such, teachers at primary and secondary levels have developed coping techniques and strategies in order for them to remain functional and reach their intended teaching and learning goals and objectives. Such strategies include code switching into the learners’ home language. The aim of this paper is to explore and understand teacher code switching consistency and precision in a multilingual mathematics classroom.

CLASSROOM CODE SWITCHING

Code switching occurs when a teacher substitutes a word or phrase in one language with a phrase or word in another language. It is defined as the use of more than one language in the same conversation or utterance (Adler, 2001; Choudhury & Bose, 2011). The perception that code-switching is a teaching and learning resource has had much attention in a range of mathematics education studies in and outside South Africa (Adler, 2001; Chitera, 2009; Halai & Karuku, 2013; Jegede, 2012; Khisty, 1995; Moschovich, 2007; Setati, 2005, 2008). It is considered a practice in which teachers and their pupils who are learning mathematics in a second language can harness their main language as a learning resource (Setati, Adler, Reed and Bapoo, 2001). While this study embraces the notion of code switching as a legitimate strategy employed by teachers, it however seeks to answer the following questions: How consistent is the teacher’s code switching in the mathematics classroom? Is the observed code-switched
terminology precisely communicating mathematical concepts and ideas during teaching? What are the best practises for teacher code switching in a multilingual mathematics classroom in South Africa?

THEORETICAL FRAMEWORK

The role of language used by the teacher during teaching and learning is particularly pertinent in multilingual South African schools such as those in rural and township environments. This is because the majority of students in these environments learn mathematics through the medium of English which in most cases is their second or third language. This study is informed by aspects of Vygotsky’s (1978) socio-cultural theory particularly with regard to the critical role of language in communication and cognitive development. Embracing a socio-cultural perspective makes it possible to view the language backgrounds of the teachers and pupils as a resource for teaching and learning mathematics (Moschkovich, 2007).

SAMPLE AND RESEARCH PROCESS

This study used a case study approach. Three Grade 11 mathematics teachers were purposefully selected from three secondary schools in the Eastern Cape Province, South Africa. Each teacher was observed for five consecutive lessons in a week teaching Trigonometry. The lessons were video recorded. At the end of each lesson, each teacher was interviewed following up on the lesson that the teacher had just taught.

The videos were transcribed and analysed for consistency in frequency of code switching into IsiXhosa across teachers, lesson categories developed from the works of Gumperz (1982) and Mercer (1995). Further, the data was analysed for consistency and precision across mathematical domains of practice as propounded by Dowling (1998). These domains are esoteric domain, descriptive domain, expressive domain and public domain. Two teacher code switching practises emerged and these were referred to as transparent code switching (TCS) and borrowing code switching (BCS). Data was further analysed for consistency using these two emergent themes.

FINDINGS

Results indicate that most of the classroom code switching was done during questioning and explaining. All the three teachers operated mainly in the public domain, implying that teachers predominantly taught in the everyday domain. This domain, according to Dowling (1998), is where not much formal mathematics is taught. The public domain does not provide students with the competency they require to effectively participate as members of the greater mathematics community (Dowling, 2010).
Results also revealed that borrowing code switching strategy was prevalently employed consistently across the participating teachers. Most of the mathematical talk in IsiXhosa was done through borrowing where teachers would attach prefixes to already existing English mathematical terms. Borrowing strategy was consistently practiced more than the transparent code switching strategy by all participating teachers throughout the teaching of trigonometry. Very little transparent code switching, which according to Meaney et al (2012), is critical for supporting students’ understanding, thinking and conceptual growth in mathematics, was evident in teacher language. In those few instances where transparency was eminent, it was mainly of those mathematical terms commonly used in the foundation and the intermediate phases. No grade 11 trigonometry terms in indigenous language were transparently code switched, all such terms were consistently code switched through borrowing. Conceptual understanding of mathematics and language practices that will aid thinking are key to achieving upswing teaching of the subject.

Based on these preliminary results it can be argued that because code switching is not being used optimally to enhance the conceptual understanding of Mathematics, performance in these schools is being compromised. The current code switching practice is not transparent and hence does not support student thinking, apart from the prefixes added, the words are still the same as in English and thus do not necessarily provide students with sufficient clues, hints or access to mathematical concepts.

**CONCLUSION**

With code switching occurring mainly in informal, everyday languages and school mathematics texts written in formal and more esoteric language, there is an urgent need to reduce this gap. This study argues that in order to introduce the formal mathematical talk in indigenous languages, resulting in orienting the teacher’s and students’ oral language towards esoteric mathematical practices (Dowling, 1998), requires the development and use of best practices and guidelines for code switching. Best practices in this paper mean all those efforts, strategies and initiatives that will narrow the gap between every day, informal and unspecialised teacher’s spoken language and the variety selected for academic purposes (Wildsmith-Cromarty, 2012).

Mathematics teachers in multilingual classrooms need to be encouraged to plan and think about how, when and where they would need to use pupils’ first language during teaching. Best practices will thus imply proper prior planning for code switching in the classroom as opposed to impromptu, inconsistent and ad hoc code switching which is a current phenomenon in many mathematics secondary school classrooms. Tanzanian High School teachers using KiSwahili observed by Mgqwashu (2004), were not using KiSwahili technical terms, but non-technical register. This did not give students access
to the concepts and vocabulary needed for understanding the subject. Systematic, regular and informed planning for code switching, where teachers refer to available multilingual resources for guidance, is regarded, in this study, as crucial and leads to code switching that is beneficial to students. Such switching is as a result of use of transparent terms in indigenous leading to code switching that is transparent.

With school mathematics texts written in formal language, and code switching practiced mainly in informal language, best practices would be those that aim to reduce this gap. To introduce formal mathematical talk in indigenous languages, resultantly orienting students’ oral language towards esoteric mathematical practices (Dowling, 1998), requires the development and use of mathematics registers in indigenous languages. Mathematics teachers in multilingual classrooms need to be made aware of, and encouraged to use available multilingual mathematics resources to aid their teaching and learning of Mathematics.

This study concludes that development of supporting mechanisms to encourage transparent, meaningful and beneficial code switching by mathematics teachers is required. I also propose that institution of best practices for code switching is required to guide and promote code switching that is precise, consistent, transparent and supportive of the conceptual understanding of strongly bounded sub-registers of Mathematics such as trigonometry in secondary schools.

REFERENCES


THE ROLE OF ADVANCED PROGRAMME MATHEMATICS IN BRIDGING THE GAP BETWEEN SCHOOL AND UNIVERSITY MATHEMATICS

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EXTENDED ABSTRACT

Advanced Programme Mathematics (APM) is an optional matriculation course in Mathematics, offered and assessed by the Independent Examination Board (IEB) in the final three years of high school in South Africa. This paper, based on du Plessis (2014) focuses on the transition from school to university Mathematics and the role of Advanced Programme Mathematics (APM) there in. At present, the South African school system does not adequately prepare learners for the transition from school to university Mathematics, and APM has been designed to address this gap. South Africa has many mathematically gifted and talented learners who obtain the required marks needed to gain access to university, but often lack the problem-solving, logical thinking, analyzing and critical thinking skills necessary for the rigour of first-year Mathematics. The research question guiding this study was: To what extent does the APM course succeed in preparing learners for the rigour of first-year Mathematics in the Science, Technology, Engineering, and Mathematics (STEM) university programmes? Very few studies have been conducted about the role of APM in South Africa, hence the contribution of this paper to that emerging discourse.

The target population (n=1199) included all first-year students who registered at a historically white university (HWU) for the Mathematics 114 (Calculus), Engineering Mathematics 115 (Introductory Differential and Integral Calculus), Mathematics 144 (Further Calculus and Algebra) and Engineering Mathematics 145 (Further Differential and Integral Calculus) modules during 2013. A convenient sample group of 441 learners was selected and the archival data of their achievement in the Grade 12 National Senior Certificate (NSC) Mathematics, NSC Mathematics Paper 3, APM examinations and the National Benchmark Test (NBT) for Mathematics were compared with performance in the first- and second-semester university Mathematics examinations.

The empirical investigation consisted of an analysis of the data to determine the difference between the performance of students who took APM, and those who did not, in terms of their achievement in the different Grade 12 examinations as well as their first year Mathematics examinations. A regression analysis was performed to determine the relationship between a) performance in NSC Mathematics and performance in first- and second-semesters of first year Mathematics b) performance in NBT in Mathematics.
and first-year performance in Mathematics. A General Regression model was used to
determine the significance of APM marks as a predictor of success in first-year
Mathematics when compared to the NBT and NSC Mathematics marks. The regression
analysis was triangulated with results from an online survey questionnaire, which was
distributed to all the first year university students who took APM in high school.
Students had to rate their experience of the transition from school to university, and the
extent to which APM course-taking helped them in handling large workloads, time
management, and to understand first year Mathematics. The larger study was designed
according to Plowright’s (2011) Framework for an Integrated Methodology (FraIM), a
version of mixed-methods approaches. Only quantitative results were considered for
this paper.

The results showed a statistically significant difference in first-semester university
Mathematics achievement between students who took APM and those who did not;
more so across the National Senior Certificate (NSC) Mathematics mark categories of
80-100%. However, this effect evened out in the second semester. The effect size
results for the APM Yes group showed that APM, NBT and NSC Mathematics marks
respectively explained 30%, 23 % and 40% of the variation in first semester
achievement but together they explained 68%. In the second semester the APM mark
was the strongest predictor explaining 37% of achievement while the NBT and NSC
Mathematics marks explained 34% and 13% respectively. The results support Ernest’s
(2002) view that those students who are more mathematically and epistemologically
empowered, by whatever curricula, will be better prepared and thus more likely to
perform better in achievement tests. Survey results also confirmed that APM course-
taking significantly eases the transition from school to university Mathematics by
boosting the students’ self-confidence in their ability to do Mathematics and building
strong domain knowledge of Calculus, thus epistemologically empowering them.
Research elsewhere (e.g. Stankov, 2014; Ackerman, Kanfer & Beier, 2013) confirms
that confidence and high school effort to acquire domain knowledge are the best
predictors of post-secondary success in Mathematics and STEM careers. Schools are
therefore called upon to provide access to APM, which is available free of charge, for
mathematically gifted students. Teachers and guidance counsellors should encourage
learners to enrol for APM which is a free exam.

REFERENCES


Ackerman, P. L., Kanfer, R., & Beier, M. E. (2013). Trait complex, cognitive ability,
and domain knowledge predictors of baccalaureate success, STEM persistence, and

WHAT CAN TEACHERS LEARN FROM COMPARING DIFFERENT STRATEGIES TO FIND THE GENERAL TERM OF QUADRATIC SEQUENCES

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Department of Mathematics and Applied Mathematics
University of the Western Cape

The aim in this paper is to give ideas to teachers on how meaning for symbols that students use in patterning activities can be developed which should result in a deeper understanding of number patterns. A further aim is to represent and analyse mathematical situations aimed at assisting students to move from specific numeric situations to develop general rules that model all situations of that type.

INTRODUCTION

Using patterns to promote and provoke generalization is seen by many as a pre-algebraic activity (e.g. Mason et al., 1985; Mason, 1996; Lee, 1996). The focus on pattern exploration is frequent in the recent approaches to the study of algebra. The use of symbols and variables that represent patterns and generalization are important components of the mathematics curriculum in many countries, including South Africa. There are now several studies about the analysis and development of pattern-finding strategies with students. For example, Stacey (1989) focused her investigation on the generalization of linear patterns with students aged 9-13 years old and reported that a significant number of students used an erroneous direct proportion method in an attempt to generalize; Orton & Orton (1999) focused their investigation on linear and quadratic patterns with 10-13 years old students. They reported a tendency to use differences between consecutive elements and its extension to quadratic patterns, by taking second differences, but without success in some cases. They also pointed to students’ arithmetical incompetence and their fixation on a recursive approach as some of the obstacles to successful generalization; Lee and Wheeler (1987) used generalizing problems (problems which can be solved by examining special cases, organizing the results systematically, finding a pattern and using it to get to the answer) involving quadratic and linear patterns in their study of generalization and algebraic thinking of Grade 10 students. They report very little evidence of students’ checking their patterns but caution that even in an interview it is often difficult to tell if checking has taken place. They conclude that “the reflex of checking the formula against the given [evidence] is not there” (pp. 112-113); Lannin (2003) examined the various strategies that students use as they attempt to generalize numeric situations and articulate corresponding justifications; Stacey (1989) found that students construct rules and generalizations too readily with an eye to simplicity rather than accuracy; Cooper and Sakane (1986) reported that once students select a rule for a pattern, they persist in their
claims even when finding a counter example to their hypothesis. Another fundamental difficulty is the lack of rigor and commitment to justifications that students demonstrate (e.g. Stacey (1989); Mason (1996); Lee (1996)).

The typical patterning activity provides a context and asks students to generate a rule or rules that could be used to determine other particular instances of the pattern. Researchers (Kenney, Zawojewski, & Silver, 1998; Stacey, 1989; Swafford & Langrall, 2000) have demonstrated that such activities encourage students to construct a variety of generalizations. However, many student generalizations represent faulty reasoning—incorrectly applying multiplication and ratio concepts (Stacey, 1989) or using a guess-and-check strategy to construct a generalization (Healy & Hoyles, 1999). The guess-and-check strategy can lead to what Mason (1996) described as “local tactics”, attempting to find a rule to fit a particular instance of the pattern rather than understanding a general relation in the problem situation. Generalization cannot be separated from justification. Considerable evidence exists that establishing the validity of a general statement is a challenging task for students (Carpenter & Levi, 1999; Chazan, 1993; Martin & Harel, 1989) with many students relying on empirical validation to support general statements (Mason, 1996; Sowder & Harel, 1998). When justifying an algebraic model, an argument is viewed as acceptable when it connects the generalization to a general relation that exists in the problem context.

**BACKGROUND**

The National Curriculum Statement (NCS) in its Curriculum and Assessment Policy Statement (CAPS: Department of Basic Education, 2011) describes Number Patterns for Grade 11 (taught in the first term) as follows:

Investigate number patterns leading to those where there is a constant second difference between consecutive terms, and the general term is therefore quadratic.

A typical examination question on number patterns of this type is the following [Question3, DBE/November 2010]:

The sequence $4 ; 9 ; x ; 37 ; \ldots$ is a quadratic sequence.

(i) Calculate $x$.

(ii) Hence, or otherwise, determine the $n^{th}$ term of the sequence.

The idea for this article was prompted by a curiosity of how learners would answer this question. It would especially be interesting to see how learners answer the second part without the answer to the first part. Based on the comments of a colleague, an experienced high school mathematics teacher, a few solution strategies for (i) and (ii) above are offered that are perhaps less familiar.
By solving the linear equation \((x - 9) - 5 = (37 - x) - (x - 9)\), we get \(x = 20\). Alternatively, \(x\) can be found by using the fact that the first difference terms form an arithmetic sequence of the form \(5 ; 5 + a ; 5 + 2a ; \ldots\), where \(a\) is the common difference. Hence the sum of the first three terms is \(5 + x - 9 + 37 - x = 5 + 5 + a + 5 + 2a\); giving \(33 = 15 + 3a\) and so \(a = 6\). By substituting \(a = 6\) into the linear equation \(x - 9 = 5 + a\), we get \(x = 20\). One can also easily find \(x\) through trial and improvement.

We show that the general term \(T_n\) of the sequence is quadratic in the variable \(n\), that is, \(T_n = an^2 + bn + c\), for some parameters \(a\), \(b\), and \(c\). The first difference terms of the sequence \(4 ; 9 ; 20 ; 37 ; \ldots\) are \(T_2 - T_1 = 5; T_3 - T_2 = 11; T_4 - T_3 = 17; \ldots\) Hence \(T_2 = T_1 + 5; T_3 = T_2 + 11 = T_1 + 5 + 11; T_4 = T_3 + 17 = T_1 + 5 + 11 + 17; \ldots\);

\[
T_{k+1} = T_1 + \frac{k}{2} (2(5) + (k - 1)(6)) = T_1 + 2k + 3k^2 = 4 + 2k + 3k^2.
\]

By substituting \(k = n - 1\) into this formula for the \((k+1)\)th term we get \(T_n = 3n^2 - 4n + 5, n \geq 1\).

It is clear that students learn very little new mathematics from answering the questions in this problem. It will be shown in this article that “in context” problems involving number patterns can provide much more mathematical insight.

**THEORETICAL FRAMEWORK**

According to Kaput (1999) algebraic thinking consists of: (a) the use of arithmetic as a domain for expressing and formalizing generalizations; (b) generalizing numerical patterns to describe functional relationships; (c) modelling as a domain for expressing and formalizing generalizations; and (d) generalizing about mathematical systems abstracted from computations and relations. The strong bonds between generalisation and mathematics (especially algebra) are quoted by numerous other researchers. Lee (1996) states that:

… it is possible to make a case for introducing algebra through functions, and through modelling, and through problem solving, quite as honestly as it is to make the case that generalizing activities are the only way to initiate students into the algebraic culture. (p. 102, our emphasis)

The various processes involved in generalization have been identified by a number of researchers; Rivera and Becker (2008) in their literature review state that the initial stages in generalization involve: focusing on (or drawing attention to) a possible invariant property or relationship, ‘grasping’ a commonality or regularity and becoming aware of one’s own actions in relation to the phenomenon undergoing generalization. Mason (1996) offers an interpretive overview of these phases by seeing them as forming a spiralling helix, which contains:
manipulation (whether of physical, mental, or symbolic objects) provides the basis for getting a sense of patterns, relationships, generalities, and so on;

the struggle to bring these to articulation is an on-going one, and that as articulation develops, sense-of also changes;

as you become articulate, your relationship with the ideas changes; you experience an actual shift in the way you see things, that is, a shift in the form and structure of your attention; what was previously abstract becomes increasingly, confidently manipulable (pp. 81-82).

One of the basic tools that one has during generalizing is visualization, i.e. a “process by which mental representations come into being” (Dreyfus, 1991b, p. 31); however, its use is not unproblematic for students who may be likely to create visual images but are unlikely to use them for analytical reasoning (Dreyfus, 1991a). Generally, from the point of view of students, coming to think algebraically is not an easy process. The ‘shift of attention’ mentioned by Mason (1996) is the activity that differentiates the professional mathematician from the novice. Thus, the transition from arithmetic to algebra is a challenging aim for teachers in the last classes of primary school and the first of secondary school; and ‘early algebra’ is now a commonly used term (Lins & Kaput, 2004), signifying the assumption that the initiation to algebraic thinking may start in primary school:

… experiences in building and expressing mathematical generalizations – for us, the heart of algebra and algebraic thinking – should be a seamless process that begins at the start of formal schooling, not content for later grades for which elementary school children are “made ready” (Blanton & Kaput, 2005, p. 35).

In order to clarify the teachers’ role in that initiation, and its consequent implications for teachers’ education, we could adopt a situated view of learning (Lave and Wenger, 1991), in which learning is seen as changing participation and formation of identities within relevant communities of practice. To put it simply, teachers should be initiated into the practices that they will initiate their students (Borko et al., 2005). Additionally, we are in line with Cobb (1994) who stresses that learning “should be viewed as both a process of active individual construction and a process of enculturation” (p. 13). In other words, we do not want to ignore the importance of engaging students in activities that are expected to promote the construction of meaningful knowledge.

A typical patterning activity

Problems such as the one given as an examination problem mentioned in the background section, contributes very little to the development of students’
mathematical insight as the focus is too much on manipulation, that is, applying rules, rather than on understanding the meaning of the symbols used. This can be addressed by selecting problems that can be solved by the investigation of patterns in the problem situation as will be illustrated next.

Consider the following problem.

In any (convex) polygon with \( n \) sides, call a line connecting a vertex to all other vertices, excluding the two adjacent vertices, a diagonal in the polygon. How many diagonals can be drawn in this polygon?

We begin to discuss different solution strategies for this problem, starting with the counting strategy.

**Counting Strategy**

This strategy describes the level of problem solving at which every student should begin, that is, by drawing a (convex) polygon \( P_n \) with \( n \) sides and by counting the number \( D_n \) of diagonals in \( P_n \).

1. Initial problem: What is the number of diagonals in a (convex) hexagon?

Figure 1. Diagonals in a hexagon.
All students draw a hexagon and count the diagonals as in Figure 1. Students can colour the diagonals while counting.
Answer: 9 diagonals

2. Count the number of diagonals in pentagons, quadrilaterals and triangles.
Answer: In triangles the number of diagonals = 0; in quadrilaterals the number of diagonals = 2; in pentagons the number of diagonals = 5.

3. Count the number of diagonals in heptagons and octagons.
Answer: All students count 14 and 20 diagonals, respectively.

4. Consider the answers given in 1-3 and answer the following generalized question: Is there a connection between the number of sides and the number of diagonals?

Answer: Diagonals connect vertices and therefore the number of sides is important.

Collect all results in Table 1 below.

<table>
<thead>
<tr>
<th>Polygon (P_n)</th>
<th>Number of sides (n)</th>
<th>Number of vertices</th>
<th>Diagonals (D_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Quadrilateral</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Pentagon</td>
<td>5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Hexagon</td>
<td>6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Heptagon</td>
<td>7</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Octagon</td>
<td>8</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

Table 1

6. Is there a connection between the number of vertices and the number of diagonals?

Answer: To answer this question, students must be guided to look for patterns in the table which will provide a systematic way to express the relationship between the numbers of sides, vertices and diagonals.

<table>
<thead>
<tr>
<th>Number of sides in P_n (n \geq 3)</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of diagonals D_n in P_n</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>9</td>
<td>14</td>
<td>20</td>
</tr>
<tr>
<td>First Difference</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Second Difference</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is clear from the above table that diagonals D_n form a number pattern in which there is a constant second difference between consecutive terms. As was explained in the background section, the general term of such a number pattern is quadratic, that is, \(D_n = an^2 + bn + c\), \(n \geq 3\) and it is shown, in the same way as earlier, that

\[D_n = \frac{1}{2} n^2 - \frac{3}{2} n = \frac{n}{2} (n - 3), \quad n \geq 3.\]

Even though this formula gives the correct number of diagonals for different polygons, it supplies very little insight into the problem situation. For example, a question can be posed to a student “Why are you multiplying \(n\) by \(n - 3\) and dividing by 2?” or “Can you construct a polygon having 40 diagonals?” Such questions will help students to
achieve a deeper understanding of the generalization. Where students produce a rule, teachers must require students to explain each component of their rules. Students tend to provide rules without any explanation of what the various parts of their rules represent in relation to the context of the situation (Lannin, 2003).

Teachers must encourage students to move beyond counting by asking questions such as “Drawing and counting would be difficult for a polygon with 30 sides. Can you use what you know about polygons with fewer sides to find a way to calculate the number of diagonals in a polygon with 30 sides?”

In this section algebraic symbols were introduced such as $P_n$ representing a polygon with $n$ sides and $D_n$ denotes the number of diagonals in $P_n$.

The aim in this paper is to give ideas to teachers on how meaning for symbols that students use in patterning activities can be developed which should result in a deeper understanding of number patterns. A further aim is to represent and analyse mathematical situations aimed at assisting students to move from specific numeric situations to develop general rules that model all situations of that type.

In the next two sections we look at strategies that will help students to make connections to a general relationship that exists for all polygons of sides 3 or more.

**Contextual Strategy**

In this strategy a rule is constructed based on information provided in the problem situation; relating the rule to a counting technique.

Consider first a polygon with 20 sides. Each vertex is connected to 19 other vertices by lines which are either sides of the polygon or diagonals. The connections to the two adjacent vertices are sides of the polygon. Thus the connections to the remaining 17 vertices are diagonals. Since there are 20 vertices, the total number of diagonals is 20 times $17 = 340$. But a polygon with 20 sides has 20 corners and there are 17 diagonals starting at each corner. Since each diagonal starts at two corners, it gets counted twice while we count corner by corner. Therefore there are $\frac{340}{2}$ diagonals.

Can we now generalize this method of counting the number of diagonals to any $n$-gon?

Notice that each vertex is connected to $n - 1$ other vertices, including the connections to adjacent vertices, which are the sides of the polygon. Therefore each vertex is connected to $(n - 1) - 2 = n - 3$ other vertices by diagonals, that is, only $n - 3$ diagonals start at each vertex.
Conjecture: There are $n$ vertices; so there are $n(n - 3)$ diagonals.

Let us test this formula for a few values of $n$.
For $n = 3$, the number of diagonals is equal to 0, which is correct.
For $n = 4$, the number of diagonals is equal to 4.
For $n = 5$, the number of diagonals is equal to 10.
For $n = 6$, the number of diagonals is equal to 18.
Notice that the number of diagonals given by the formula is an overestimation of the number that we got by counting. What could be the reason for this? Look at the case for $n = 20$.

Answer: The reason is that each diagonal is counted for two vertices; hence there is duplication in the counting process.

Can you now formulate a general statement of the number of diagonals?

Any $n$-gon has $\frac{n(n-3)}{2}$, $n \geq 3$ diagonals.

Test this formula for $n = 3$; $n = 4$; $n = 5$; etc
Calculate the number of diagonals in a nonagon and a decagon.
Can you justify the formula for the number of diagonals?

Answer: Each $n$-gon has $n$ corners and there are $(n-3)$ diagonals starting at each corner. Since each diagonal starts at two corners, it gets counted twice while we count corner by corner. Therefore there are $n(n-3)/2$ diagonals.

This generalization is one of many such explicit rules that could be constructed for this situation. For example, consider the polygon $P_n$ with $n$ vertices. Moving around the polygon in an anti-clockwise fashion and numbering the vertices 1, 2, 3, etc. and counting the number of diagonals (without duplication). Remember the sides of the polygon connect adjacent vertices and are not diagonals. For example, for $n = 6$, vertex 1 is connected to 3 other vertices (excluding adjacent vertices) giving 3 diagonals; vertex 2 is connected to 3 other vertices (excluding adjacent vertices) giving 3 diagonals. Why? The reason for this is that the two adjacent vertices 1 and 2 do not have any diagonals in common. Vertex 3 is connected to 2 other vertices (excluding adjacent vertices) giving 2 diagonals. Vertex 4 is connected to 1 other vertex (excluding adjacent vertices) giving 1 diagonal. Therefore the total number of diagonals $D_6 = 3 + 3 + 2 + 1 = 9$. In the same way, $D_7 = 4 + 4 + 3 + 2 + 1$, $D_8 = 5 + 5 + 4 + 3 + 2 + 1$.

In general,

$D_n = (n-3) + (n-3) + (n-4) + (n-5) + \ldots + [n-(n-2)] + [n-(n-1)]$

$= (n-3) + (n-3) + (n-4) + (n-5) + \ldots + 2 + 1$

$= (n-3) + \frac{n-3}{2} [2 + ((n-3) - 1)]$
\[(n-3) + \frac{n-3}{2} (n - 2)\]

\[= \frac{n-3}{2} (2 + n - 2)\]

\[= \frac{n-3}{2} (n), \quad n \geq 3.\]

This rule for the number of diagonals in a polygon with \(n\) vertices is exactly the same as the one obtained earlier but it is now much clearer what the components of the rule mean. In the next section we discuss the recursive strategy for counting the diagonals.

**Recursive Strategy**

A student using a recursive strategy has constructed a relationship for finding the number of diagonals of a polygon with a given number of sides from a polygon with one side less than the desired polygon.

For example, a polygon with 3 sides has no diagonals.

\[D_3 = 0\]

For a polygon having 3 sides, pick one point outside the triangle.
Join this point to all other vertices. The line connecting the vertices 1 and 3 now becomes a diagonal. Thus, adding an additional point result in 2 diagonals. Therefore $D_4 = 0 + 1 + 1 = 2$. 
For a polygon having 4 sides, pick one point outside the quadrilateral.

Join this point to all other vertices. The line connecting the vertices 1 and 4 now becomes a diagonal. Thus, adding an additional point result in 5 diagonals.
For a polygon having 5 sides, pick one point outside the pentagon.

\[ D_5 = 5 \]
Join this point to all other vertices. The line connecting the vertices 1 and 2 now becomes a diagonal. Thus, adding an additional point result in 9 diagonals.

$$D_6 = 9$$

By continuing in this way, we arrive at

\[
D_5 = D_4 + 2 + 1 = 5 \\
D_6 = D_5 + 3 + 1 = 9 \\
D_7 = D_6 + 4 + 1 = 14 \\
D_k = D_{k-1} + (k - 3) + 1 = D_{k-1} + k - 2 \\
D_k = 2 + 3 + 4 + \ldots + k - 2 \\
= \frac{k-3}{2} [4 + ((k - 3) - 1).1] = \frac{k-3}{2} (k), \ k \geq 4.
\]

This rule for the number of diagonals in a polygon with \( n \) vertices is exactly as before, but it is now much clearer what the components of the rule mean.
The final strategy offers no connection to the context or the number sequence generated for the number of diagonals in an $n$-gon. A student who uses this strategy might state, “I tried a few rules and came up with multiply $n$ by $n−3$ and divide by 2”. Although the rule is correct for this situation, it provides no insight into the relationship between the rule and the context and, therefore, is difficult to justify.

**Conclusion**

Problems such as the one given as an examination problem mentioned in the background section, contributes very little to the development of students’ mathematical insight as the focus is too much on manipulation, that is, applying rules, rather than on understanding the meaning of the symbols used. It is recommended that teachers should focus on patterning activities that provide a context such as the problem of finding the number of diagonals in an $n$-gon that was discussed in this article. We showed that both the contextual and recursive strategies link the rule to a general relationship that exists in the problem situation. Teachers must then help students to recognize the importance of linking their rules to the context of the situation (Lannin, 2003). Another issue that arises is that students do not have the experience of justifying statements and often resort to justification through the use of examples (Lannin, 2003). According to Hoyles (1997) a common error that occurs throughout Grades K – 12 is that students try to justify a general situation by demonstrating that the rule results in the correct values for a few individual cases. It is important that teachers should emphasize the need to adequately justify all generalizations. This is most often done using proof by induction. Since the contextual and recursive strategies are so intimately linked to the general relationship that exists in the problem situation, they qualify as acceptable justification for the rule arrived at (Lannin, 2003).

Generalizing numeric situations can create strong connections between the mathematical content strands of number and operation and algebra as well as other content strands such as Graph Theory (the polygon with $n$ sides together with its diagonals form a complete graph) as was demonstrated in this article.

**REFERENCES**


 APPLICATION OF A PSYCHO-PRAGMATIC APPROACH TO THE TEACHING AND LEARNING OF MATHEMATICS.

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It was Richard Skemp who, back in the 1970s, introduced the idea of “the psychology of learning mathematics” while in the 1990s Freudenthal supported the notion of ‘mathematising’ instead of ‘learning mathematics’. In both theories ‘teaching and learning’ are viewed as happening simultaneously even when one ‘teaches his/herself’. Cognitive Psychology and Behaviourism and their variations, have also contributed to the improvement of the teaching-learning situation with some success. Then Giannakopoulos (2012) introduced the “psycho-pragmatic” approach, which is a combination of cognitivism, behaviourism (psychological) and pragmatism (pragmatic).

This article makes use of the psycho-pragmatic approach to apply it in the context of teaching and learning of mathematical concepts and problem solving.

The Psycho-Pragmatic Approach To Teaching And Learning Of Mathematics

This paper is based on a new teaching-learning paradigm, the psycho-pragmatic approach, to improve the understanding of mathematics at any level in education. The psycho-pragmatic approach was introduced by Giannakopoulos (2012) and it recognizes the importance of the psychological aspect of the human being with respect to learning but also its relationship to the outer reality. This paradigm requires that learning is approached in a bottom-up manner (see, Figure 1). This means that first we identify the knowledge to be acquired and relate that to various principles involved and identify the necessary concepts. The Psycho-pragmatic approach involves around the acquisition of knowledge and skills (cognitive, such as problem solving and critical thinking, and physical), which are the products of learning (see Figure 1). Many questions arise when one tries to answer the questions ‘how do human beings learn?’ or ‘how do human beings think and act?’, ‘what skills are necessary?’, especially at a higher learning institution.

Skills include trade or craft skills, professional skills, social and sporting skills. Skills become better through practice and training.
Figure 1: The Act of Learning (Adapted from Giannakopoulos, 2012:9)

Act of Learning

Thinking (theory) → Doing (practice)

- **Step 1**: Abstracting
- **Step 2**: Concept formation
- **Step 3**: Principles (concepts + relationships)
- **Step 4**: Structures of knowledge
- **Step 5**: Formation of skills
- **Step 6**: Knowledge + Formation of skills
- **Step 7**: Problem solving

LEVEL III

Steps 1-7
Many researchers (Barrows, 1986; Bensley, 1993; Lombard & Grosser, 2004; Paul, 1993; Sternberg & Ben-Zeev, 2001; Tiwari, Lai, So & Yuen, 2006) agree that cognitive skills such as critical thinking and problem solving can be “taught”, although they differ on “how”, while others (e.g. Brookfield, 1987) disagree. Giannakopoulos (2012) adopts the idea of Pascarella (1999), Feldman and Newcomb (1969, cited in Gellin, 2003) and Gellin (2003) who prefer to use the word “enhanced” rather than “taught”. The reason being that, at a tertiary institution, students already possess cognitive skills to a greater or a lesser extent. It is the institution’s duty to enhance such skills intentionally and systematically.

While human beings have limited experience at the early stages of education when the learner arrives at a higher learning institution on the one hand many concepts and misconceptions have already been formed on the other hand the learner also has general knowledge from experience. The conceptual framework (Figure 1) is based mainly on a cognitive approach to learning with behaviourism playing a complementary and necessary role. The pragmatic approach deals with the application of theory: the doing. This practical and useful way of knowledge does not reject the psychological or philosophical findings, on the contrary they form the basis of praxis. Such view is necessary especially in vocational education where the content to be learned will dictate the teaching approach to be used (Hammer, 1997, cited in Reddy, Ankiewich & de Swardt, 2005).

The psycho-pragmatic approach is illustrated in ‘The Act of learning’ (see, Figure 1), which implies the existence of thinking and doing (thinking while we are doing and doing while we are thinking). It is act(ive) learning. It is praxis. Translating theory into practice and reflecting on practice we develop theory. It promotes conceptual understanding as knowledge in the end is a synthesis of concepts and their relationships.
and principles. The psycho-pragmatic approach is discussed in detail later in this paper as it is applied in the learning of mathematics.

**Application of the psycho-pragmatic approach to the teaching and learning of mathematics**

It cannot be disputed that the teaching–learning situation of any domain of knowledge is a complex phenomenon and studying this phenomenon requires a multitude of synthesized research methods. When the ‘Act of Learning’ is involved Giannakopoulos (2012) established that approaching the teaching and learning of mathematics from a psycho-pragmatic perspective using a problem based approach a) critical thinking plays a central role; b) The teacher has to be diagnosing rather than ‘teaching’ (a diagnostic teacher as defined by Solomon (1999) and is explained later in the paper); and c) problem solving must be preceded by diagnosing.

In any teaching and learning situation (see, Figure 2) there is the teacher, the learner and the subject content in form of information that needs to be converted to knowledge and become part of the cognitive structure of the learner. This conversion is taking place in a certain cultural environment and within a certain context. Learning is therefore situated (Lave & Wenger, 1991). Many factors, personal (learner and teacher) and environmental (social, education environment) contribute to the successful or unsuccessful conversion of such endeavor. Research has identified numerous factors and teaching strategies and modification of the environment had been developed to minimize the negative effect of such factors.

Figure 2: The teaching-learning situation
However two indisputable problems, especially in higher education, still exist: In general, low graduation rates and inability by many graduates to solve real world problems. In particular, learning of mathematics at any level appears to be elusive by the majority of learners. Many authors (Astin, 1993; Pascarella, 1999; Tinto, 1993; Tiwari et al. 2006) agree that there is a discontinuity between the classroom setup and the real world. The reasons could lie in the teaching styles (Sterenberg & Ben-Zeev, 2001), promotion of mathematics as an abstract subject (Rodd, 2002) and thus not connected to the real world.

Mathematics is a paradoxical subject. On the one hand it is characterized by absolutism (right/ wrong) on the other hand it lends itself to “abuse” of mathematics by learners when they write non-sensical answers (e.g. $2x+x = 2x^2$) or “creative thinking” by learners who due to misconceptions “create their own mathematics”. Any experienced mathematics teacher when analyzing misconceptions of learners of a specific concept year after year encounters new misconceptions by the new learners. Skemp (1977) having recognized this went as suggested “the psychology of learning mathematics”, psychology developed especially for the learning of mathematics. Giannakopoulos (2012) went as far as to say that mathematics is a problem on its own and before it is
used to solve problem one must solve the mathematic problem which is to understand
the nature of mathematics.

Learning and understanding of mathematics in the teaching-learning situation is of complex
to due the variability of the various components. However consistency can be used to overcome this problem. For example the environment can be fixed, the teacher can define his or her style, method, and pedagogy, the content is predetermined, and the learner’s cognitive development, style and personality can be defined too. The teacher’s role though has to be one of what Solomon (1999) called a diagnostic teacher, which is discussed briefly later; the teacher to be ‘diagnosing’ rather that ‘teaching’.

Presently teachers try to achieve this by using “word problems” in order to apply the theory into practice. However, research (Zevenbergen,2001) has shown introducing word problems creates a new set of problems: Misreading the problem, misunderstanding or not understanding the problem, lack of problem solving skills, problem modeling and so on. Zevenbergen (2001) added ‘lexical density’ (number of lexical [significant] words divided by the number of clauses). Furthermore, word problems are not necessarily connected to real world problems. As an example, it is assumed that if somebody can solve simultaneous equations such as if \( x + y = 5 \) and \( 2x + 3y = 13 \) will be able to solve the problem of Type A:

\[
\begin{align*}
\text{The sum of two numbers is } 5. & \quad \text{If the sum of twice one number and three times the other number is } 13 \text{ determine the two numbers.} \\
\text{OR} \\
\text{If the sum of the ages of two young people is } 5 \text{ and twice the age of the younger of the two and three times the age of the older add up to } 13 \text{ determine their ages.}
\end{align*}
\]

These are just two typical word problems which are used to apply the knowledge gained in solving simultaneous equations. The processes involved to solve the word problem and the symbolic representation of the problem differ radically. Alternatively word problems can be constructed using problems from other subjects that the learner has completed or busy with them. For example, using chemistry, Type B:

\[
\begin{align*}
\text{A solution weighs } 5 \text{kg and contains substance A and B. When the mass of A is doubled and the mass of B is tripled the solution weighs } 13 \text{kg. Determine the original weight of the substances.} \\
\text{OR} \\
\text{A solution weighs } 5 \text{kg and contains substance A and B. If the mass of substance A needs to be doubled how much of substance B should be used if the total mass of the solution needs to be } 13 \text{kg if it is known that the mole ratio of substance A to B is } 2:3?
\end{align*}
\]
This second type of word problem is closer to reality as a problem like this can occur in chemistry.

However pragmatism requires that application should involve real world problems (Type C) which are unstructured and could contain different disciplines such as designing a steel bridge. It requires the use of strength of materials and differential equations. This is where a new problem could arise since: a) most of the world problems are unstructured, and b) it is difficult to find real world problems that fit with abstract mathematical symbols. In this particular case for a teacher to find real world problems that fit this case, he or she must be well versed in many disciplines and have a very good general knowledge.

What is suggested by “the Act of Learning” is that application should contain all three types of problems: a) Structured but “homemade” problems (like Type A); b) Structured but connected to other subjects (Type B), and c) Unstructured, real life problems, but analogous to the previous types (Type C).

Before the teaching-learning situation is discussed it is necessary to discuss the educational foundation of the ‘Act of learning’. For simplicity’s sake the ‘Act of Learning’ is of cyclical nature as it was stated earlier as it goes through various levels of knowledge acquisition. The levels of Van Hiele (1986) and Bloom (1979) are used to illustrate the learning of any concept and here mathematical concepts and subsequent acquisition of mathematical knowledge. Van Hiele’s levels and Bloom’s levels are superimposed.

Van Hiele

Level 1: The visual level  Knowledge (information)

Level 2: The descriptive level (concrete semi abstract) Comprehension and application

Level 3: The theoretical level  Analysis

Level 4: The formal logic level  Synthesis

Level 5: The nature of logical laws (abstract).  Evaluation
For Van Hiele (1986: 57) to move from level to level there are certain intermediate stages that can (should) be followed:

Stage 1 The information stage

Stage 2 Guided stage

Stage 3 Explicitation stage

Stage 4 Free orientation stage

Stage 5 Integration stage

These stages could be simplified using the “Act of Learning”, whereby the implicit has to be made explicit through thinking, abstracting, synthesising and concept formation and integrating them in the existing cognitive structure.

Skemp (1977: 20) reduced these levels into two: In level I the learner uses the concrete, the experiential and through abstracting, “an activity by which we become aware of similarities among our experiences”, concepts are formed. An abstraction is some kind of lasting image. Through further abstracting on the concepts level II is attained.

From Figure 2 the four parts, environment, learner, teacher and content are now discussed in conjunction with the ‘Act of Learning’ and the levels of knowledge acquisition. A concept of partial derivatives is used to demonstrate the ‘Act of Learning’ where conical flasks of certain volume (V), radius (r) and depth (h) are filled with liquid running out at a certain rate (see Figure 3). The volume \( V = \frac{1}{3} \pi r^2 h \).

![Figure 3: Partial derivatives example](image-url)
The learner

Level 1 (Figure 1, cycle 1) is related to the first encounter with the new concept. From a psychological perspective, the learner “thinks” about the concept in an intentional as well as subconscious manner (psychological aspect). The learner becomes actively involved by relating the concept to a real life specific situation (pragmatic aspect). Through this experience the learner begins to develop tacit knowledge and takes ownership of such knowledge. The encountered concept is perceived to be abstract or concrete. Initially the learner uses low order thinking as no insight has been achieved. By connecting it to a real life situation the learner “concretizes” the abstractness through higher level of thinking and begins to gain insight. Here the learner identifies the variables, tabulates the results and tries to make predictions as to what will happen to the volume if one of the independent variables $r$ or $h$ are kept constant.

The concept could be a primary concept or a secondary concept. The learner has to recognize which type it is and act on it accordingly. Secondary concepts are byproducts of primary concrete or abstract concepts and they are more difficult to learn. At this point the learner could use existing cognitive structures to assimilate the new concept (conceptualization of concept/abstracting/reflecting) and could attempt to apply it in a domain specific situation or in a general situation. As the concept is assimilated and applied it forms part of the cognitive structure but at a low level. At this point learning of the concept could be considered to be “surface learning” or as Skemp (1977) puts it the learner learned the “name” of the concept. The concept is still in a fluid state as the learner is not certain about its meaning. The concept partial derivative is still abstract but the learner notices the changes in the volume as $r$ or $h$ changes.

Once the first cycle has been completed Level 2 begins. The learner now has a fair idea as to what the concept is about. The visual (concrete) is followed by the ability to describe what the concept is. Again through thinking (abstracting/reflecting) and applying the concept in diverse situations a better understanding is attained. The learner begins to feel confident and becomes more consistent in the application of the concept. At this point restructuring of the cognitive structure is taking place. From a pragmatic perspective the learner seeing the utility of the concept may go as far as to intentionally restructure existing knowledge which could be contrary to the newly acquired concept. Now the learner begins to understand the abstract concept by establishing that there is a certain relationship between $V$ and $r$ and $h$. Further discussions about the concept take place.

In Level 3 the learner does not any longer need the concrete presentation but through further abstractions if the concept was perceived to be of abstract nature it begins to “solidify” in the cognitive structure and become easily accessible. This process
Giannakopoulos (1992) called it ‘concretisation of the abstract’. The learner now can differentiate between various occurrences and contexts of the concept. A degree of automatism of the usage of the concept has also been achieved. Now the learner does not need the concrete example to do calculations as partial derivative has been accommodated in the cognitive structure.

Level 4 is characterized by logic. The learner operates mostly on the abstract thinking level and is capable of synthesizing the concept with other related concepts. Everything begins to make sense. The learner can argue about the concept as he or she has become more confident at the theoretical level. At this stage the symbols $\frac{\partial V}{\partial r}$ and $\frac{\partial V}{\partial h}$ are introduced and contrasted to the related concepts $\frac{dV}{dr}$ and $\frac{dV}{dh}$.

Finally in Level 5 the learner can now operate in the abstract level as he or she can evaluate situations that the concepts involved and support his or her understanding with evidence. The learner can confidently say “I know”, and knowledge has been acquired as it has been at the highest thinking level. The learner can perform now symbolic partial differentiation and knowing the difference of partial derivatives to total derivatives.

Once the concept has been achieved in this level (cycle 5) the learner can use the concept in diverse but analogous situations which could be domain specific or general. This is achieved through conceptualization. Conceptualisation could be seen as the addition of “something” by the individual. The ability to conceptualise leads to the ability to think critically, a prerequisite to problem solving. The learner now has the choice of applying his knowledge in concrete situations (move to the start – doing) or theorise further (move to – thinking) or begin with a new concept.

*The teacher*

The role of the teacher in a psycho-pragmatic approach has to be well defined. The teacher has to intentionally modify his or her total behaviour to be able to implement such approach. From a pragmatic perspective the teacher is always faced with a non-ideal situation: on the one hand his or her success is measured in pass rate in his or her subject on the other hand the teacher has to promote “deep learning” which aims at developing in the learner life skills which enable him or her to cope with the ever changing world.

Trying to enhance critical thinking and teach problem solving skills, although necessary for general knowledge it might not make a difference to the pass rate. As the diagram (Figure 2) shows the teaching style and method must be adjusted accordingly. The approach requires the teacher to use an appropriate method depending on the new
concept to be taught. For example, certain concepts require the traditional method. Others could require a constructivist approach and so on. When it comes to problem solving Giannakopoulos (2012: 277) suggested a problem solving model which is a holistic model to problem solving.

Furthermore, the teacher has to become a diagnostic teacher, as stated earlier, according to Solomon (1999). She sees diagnostic teaching one of:

- Determining thinking levels;
- Diagnosing = medical diagnosis. What is wrong or right with the learner? What are the misconceptions and how to correct them?;
- Diagnostic teaching shifts the emphasis from teachers’ practices to learners outcomes, understandings or misconceptions; and
- The diagnostic teaching then demands from the teacher to take the different types of knowledge into account and determine which type/s of knowledge the learner is lacking or rather which type/s the learner possesses and how to enhance the various existing skills for further knowledge acquisition.

Diagnosing is often confused with testing what learners already know; a kind of confirming their pre-knowledge. However, diagnosing should be understood in the same context as the medical interpretation of the term. The doctor determines what is wrong with one. For example if a learner cannot perform certain mathematical operations, the teacher must determine “why”, “what is wrong with the learner”. But to make a correct diagnosis the teacher should be conversant with more than just the content of his/her subject.

Solomon (1999: xvi) describes a diagnostic teacher as one “who casts oneself as an observer, scrutiniser, and assessor, as well as an engaged teacher.” The author further states that a diagnostic teacher should develop assessment capacities in three domains: (1) seek to know students’ current understandings and misconceptions; (2) deepen their own subject area knowledge and choose what is worth teaching; and (3) assess their own beliefs and practices, selecting, designing, and redesigning appropriate pedagogical strategies (Solomon, 1999). What should be added here with respect to knowledge possessed by the student, is that the teacher should not assume that the students only possess pre-knowledge (with respect to content previously taught) but also other cognitive abilities and content beyond the one that was taught.

The traditional idea that learners are like “empty containers” that need to be filled by the teacher should not feature in any teacher’s repertoire. The learner should be viewed as a capable being, that (s)he has what it takes, in collaboration with his/her peers and
teacher to keep conquering the unknown by constructing his/ her own meanings. In this case the learner occupies the central role in learning as an active participant and gains knowledge. Situated learning or constructivism is based on this premise. It is a learner centred approach in line with outcomes based education (OBE) (Curriculum 2005).

The environment

Here the environment has to be seen from a context perspective (primary, secondary, tertiary) as well as from a content perspective (the classroom where the new content is introduced). This study is primarily about tertiary, vocational context. Tinto (1993) and Astin (1993) highlighted the importance of ‘student fit’ in higher education. Everything that is involved with the institution could be detrimental to the student’s progression to graduation. Tinto (1993) went as far as to say that “it is the institution that fails the student” since if the student leaves before graduation (s)he did not get what they went there for: Graduate. Therefore environmental factors are crucial to the student’s progression and retention but also play a motivational role in learning.

Once this environment is conducive to attendance, then the classroom environment also needs to be conducive to the teaching and learning. If this is accepted then neither behaviourism or cognitivism (or constructivism) is an appropriate learning theory to be used in a vocational field. What is suggested here is that learning in the vocational context is approached from a psycho-pragmatic perspective, which is a combination of neocognitivism (for the theoretical aspect of learning) and pragmatism (for the practical aspect of learning).

CONCLUSION

The teaching-learning situation is a complex phenomenon and attempts have been made by educationalists, psychologists and scientists in general to understand that phenomenon. Behavioural and Cognitive psychologies have made great contributions in understanding the education phenomenon and improve the teaching-learning situation. However what they lack of is pragmatism. Combining the two with pragmatism gives rise to the psycho-pragmatic approach and pragmatism brings the two former theories into life through practice. Pragmatism brings mathematics to life as it connects it to reality.

Applying the psycho-pragmatic approach to the learning of mathematics benefits the teacher and the learner as there is a continuous move from the real to the abstract and vice versa. It is true that the approach is learner centred which places a great responsibility on the teacher’s shoulders. But at the same time can be very rewarding.
as the teacher will be a learner by himself/herself till the end of their lives and thus they will always learn new things thus they won’t get bored and will enjoy teaching.

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Mathematics by its nature is abstract from the minute a learner encounters numbers (e.g. 2 is abstract for a 5 year old) and becomes even more abstract when that learner goes to university. The use of symbols to generalise concepts or model problems, makes Mathematics a very complex subject. As various abstract symbols are combined to form concepts the learner has to continuously, what Giannakopoulos (1992) called, ‘concretise’ in order to make sense of those symbols. One section in which university students have difficulty with is the differentiation of functions. This could possibly be due to students’ misconceptions of mathematical structures and a lack of critical thinking. This paper explores a new method in which differentiation (at first year level) could be taught and data from the preliminary results are more than encouraging.

The Nature Of Mathematics And Critical Thinking

The role that critical thinking plays in the learning of mathematics has been studied by a number of authors (Kwon et al. 2005; Pushkin, 2007; Hiebert, J., & Lefevre, 1986) and most recently by Giannakopoulos (2012). For example Giannakopoulos found that critical thinking is a prerequisite to mathematics knowledge acquisition and application and thus problem solving. The learning of mathematics, due to its abstract nature, necessitates an approach that promotes high level thinking from early stage of cognitive development.

The nature of mathematics

First, Mathematics can be considered as a very concise and precise subject allowing no room for multiple interpretations or use of redundant words, which implies that almost every word or symbol in a mathematical expression conveys meaning. Missing one word or a symbol could lead to misinterpretation or misunderstanding (Zevenbergen, 2001: 17). For example, 2 (x + y) is not the same as 2x + y (missing brackets, misinterpreting the distributive property).

Second, mathematics is abstract right from the start and its abstractness increases as the learner moves from one grade to the next. Rodd (2002) calls mathematics “hot and abstract” rather than “cold and abstract”. It receives its “coldness” from individuals that view mathematics as a purely, abstract, made up of meaningless symbols and abstract rules, and their inability to understand mathematics for whatever reason. But it also receives ‘hotness’ from the fact that the person that understands mathematics sees
the aesthetic side of mathematics too as he/she does mathematics for enjoyment and enjoyment is highly correlated with skill (Rodd, 2002).

Third, Schaffler (1999: 1-6) states that mathematics maybe understood to represent ‘internal’ logical relationships among concepts or very abstract, though still empirical and generalisations based upon experiences. Mathematical truths are not dependent on experience, though an awareness of them may be suggested by experience.

A fourth reason is that mathematics is perceived by many (MacLelan, 2005; Giannakopoulos, 1992; Hiebert., & Lefevre, 1986) as a subject of meaningless abstract symbols which can be learned by memorising abstract rules and algorithms and reproducing them. A kind of “surface learning” takes place, void of any “deep understanding” (MacLelan, 2005) or what Skemp (1977) called instrumental and relational. Surface learning is predominantly algorithmic (instrumental) where the learner is attempting to solve a problem using rules without any understanding of the meanings attached to the rules.

Finally, the absoluteness of mathematics which is based on its constancy, precision, and universality requires a particular way of teaching which culture dependent is, paradoxically. For example if the goal is to teach a child, in any place in the world, that \( 3+2 = 5 \), the means to be used can differ from culture to culture. Thus universality is locally (unique culture) dependent. Educators should be aware of this and adjust teaching accordingly.

At the centre of the above points made for the nature of mathematics, lies its indisputable abstractness from the start. This implies from an early age human beings must possess a certain amount of abstract thinking in order to learn mathematics. One way to overcome such abstractness is through critical thinking as it was claimed by Giannakopoulos (2012) and others (Pushkin, 2007; Elliott., Oty., Mcarthur & Clark; Brown, 1998).

Critical thinking and mathematics

Critical thinking (CT) is a particular human activity and a complex way of thinking. CT is considered to be a higher order type of thinking, non-algorithmic, complex mode of thinking that often generates multiple solutions (Barak., Ben-Chaim & Zoller, 2007). CT is a complex human activity which involves “much more than skills and logical analysis” (Brookfield, 1987: 1). Pushkin (2007) and others (e.g. Alonso, 1992; Forinash, 1992 cited by Pushkin, 2007) found that CT is connected to mathematics, scientific problem solving, and physical sciences among other fields.
The complexity of CT makes the definition of it difficult as there is no consensus among educationalists and psychologists as to what CT really is (Pushkin, 2007; Martinez, 2007; Lombard & Grosser, 2004; Brown, 1998; Halpern, 1997; McPeck, 1990). Many authors however, developed working definitions in order to shed some light into the understanding of CT.

Vaughn (2008: 4) defines CT as “The systematic evaluation or formulation of beliefs, or statements, by rational standards.” The author adds that CT is not about what we think but how we think (2009: 4). It is systematic because it involves procedures and methods. It is used to assess existing beliefs and formulate new ones; and it operates according to rational standards in that beliefs are judged by how well they are supported by reason. It involves a number of higher order thinking such as, analysis, synthesis, make choices (use appropriate formulae), classifying, pay attention to detail and other higher order thinking skills.

Halpern (1997: 4) gives a more detailed definition of CT though she admits that there are several definitions but they all have similar content. The author gives a working definition whereby she states that “CT is the use of those cognitive skills or strategies that increase the probability of a desirable outcome. It is used to describe thinking that is purposeful, reasoned and goal directed.” The author states further that “When we think critically, we are evaluating the outcomes of our thought processes” (1997: 4).

Critical thinking can be assumed to be a purely cognitive ability (like thinking, logic). However CT can be represented both as an ability as well as a cognitive disposition (Williams, 2005; Leader & Middleton, 2004; Paul, 1993; Facione, Facione & Giancarlo, 1996 in Barak et al., 2007; Ennis, 1985). CT involves experiences, thoughts and feelings (Vaughn, 2008; McPeck, 1981) and ultimately leads one to understanding, knowledge and empowerment (Vaughn, 2008). Recent publications on CT have focused among other fields, in mathematics and science education (Elliot, et al., 2000).

In vocational mathematics where it is a mixture of operating on symbolic functions and solving word problems, the need to look at mathematics content in a critical way is absolutely necessary as mathematics requires all the attributes of critical thinking in order to understand and apply mathematics.

**Vocational mathematics**

Before we explore the differentiation of functions it is necessary to give a brief description of vocational mathematics and what the content is aiming at. First, the vocational mathematics does not contain mathematical proofs. The students are given a formula book and the teaching is directed at training the students to “apply” these
formulae in solving “abstract problems”. Very little connection is made between the symbols and the real life problems. The solution of problems is predominantly algorithmic. For example it is given that if \( y = \sin^{-1} f(x) \) then 
\[
\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-(f(x))^2}}.
\]
The student now must be able to find the derivative of \( y = \sin^{-1}(\cos 3x) \) and so on.

Some Engineering fields such as Mine Engineering, Extraction Metallurgy do mathematics in the first semester (S1) and second semester (S2) while others also do mathematics in the 3rd semester (S3) (e.g. Chemical Engineering). This study concentrates only on S1 and S2 mathematics. S1 mathematics consists of algebra, mostly revision of matric mathematics, with Cramer’s rule, binomial theorem, and complex numbers as ‘new concepts’. It also contains trigonometry which is also mostly matric level and contains differential calculus (differentiation and integration) at low levels. The S2 course again contains differential calculus as well as partial derivatives and simple differential equations which can be solved provided they are of a certain type (separating variables, integrating factor, homogeneous, Bernoulli’s).

Vocational mathematics is mostly symbolic and thus most of the content requires acquisition of procedural knowledge, to increase the pass rate the teaching method must fit with ways appropriate for the student to acquire such knowledge. It has been argued at length by Shavelson et al. (2005) that other types of knowledge are interlinked with this type of knowledge. The assumption is that if the teaching method promotes the other types of knowledge, two goals could be achieved:

a) It will enhance problem solving and critical thinking in the students.

b) By designing the course in such a way that the different types of mathematical knowledge are acquired, it will not only increase the pass rate but also make the student more proficient in other mathematics related subjects.

The proposed teaching method here, which is illustrated later in the paper, demands that the teacher/lecturer to be a ‘diagnostic teacher’ (Solomon, 1999). This is vital for the success in the course which is a shift from teaching to diagnosing. Being a diagnostic teacher rather than a traditional teacher, misconceptions can be identified early. This can be achieved, in short, by tracing the misconceptions back to the pre-requisite knowledge.

Achieving the two goals, (a) and (b) above, is the ultimate aim of this study. In this study it is hypothesised that mathematics content (knowledge) combined with critical thinking, can successfully be transferred to various situations (mathematical or
mathematically related) which will ultimately lead to academic success and thus the throughput will increase.

Differentiation as a predictor for mathematics achievement in S1 and S2 courses.

Giannakopoulos (2012) found that differentiation can be used as a predictor for mathematics achievement in S1 and S2 course. With respect to S1 course, differentiation makes up of about 30% of the course. Obviously obtaining 100% in this section is not sufficient to pass the course. However, on the one hand it is a prerequisite for success in S2 course and on the other hand, a number of the other sections are closely related to school mathematics as a result the student can make up the rest of the marks from them and pass the S1 course.

The S2 mathematics course comprises of more than 50% of differentiation as it is used in the other three main sections, namely, partial derivatives, integration, and simple differential equations. Although the basis is set at S1 it happens very often that different lecturers teach S1 and S2 groups, as a result S2 teaching should be preceded by diagnosing of the students misconceptions. Furthermore a lot of students in S1 chose certain sections that they feel comfortable with and left out others, as an example, integration and some differentiation. The lecturer should keep in mind that success in S2 mathematics course is related closely with overall achievement in the National Diploma.

Since vocational mathematics is predominantly symbolic (although most of mathematics is represented in symbolic form from the beginning of Algebra), it stands to reason to examine the structure of the symbols which combined give rise to mathematical structures. In order to understand mathematical structures it presupposes the existence of a number of types of knowledge as defined by Shavelson, Ruiz-Primo and Wiley (2005) and others (Stolovitch & Keeps, 2002; Kwon., Allen & Rasmussen, 2005; Hiebert & Lefevre, 1986). Anderson and Krathwohl (2001) also distinguished factual and meta-cognitive knowledge.

Shavelson et al. (2005) (see Table 1) classified knowledge into declarative knowledge that includes conceptual knowledge other than facts, procedural knowledge, strategic knowledge- knowledge of when, where and how their knowledge applies and to check if their application of this knowledge is reasonable. The authors add schematic knowledge – “knowing why”- to the other types of knowledge. They also add another dimension to these types of knowledge (the length of knowledge), and that is the “depth” which deals with proficiency and that is the extent, structure and others. These types of knowledge are affected by emotions and motivation. In Table 1 they give a conceptual framework of the two dimensions.
Table 1: Conceptual framework of knowledge (Shavelson et al., 2005: 415)

<table>
<thead>
<tr>
<th>Proficiency</th>
<th>Declarative Knowledge</th>
<th>Procedural Knowledge</th>
<th>Schematic Knowledge</th>
<th>Strategic Knowledge</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extent</td>
<td>Knowing “that”</td>
<td></td>
<td></td>
<td>Knowing “where, when and how”</td>
</tr>
<tr>
<td>(how much?)</td>
<td>Domain-specific content:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Structures</td>
<td>Facts</td>
<td>Knowing “how”</td>
<td>Principles / schemes / Mental models</td>
<td></td>
</tr>
<tr>
<td>(how is it organised)</td>
<td>Definitions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Others</td>
<td>Descriptions</td>
<td></td>
<td>Strategies / domain-specific heuristics</td>
<td></td>
</tr>
<tr>
<td>(How efficient? How precise? How automatic?)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This table shows the different types of knowledge and the second dimension which has to do with proficiency in the various types of knowledge. This means that although learners might possess the different types of knowledge they can differ on the extent to which such particular type of knowledge has been acquired, how well organised is in the cognitive structure so that retrieval is automatic and how accurate and efficient it is.

Irrespective of the types of knowledge discussed above, it stands to reason that they all refer to some content or subject matter. Chinnapen (2003) states that content knowledge refers to: the knowing about a subject, and the disciplinary knowledge of a subject. With respect to Mathematics; ‘[m]athematical content knowledge includes information such as mathematics concepts, rules and associated procedures for problem solving’ (2003:1). If this definition is accepted then Shavelson’s et al., (2005) types of
knowledge are all included in the content knowledge; And to understand the content it is necessary to possess all these types of knowledge, otherwise the content will be demoted to mere information.

But all mathematical concepts (the content) are composed of symbols and operations on symbols or transformation of concepts which give rise to structures. These structures demand critical thinking, contain all types of knowledge and once the student can apply them successfully it could lead to the enhancement of critical thinking.

The study of mathematical structures

If we accept that mathematical structures comprise of functions and equations (functions containing dependent and independent variables, operations, transformations), then a mathematics symbolic superstructure could be analogous to a chemical substance. Numbers, constants and variables could be like the ‘atoms’, while the ‘bond’ is made up of the various operations (+,-,x,÷). The following axioms are used (Giannakopoulos, 2012):

a) Variables, constants, and numerals are the simplest mathematics structures (units) (the atoms…the bricks)

b) Operations (+,-,÷, x, Brackets ( )) separate variables, constants, and numerals. Operations are the ‘bonds’ or the ‘mortar’ of the structure. Brackets are used to create new structures.

c) There can be no new structure comprising of two or more consecutive units or operations. i.e. xy or ++ e.t.c xy MEANS x multiplied by y for example.

d) A final generic structure in its SIMPLEST FORM cannot be broken down to individual units. For example, log x, sin y.

e) A transformation is defined as an action on a structure to produce another structure which is related to the given structure. For example, differentiate with respect to x if y = sin 2x. The action here is ‘differentiate’ and the new structure is the ‘derivative’ which is related to the given function as being the ‘gradient of the tangent to the given function’.

f) All substructures comprise of factors (separated by x,÷) and combined with a ‘+’ or a ‘-‘ give rise to terms.

Substructures can become very complex if operations on operations are involved.

Given any mathematics substructure the student must be able to analyse and synthesise units and generic structures and perform operations on operations as well as transformations and these are aspects of critical thinking. In simple terms:

Figure 1: Performing symbolic operations
Only one example here is used to illustrate the formation and the analysis of a substructure. The rectangles here (see Figure 3) are used to indicate that it is one structure. Consider the expression

$$\sin^2(3x+5) + \sqrt{1+3\cos(2x)}$$

$$\log(5x^2)Z$$

Figure 3: A structural approach to differentiation

Using the structural approach in analysing the structure we arrive at the simplest units that the structure is comprised of. Using a ‘bottom-up’ approach we can see how the given expression was constructed. It consists of the most basic units (numbers, variables), operations, and transformations (actions). For example, for \(\sin^2(3x+5)\):

a) Start with 3 and x and create 3.x
b) Add 5 to get \(3x + 5\)

c) Perform the action…take the sine of \(3x + 5\), create \(\sin (3x+5)\)

d) Perform the action … square this function …\(\sin^2 (3x+5)\).

By analysing the given function the student gets a deeper understanding of the structure, and can be asked also to reverse the process by building the structure from its individual elements as shown in steps a-d above and continue with the rest.

Developing symbolic mathematical structures, at any level of mathematics education, and transforming them could enhance understanding of mathematics and critical thinking (Giannakopoulos, 2012) since the learner is forced to analyse and synthesise and evaluate, which all require higher levels of thinking.

In order to test the hypothesis that using the structural approach students perform better in differentiation a sample of 175 students were exposed to this method during normal lectures. The students were given four functions to differentiate at the end of the section of differentiation (when algebraic, trigonometric, exponential and logarithmic had been covered). The four functions were: \(y = \csc 2x\), \(y = (\csc 2x)^2\), \(y = \ln^2 \csc 2x\) and \(y = 2^{\csc 2x}\). They could use either the structural approach or the traditional or ‘short cut’ approach or even both approaches so that their answers could be confirmed if the two answers are the same. By traditional way here is meant that the student looks at the function and proceeds with differentiation. In the last case if either approach used was correct they would get 100%. If both were correct they would get 110% (that was an incentive).

Table 2 shows the performance of students in differentiation using either methods or both.

Table 2: Results of differentiation

<table>
<thead>
<tr>
<th>Method</th>
<th>No of Students (n)</th>
<th>%</th>
<th>Correct responses</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>No of students</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>%</td>
</tr>
<tr>
<td>Structural only</td>
<td>87</td>
<td>50</td>
<td>52</td>
</tr>
<tr>
<td>(S)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Traditional only</td>
<td>52</td>
<td>30</td>
<td>12</td>
</tr>
<tr>
<td>(T)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
What is evident from Table 2 is that 123 (70%) students used the structural method while 88 (51%) used the traditional and the success rates were 60% (74) for structural method and 16% (14) for use of the traditional method.

What was even more surprising is that more than 40% (72) of the students who got 100% had term marks less than 50% and that more students preferred to use either the structural or both methods. Normally if a student has a term mark less than 50% the student performs most of the time worse in an exam.

When students (N = 175) were given a questionnaire and were asked of their opinion of the new method they had to choose from predetermined answers, as many as they liked. The answers contained positive and negative comments about the use of the new method. Else they could write their own answer. Some of the comments they chose with the highest frequencies (in brackets) were:

“The method is too long. We rather use the formulae”, “We prefer short cuts” (35).

“Now I can see why I could not differentiate” (68).

“I understand algebra better” (15).

“I look at functions in a different light” (36).

“The method demands that you have to ‘pause and think’ before you differentiate” (45).

“The method promotes understanding of differentiation” (69).

“Now I can apply the rules of differentiation more accurately” (56).

“I find the method confusing” (11).
“I cannot see the point having to go through all those analytical ways”(15).

“I prefer the ‘short cut method’ rather than the ‘structural method’”(21)

“I prefer the ‘structural method’ rather than the ‘short cut method’” (82).

From the above answers it can be inferred that the majority of the students preferred the ‘structural method’ over the ‘short cut’ method.

**Implications to teaching and learning**

As the structural method is based on critical thinking, where the student has to analyse, synthesise, make choices (use appropriate formulae), classify, pay attention to detail, higher order thinking skills and so on, then the method could be applied at the beginning of algebra. In the above Figure 3 for example, breaking down the functions, putting together individual derivatives, using the right formula, paying attention to detail such as identifying all operations and so on are pre requisites to critical thinking. Even experienced teachers could still be frustrated when they introduce algebra for the first time as they discover, in a kind of predictable way, that only about a 40% of the learners could operate abstractly with abstract symbols. By applying the structural methods the abstract symbols are ‘concretised’ into a sensible structure.

Many teachers assume that by using concrete examples like 2 apples + 3 apples it has to be 5 apples and not 5 potatoes the learners will be able to say that $2x + 3x = 5x$. To their horror they don’t. However, it is possible to explain the answer using other simpler abstract concepts, for example numbers, or place values. Skemp (1977) and Kincheloe and Weil (2004) promoted this idea when the stated that one cannot transform an abstract idea to a concrete one as it loses its essence (abstractness). However, concentrating on the structure of the problem through analysis and synthesis and using other abstract ideas (for example numbers which on their own are abstract) a better understanding can be achieved.

Furthermore, if it is accepted that abstract concepts (like symbolic functions) cannot be taught by definition (Skemp, 1977) but the definition should first be constructed by the learners guided by the teacher, then the structural approach can assist the teachers building the mathematical structures.

Giannakopoulos (2012) also added that the teachers should become diagnostic teachers. Here diagnosis has the same meaning as the medical diagnosis and new concepts should be taught using a mixture of psychological (cognitivism, constructivism, behaviourism) in a pragmatic (realistic) way. For example before symbolic differentiation is
introduced the teacher/lecturer should re-visit the concept differentiation which was
introduced at matric level and establish any misconceptions that the students have
brought with (diagnosis). Use real example from everyday life even an experiment
(pragmatism) related to rate of change (differentiation). Connect theory into practice.
The structural method fits this paradigm.

CONCLUSION AND RECOMMENDATIONS

The method described above in approaching differentiation can be used for any level
of mathematics starting from the introduction of algebra. Obviously it is directed at
operations with symbols (one of its limitations). However we all know that most of the
algebra is symbolic. When the learners encounter symbols for the first time they are not
aware of their abstractness. Therefore to understand it a systematic, abstract way is
necessary which is in line with Skemp’s (1977) recommendation. It is an assumption
that we can use concrete examples to teach abstract concepts but we can use concrete
concepts to generalise to develop the abstract concept. For example giving the rule \( a^{m+n} = a^m \times a^n \), to apply it to \( x^2 \times x^n \) is so the same as if the rule is derived from different powers
using numbers (concrete) and deriving the general formula and then apply it.

What needs to be borne in mind is that the teacher/lecturer must move towards being
a diagnostic teacher/lecturer so misconceptions like the above, \( 2x^2 \) can be avoided as
preventative measures will have been taken by establishing the existence of the various
types of knowledge and content.

It is true that the method is a time consuming method. But the teacher has to decide
whether he/she wants to promote everlasting understanding or temporary ‘surface
learning’ that disappears with the blow of a slight wind.

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ON WORKING WITH MATHEMATICS TEACHERS FROM HISTORICALLY DISADVANTAGED HIGH SCHOOLS THROUGH A CONTINUOUS PROFESSIONAL DEVELOPMENT INITIATIVE

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In this paper I discuss the main arguments that underpin working with mathematics teachers through a continuous professional development (CPD) initiative. In distinguishing between the mathematics teachers and me as the participating mathematics teacher educator, my aim is to highlight ways I understand opportunities that arise during teacher institutes and workshops in the initiative. By providing details on the education context in relation to the model of the initiative, I assess the extent to which this context and its attendant pressures – such as the high-stakes National Senior Certificate Mathematics examinations and the Annual National Assessments – lay the groundwork for working with the mathematics teachers. The paper is structured in three sections. The overview of the particular CPD model and its context is followed by a literature review that illuminates ways of thinking about working with the mathematics teachers in the case of (quadratic) functions and classification of quadrilaterals. An analytical framework is then used to analyse two incidents in which the mathematics teachers and I worked on activities related to the initiative. The paper argues for a constant awareness of opportunities for learning what it takes to work with the mathematics teachers.

INTRODUCTION

This paper focuses on some of the main arguments that underpin a continuous professional development (CPD) initiative (Local Evidence-Driven Improvement of Mathematics Teaching and Learning Initiative (LEDIMTALI), located at the University of the Western Cape with special reference to my participation as a mathematics teacher educator (MTE) involved in working with the mathematics teachers. These arguments address the different levels of expertise among the participants involved in the initiative and the way that they think differently about the teaching and learning of school mathematics. The participants are from different higher education institutions in the Western Cape, the provincial Department of Education and a selected number of schools located in areas that are historically and socio-economically disadvantaged. The participants include a mathematician who is a professor of mathematics in a mathematics department at one of the higher education institutions in the Western Cape, mathematics (teacher) educators, mathematics teachers who teach in Grades 10 to 12 (Further Education and Training, i.e. FET), curriculum specialists and mathematics curriculum advisors. Mathematics teacher educators are those participants who teach pre-service and practising teachers in teacher
education programmes. On the other hand, mathematics educators refers to those who teach mainly mathematics courses aimed at engineering, science and arts undergraduate students, i.e. students who are not in teacher education programmes. These mathematics educators, however, do work with the teachers through the initiative. The initiative focuses on the ‘co-development of teaching Mathematics at the FET level’ that can lead to ‘effective learning of Mathematics.’ In the initiative, for example, there is a focus on the development of an ‘intentional teaching’ model (Julie, 2013) aimed at getting more learners from the mentioned schools to pass the high-stakes Annual National Assessments (ANAs) and the National Senior Certificate (NSC) Mathematics examinations. ‘High-stakes’ in this case means that the schools, principals, teachers and learners are judged by the outcomes of these assessments and examinations. The initiative has to contend with the dilemma of ‘predicting or controlling’ the impact of the high stakes on professional development (Wall, 2000). LEDIMTALI thus organises various teacher institutes and workshops where the different participants interact around issues of designing, testing and fine-tuning teaching modules aimed at different grade levels, a classroom observation instrument, spiral revision (Julie, 2013; Selter, 1996) and examination practice units, for example. Also, there are classroom visits aimed at communicating developments, ideas and goals related to the initiative to the teachers. Figure 1 is an attempt to represent the CPD model and its various participants and the way that they relate to each other.

Figure 1: A simplified outline of the differentially distributed expertise among participants in the CPD model

The figure has been adapted from one of the publications of the CPD initiative (LEDIMTALI, 2013). What should be noted is that the participants work at institutions
that are physically separated, e.g. universities, schools and the Provincial Department of Education offices. The participants thus have different time tables, different holidays and different reward systems as well as different ways of knowing, talking and doing school Mathematics. Collegial relationships among the participants are fostered mainly by the activities organised by the initiative.

Any thinking around working with the teachers has to be mapped in relation to the education context, especially with respect to mathematics (teaching) in the last three years of compulsory schooling. The teachers teach in high schools that are located in segregated neighbourhoods that were historically enforced under Apartheid. The teachers encounter pervasive deficit discourses (Keitel, 2005) and denigration (Slonimsky and Brodie, 2006). The teachers have to contend with the performance of their schools in the high-stakes NSC Mathematics examinations through platitudes stating that they have to ‘work harder’ (*Die Burger*, 2011), for example. In other words, the teachers are ‘put to the test’ (Smith, 1991). Given this high-stakes education context it is thus not surprising that the teachers take NSC Mathematics examinations as well as the ANAs as their ‘reference point’ (Boardman and Woodruff, 2004). The CPD model incorporates these issues and tries to respond accordingly by organising teacher institutes and workshops that focus on developing spiral revision units, learning modules per grade level and ‘examination practice’ modules, as mentioned in the above. All of these are aimed at expounding ‘intentional teaching’ with respect to the mathematics content in the last three years of compulsory schooling which for most learners culminates in the high-stakes NSC Mathematics examinations.

The problem statement and research question for the paper read as follows. Expertise among the CPD initiative participants is differentially distributed. It is thus of interest to the mathematics (teacher) education community to know how participating MTEs ‘work with’ the mathematics teachers. The research question is: What can we learn from interactions between me as the MTE and the teachers during teacher institutes and workshops organised by the CPD initiative? Two data incidents (1 and 2) are analysed to answer this question. The first occurred during a Grade 12 ‘examination practice’ on quadratic sequences/series, while the second happened during a Grade 9 lesson design on straight-line geometry dealing with parallelism and the sides of quadrilaterals. The incidents occurred during two separate teacher institutes. The paper should be viewed in the light of MTEs who write from their perspective on ‘working with’ teachers (Setati, 2005).

The section below reviews the literature on different kinds of mathematics with reference to school mathematics and the ways that MTEs think about this mathematics in relation to the (quadratic) function concept and classifying quadrilaterals. This is followed by the outline of an analytical framework that will be used to analyse the two
data incidents. The paper ends with a discussion of the implications for CPD initiatives operating in similar education contexts.

LITERATURE REVIEW

When I work with the teachers I have to consider the different kinds of mathematics, the didactic transposition, the situation of the (school) classroom and the ways I understand and interpret my position in relation to mathematics (teaching) when interacting with the teachers.

In working with the teachers there are different ‘kinds of mathematics’ such as those represented in high-stakes assessments, school textbooks and in the journals that mathematics teacher educators read, for example, that come into play. School mathematics is a particular or special kind of mathematics (Julie, 2002; Watson, 2008: 4) that has been institutionalised in the school and is subject to its institutional and epistemological contexts (Chevallard, 1991). In the high-stakes education context described earlier, ways of knowing or the epistemology with respect to school mathematics content, is legitimised by tests and examinations. Put differently, the content of the NSC Mathematics examinations and the ANAs becomes ‘high-stakes.’ Drawing on Kvale’s (1993: 222) analysis of the knowledge associated with examinations, Julie (2012: 1) makes a case for the notion of ‘legitimate school mathematics knowledge’, which is the knowledge associated with high-stakes, time-restricted examinations. On the other hand, MTEs bring a particular perspective to school mathematics that can be characterised as the mathematics-education-researcher’s (MER) mathematics (Sfard, 1998). This mathematics draws on insights from learning theories such as constructivism or socio-constructivism, and pedagogy, i.e. from different strands of literature emanating from and related to the domain of mathematics education. MTEs may therefore want to emphasise the importance of representing the function concept graphically, algebraically and through the use of a table, i.e. as a ‘unifying concept.’ This same (linear or quadratic) function concept can be conceptualised and represented over the set of integers or the set of real numbers. There are numerous studies on the centrality of the function concept (Freudenthal, 1983; Nyikahadzoyi, 2006) in school mathematics. To date we know about high school teachers from similar education contexts who work with the ‘function concept’ (Gierdien, 2014). A focus on this as a ‘unifying concept’ may not go well in school mathematics, because it would require vertical coordination across the grade levels (Watson, 2008: 6). For the sake of mathematical/geometric connections MTEs may also want to work with teachers by brainstorming a series of lessons where there is an emphasis on classifying quadrilaterals hierarchically or through partitioning (De Villiers, 1994) in line with the intentional teaching model. When working with mathematics teachers during teacher institutes, MTEs will thus have to look for
opportunities where there can be overlaps with their MER mathematics and ways the
teachers can come to think about the intentional teaching model.

While working with the teachers, the MTE has a particular ‘researcher positionality’
(Milner, 2007: 388) that she has to ‘work with’ and ‘work through.’ The MTE cannot
view the teachers’ ‘reference point,’ described earlier, as categorically negative. The
teachers are influenced by a didactical transposition (Brousseau, 1997: 21) of the school
and its high-stakes education context as well as by the CPD through its teacher
institutes. From the MTE’s position as a researcher, these interactions are about
‘working with’ and ‘working through’ opportunities for the teachers to see possibilities
for ‘spiral revision’ (Selter, 1996), where there is ‘recurrent practise of work covered’
(Julie, 2013) before and ways of enacting the intentional teaching model for school
mathematics.

To answer the research question an analytical framework will be used. This framework
has to include the different kinds of mathematics mentioned above. It also has to
indicate the CPD education context, whether it is a teacher institute or workshop or the
high-stakes education context described earlier. The simplified diagram below (Figure
2) represents this framework.

<table>
<thead>
<tr>
<th>High-stakes education context</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mathematics</strong></td>
</tr>
<tr>
<td><strong>teachers’ reference point:</strong></td>
</tr>
<tr>
<td><strong>Mathematics</strong></td>
</tr>
<tr>
<td><strong>teachers’ reference point:</strong></td>
</tr>
<tr>
<td><strong>Mathematics teacher</strong></td>
</tr>
<tr>
<td><strong>educator (MTE)</strong></td>
</tr>
</tbody>
</table>

| **A focus on Mathematics** |
| **content driven by high-stakes examinations** |

| **Mathematics content driven by high-stakes examinations** |
| **Support environment:** |
| **Teacher institutes/workshops focusing on developing examination practice, and lessons for purposes of intentional teaching** |

| **MTE ‘working with’ and ‘working through’ MER mathematics in relation to mathematics teachers:** |
| **quadratic functions and classifying quadrilaterals** |

Figure 2: An analytical framework for thinking about ‘working with’ the mathematics teachers.
In the left-hand column there are issues that the mathematics teachers are concerned with, i.e. their reference point(s). The middle column gives an idea of the support offered through teacher institutes and workshops. The right-hand column gives an indication of dilemmas and issues that I as the MTE bring to the workshops and teacher institutes.

**METHODOLOGY**

The methods for data collection needed to answer the research question are qualitative and are drawn mainly from the literature review. The method of data analysis is based on comparing the way that the teachers understand and refer to intentional (mathematics) teaching and the ways that I think and share my inputs on similar mathematics (teaching) issues. We can thus think of the analysis as being a type of ‘constant comparison’ (Glaser & Strauss, 1967: 102) between what the teachers do and say, and the ways that I communicate with respect to what they do and say. Data excerpts showing evidence of ‘working with’ the teachers serve as the unit of analysis. ‘Working with’ here means that during workshops and institutes the teachers are assigned to groups that include either an MTE, a mathematics educator, a curriculum advisor, a curriculum specialist or a mathematician. Who gets assigned where depends on the nature of the activity in the workshop or institute. The materials produced during workshops or institutes are collected with the goal of improving and refining them so that they can be distributed to the teachers at the various schools, sometime afterwards. The improved materials take the form of examination preparation worksheets or spiral revision exercises, for example.

Central to a workshop or institute are incidents where the teachers work with legitimate school mathematics knowledge in relation to MER mathematics, i.e. inputs from me as the MTE. In particular, there will be a data excerpt (Incident 1) showing my recount of working with the teachers in relation to the quadratic function as it appears in a teacher institute on examinations practice. There is also an artefact (Incident 2) showing how the teachers captured inputs with respect to classifying quadrilaterals. This incident, in which I was a participant, occurred during a discussion on developing a series of lessons on straight line geometry dealing with parallelism and the sides of different quadrilaterals.

**FINDINGS**

*Incident 1: Examination practice: Finding a (quadratic) function*

During one of the teacher institutes on a Grade 12 examination practice module I was part of group of teachers who discussed approaches to recognising and to drawing up
questions and answers on quadratic series and sequences. This module development is central in terms of working with the teachers and the pervasive ‘high-stakes’ mathematics examinations as well as legitimate school mathematics knowledge. I saw an opportunity for working with the function concept as a possible unifying concept for spiral revision and productive practising in the case of quadratic series and sequences. There are mathematical connections between quadratic series and sequences and parabolas. In what follows I reconstruct what transpired during this incident with the teachers. Our task was to write different types of questions and answers for this module. The first question we addressed was to identify whether a sequence/series is quadratic. In one example we dealt with the following sequence:

\[-4, 4, 10, 14, 16…\]

From my perspective, the same sequence can be tabulated in the following way:

<table>
<thead>
<tr>
<th>Term number ((T_n))</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>sequence</td>
<td>-4</td>
<td>4</td>
<td>10</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>1st difference</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2nd difference</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: A quadratic sequence showing its first and second differences

We concluded that the sequence is quadratic because the 2\(^{nd}\) differences are constant.

Through our calculations we found the expression for the quadratic sequence to be: \(-n^2 + 11n - 14\). In line with the principle of working with the teachers I showed that in my extension of the table in the above there is in fact a ‘turning point’ in the sequence. I pointed out to the group that the quadratic sequence can be considered as the ‘equation’ for a parabolic or quadratic function. In the words of Freudenthal (1983: 492) we can speak of ‘the equation of a function,’ i.e. a quadratic function in this case. I showed my group the table below and an accompanying hand-drawn ‘sketch graph’ of the ‘parabola.’ In other words, I broached what I saw as mathematical connections between quadratic series/sequences and the graphic and numeric representation of the parabola.
Table 2: MTE input (italicised): an extension of the quadratic sequence in table 1.

I drew a ‘sketch graph’ as a way to connect with the teachers’ classroom and not the MER mathematics reference to a graphical representation of any particular function.

What the incident in the above illustrates is the possibility of spiral revision in terms of the equation of the quadratic function and its graph. In the examination practice the questions are chosen from past papers as found in the NSC Mathematics examinations. This mathematics undergoes a particular institutional impact where functions, for instance, appear in questions on quadratic sequences/series as well as in questions where learners have to factorise trinomials and find the roots of, and plot, quadratic equations. The answers to questions on quadratic sequences/series are restricted to the rational numbers. The questions on the factorization and plotting of quadratic functions and equations in the NSC Mathematics examinations deal with single variable trinomials. In this case the factors, for example, can be in the domain of integer \((\mathbb{Z}[x])\), rational \((\mathbb{Q}[x])\), real \((\mathbb{R}[x])\) or complex \((\mathbb{C}[x])\) numbers. Also, we can think of instances where learners have to say whether the roots of a parabola are non-real (complex) or irrational numbers. I may not have capitalised adequately on the opportunity to make the participating teachers aware of this function connection between quadratic sequences/series and plotting the parabola and/or findings its roots. During examination practice modules teachers have to contend with the strong influence of the didactic transposition and situation of the high-stakes NSC Mathematics examinations content which shapes their reference point.

**Incident 2: On designing a series of lessons on and about classifying quadrilaterals**
Incident 2 is captured in the form of an artefact that serves as evidence of ways the teachers thought about classifying different quadrilaterals. From an MTE’s perspective it can be argued that classifying quadrilaterals hierarchically or through partitioning is a prerequisite for designing any intentional teaching lessons on parallelism and the sides of quadrilaterals (kites, parallelograms, rectangles, rhombi and square). This incident happened during one teacher institute where inputs, questions and comments revolved around the properties of ‘quads’ and relationships within and between and them with respect to angles, parallel sides and vertices. In the group we asked, for example, how are the rhombus and the kite related? When does the kite become a rhombus and vice versa? We debated the importance of comparing and contrasting as a way to find the uniqueness of each quadrilateral and how the one can become the other. We discussed whether learners should know properties beforehand or whether they should ‘discover’ them. From my perspective I saw our exchanges as being about classifying quadrilaterals where we point out similarities and differences and especially how the one ‘quad’ becomes a different ‘quad’ through particular changes in angles and sides. One teacher noted how a rhombus becomes a kite ‘if you move the point downwards.’ He used AutoCAD on the laptop computer we had at our disposal to show how ‘dragging’ the vertex of a rhombus changed it into a kite.

The artefact (see Figure 2) is a ‘pencil-paper’ summary of the discussion we had about ways the one ‘quad’ becomes a different ‘quad.’ The top three quadrilaterals show a rhombus, square and a kite. The latter shows the marking of equal sides, how the rhombus becomes a kite. The bottom four quadrilaterals show some progression of a rectangle being ‘dragged,’ indicated by the arrow (see second quadrilateral from left). The second quadrilateral from the right is the rhombus that becomes a kite when its bottom vertex is ‘dragged’ down as shown by the pencil-drawn arrow. The ‘quad’ in the middle in the bottom row is the typical ‘parm’ or parallelogram. ‘Classify’ is in inverted commas because this is how I interpreted what the teachers and I were doing. They, in particular, were concerned with, and debated, whether learners ‘should know the properties beforehand’ or whether they should ‘discover’ them.
Figure 2: A pencil-paper summary of ‘classifying’ the different quadrilaterals

CONCLUDING REMARKS

The two data incidents have implications for MTEs working with mathematics teachers through CPD initiatives in high-stakes education contexts. These implications are about predicting and controlling ways that professional development can go (Wall, 2000).
First, there has to be an awareness of the differential distribution of expertise that participants bring to the particular CPD model. Depending on the activity in the teacher institute or breakaway session, the mathematics teachers will invariably bring to the fore peculiarities related to the didactic transposition of legitimate school mathematics knowledge. Put differently, they will be concerned with ways the mathematics content in the initiative can be used in their classroom teaching. Here the input of the MTE is crucial and hence the cornerstone intentional teaching model of the present initiative. From the two data incidents it appears that the lower the grade level in relation to the high stakes NSC Mathematics examinations, the more amenable the teachers might be to run with the input from the MTE in terms of organising the school mathematics content. For example, the teachers appear to be receptive to pursuing a hierarchical or partition classification of quadrilaterals based on overlaps or commonalities, where systematic changes in lengths or vertices can lead to one ‘quad’ becoming a different ‘quad.’

Noting my positionality with respect to interactions with the teachers, I tried to ‘work through’ these classifications with the teachers. They spoke of having their learners ‘discover’ and ‘investigate’ the properties of quadrilaterals where there is a systematic exploration of the role of parallelism and even the idea of using software such as AutoCAD to explore ‘what happens if’ the vertex of a rhombus is extended or lengthened in relation to the other vertices. On the other hand, there was near silence on input about mathematical connections between quadratic series/sequences and the drawing of a ‘sketch graph’ of the accompanying parabola, where the turning point is a maximum. In this incident I tried to draw attention to the quadratic function over the set of rational numbers and over the set of real and/or complex numbers. The teachers’ near silence can be explained in terms of the strong didactic transposition of the NSC Mathematics content, i.e. ways the (quadratic) function concept is institutionalised in the different questions. We do know of teachers from similar education contexts who attempt to work with the function concept as a flexible, unifying underlying concept. Overlaps and ‘non-overlaps’ between different quadrilaterals and different representations of the (quadratic) function concept are instances where further research is needed.

In the current education context it is in fact necessary to bring to the fore ways of ‘working with’ seemingly different frames that arise during CPD teacher institutes.

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THE ASSESSMENT OF QUADRATICS IN GRADE 12 MATHEMATICS EXAMINATIONS IN SOUTH AFRICA: IMPLICATIONS FOR TEACHING AND LEARNING

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Eastern Cape Department of Education &
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Quadratic equations and functions (generally called Quadratics) form a critical part of the FET Mathematics curriculum in South Africa. This paper focuses on the assessment of quadratic equations and functions in the Grade 12 Mathematics papers. A detailed analysis of the November 2014 examination papers shows that questions on these sections form a significant part of both paper 1 and paper 2 and involve “procedural” or “conceptual” understanding. This has major implications for the teaching and learning of these and other sections of Mathematics. Some recommendations, in this regard, are given at the end of this paper.

INTRODUCTION AND BACKGROUND

Mathematics is an important school subject in South Africa and in other countries. Since democracy was achieved in South Africa, there have been various changes to the Mathematics curriculum, with the latest change being implemented in the Further Education and Training (FET) phase in grade 10 in 2012. This curriculum, commonly known as “Curriculum and Assessment Policy Statement” (CAPS), is called the “National Curriculum Statement (NCS) Grades R -12” and is implemented in all nine South African provinces.

Quadratic equations and functions, generally referred to as Quadratics, has always been an integral part of the Mathematics curriculum in South Africa. In the FET, it is located in two content areas, “Algebra and Equations (including inequalities)” and “Functions and graphs” which mainly forms part of Mathematics paper 1 (DBE, 2011).

An understanding of quadratic functions may allow learners to explore important physics concepts such as motion and gravity and mathematics concepts such as instantaneous rates of change. This would enable learners to become better prepared to learn the fundamental ideas of calculus. Quadratic functions are also a gateway into understanding increasingly dynamic and applicable mathematical models (Eraslan, 2005).
Some literature on the teaching of Quadratics

Almy (2011) reported that Quadratics is widely considered, by teachers and learners in the USA, to be the most difficult topic encountered in Algebra 1. In this regard, the study of Quadratics may provide roadblocks for many students who are interested in science, technology, engineering and mathematics (STEM) careers.

Teaching is a dynamic (Wetzstein & Broder, 1985) and complex process (Reimers, 2006). This is especially true in a subject like Mathematics which is generally regarded as a difficult subject to teach. In this regard, the role of the teacher cannot be overstated. This is reflected in the next two studies which focus on the role of the teacher when teaching quadratic equations. The first one was by Xuhui Li (2011) which focused on a teacher teaching quadratic equations at an urban school in one of the largest and most diverse school districts in California. The teacher’s knowledge of the equation solving routines played the most crucial role in shaping lessons on quadratic equations. The findings suggest that mathematics teacher preparation and professional development programmes should provide more opportunities for teachers to revisit in-depth the features and applicability of various mathematical routines, and develop skills in making class instructional decisions (Li, 2011).

The second one by Makgakga in South Africa also emphasises the role of the teacher. This research was conducted in the Capricorn district of Limpopo Province and its main aim was to diagnose errors learners made in solving quadratic equations by completing a square and to determine the reasons why these errors occurred. Makgakga’s findings revealed that the approaches used by teachers contributed towards learners’ errors and the misconceptions they possessed. Learners were also not given an opportunity to discuss the concepts and they did not have enough time to practice their work (Makgakga, 2013).

The above two studies could not be more contrasting but equally relevant to this study. The one in the USA emphasises teacher knowledge, teacher preparedness and teacher development programmes, while the South African one puts the blame on learner errors and misconceptions at the teaching approaches being adopted.

The poor teaching approaches as detected in the South African study could be addressed through teacher development programmes which emphasise teacher content knowledge and teacher preparedness for their classrooms. A well-prepared and knowledgeable teacher will ensure there is ample opportunity for classroom discussion on the different concepts and for learners to work through the examples on their own. In this way learners’ errors and misconceptions could be kept to a minimum.
The location of Quadratics in the South African curriculum is discussed in more detail in the next section.

**Quadratics in the South African Mathematics curriculum**

As indicated earlier, Quadratics in the South African Mathematics curriculum is located in two content areas. In terms of grade 12 assessment, the final external exam mark breakdown for these content areas is indicated in the two tables that follow:

**Table 1: Mark breakdown: Algebra and Equations**

<table>
<thead>
<tr>
<th>Grade</th>
<th>Mark breakdown</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>30 ± 3</td>
</tr>
<tr>
<td>11</td>
<td>45 ± 3</td>
</tr>
<tr>
<td>12</td>
<td>25 ± 3</td>
</tr>
</tbody>
</table>

(DBE, 2011)

**Table 2: Mark breakdown: Functions and Graphs**

<table>
<thead>
<tr>
<th>Grade</th>
<th>Mark breakdown</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>30 ± 3</td>
</tr>
<tr>
<td>11</td>
<td>45 ± 3</td>
</tr>
<tr>
<td>12</td>
<td>35 ± 3</td>
</tr>
</tbody>
</table>

(DBE, 2011)

If one has to combine both content areas, into one table (without the approximation), the table will look as follows:

**Table 3: Mark breakdown: Algebra and Equations & Functions and Graphs**

<table>
<thead>
<tr>
<th>Grade</th>
<th>Mark breakdown</th>
<th>Composition of Mathematics P1</th>
<th>Composition of total</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>60</td>
<td>60%</td>
<td>30%</td>
</tr>
<tr>
<td>11</td>
<td>90</td>
<td>60%</td>
<td>30%</td>
</tr>
<tr>
<td>12</td>
<td>60</td>
<td>40%</td>
<td>20%</td>
</tr>
</tbody>
</table>

While table 3 confirms the importance of these two content areas, table 4 below shows the number of teaching weeks allocated to these content areas, including percentages of the total.
Table 4: Teaching weeks: Algebra and Equations & Functions and Graphs

<table>
<thead>
<tr>
<th>Grade</th>
<th>Number of teaching weeks: Algebra and Equations</th>
<th>Number of teaching weeks: Functions and graphs</th>
<th>Combined: Algebra and equations &amp; Functions and graphs</th>
<th>Total number of teaching weeks (excluding exams)</th>
<th>Percentage of total</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>7</td>
<td>4</td>
<td>11</td>
<td>35</td>
<td>31,4%</td>
</tr>
<tr>
<td>11</td>
<td>6</td>
<td>4</td>
<td>10</td>
<td>35</td>
<td>28,6%</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>29</td>
<td>13,8%</td>
</tr>
</tbody>
</table>

A comparison of table 3 with table 4 shows that there is almost a one-to-one correspondence between teaching weeks and overall mark allocation in the examinations in both grades 10 and 11. In grade 12, there is very little allocation of teaching time to these content areas as the bulk of the work would have been completed in grades 10 and 11. However, it should be noted that provision should be made for revision of these sections of the work in grade 12. This would then increase the time allocated.

A close scrutiny of tables 1 to 4 shows that Quadratics comprises a very significant part of the examinations in grade 10 -12 and this is reflected in the teaching time which is allocated to this key part of the mathematics curriculum.

RESEARCH QUESTION

This research investigates the assessment of Quadratics in the final grade 12 examination in South Africa. The November 2014 examination papers are used as a guide. Thus, in the light of the previous discussions on the teaching and learning practices with respect to Quadratics, the following research question was formulated for this study.

“How is Quadratics assessed in grade 12 examinations in South Africa?”

To answer this research questions, the following sub-questions were included:
What are the types of questions on Quadratics which feature in grade 12 Mathematics papers in South Africa?
Do these questions assess conceptual understanding or procedural understanding?
What are the implications for the teaching and learning of Quadratics?

THEORETICAL FRAMEWORK

This research focuses on the key topics of quadratic equations and functions in the South African Mathematics curriculum. These key topics form part of two content areas and involve a significant portion of teaching time. This significance is also felt in the assessment of these topics as indicated in tables 1-3.

In the USA, the National Research Council (2001) list five strands which learners have to satisfy to be mathematically proficient and are in keeping with the suggestions by the NCTM (2000) in its learning principles. These strands are intertwined and are shown below:

- Conceptual understanding: comprehension of mathematical concepts, operations, and relations
- Procedural understanding or fluency: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- Strategic competence: ability to formulate, represent, and solve mathematical problems
- Adaptive reasoning: capacity for logical thought, reflection, explanation, and justification
- Productive disposition: habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

Two of the stands mentioned “conceptual understanding” and “procedural understanding” are relevant to this study. Hiebert and Lefevre (1986) distinguish conceptual knowledge from procedural knowledge by saying that the former is identified by relationships between pieces of knowledge whereas the latter is identified as having a sequential nature.

Rittle-Johnson, Sieglar, & Alibali, (2001) define procedural knowledge as “the ability to execute action sequences to solve problems. This type of knowledge is tied to specific problem types and is not widely generalizable”. In contrast to procedural knowledge, they define conceptual knowledge as “implicit or explicit understanding of the
principles that govern a domain and of the interrelations between units of knowledge in a domain”. Conceptual knowledge is also flexible and can be applied to any problem type and is generalizable. (Rittle-Johnson, B, Sieglar, R.S. & Alibali, M.W. , 2001: 346 -347)

Although “procedural knowledge” and “conceptual knowledge” are different, Rittle-Johnson, et al (2001) stated that conceptual and procedural knowledge influence one another during a learners’ mathematical development. Both procedural knowledge and understanding as well as conceptual knowledge and understanding are relevant to this study as the analysis of the various questions on Quadratics would be done using elements of these knowledge and understanding types.

NB: The terms conceptual understanding and conceptual knowledge are used interchangeably; the same holds true for procedural knowledge and procedural understanding

November 2014 Examination questions

Mathematics Paper 1 November 2014

In the November 2014 paper (DBE, 2014 a), nearly the entire question 1 focussed on quadratic equations. In 1.1.1 the question \((2-x)(x+4)=0\) required learners to recognise that when the product of two factors is 0, then we equate each of the factors to 0. This question involved “procedural knowledge”. In 1.1.2 the question \(3x^2 - 2x = 14\) has to be solved “correct to TWO decimal places”. The procedure is given so once again “procedural knowledge” is used in this question.

Question 1.2 involved solving two equations simultaneously.

\[
\begin{align*}
"x &= 2y + 3 \\
3x^2 - 5xy &= 24 + 16y"
\end{align*}
\]

This question should be fairly straight-forward. Learners do know what procedure to use. They know they would have to work with the linear equation \(x = 2y + 3\) first by taking the \(x\) value and substituting this value into the quadratic equation, thus obtaining a quadratic equation in the variable \(y\). This question involves both “procedural knowledge” and “conceptual knowledge”. It is only through the understanding of concepts such as linear equation, substitution, variables and quadratic equation that learners would be able to use the correct procedure. However, by the time learners write the examinations it is likely that these concepts would become “second
nature” to them, thus ensuring that this question would involve mainly “procedural knowledge”.

The following two questions were set for 1.3 and 1.4

“1.3 Solve for $x$: $(x - 1)(x - 2) < 6$

1.4 The roots of a quadratic equation are: $x = \frac{3 \pm \sqrt{-k - 4}}{2}$

For which values of $k$ are the roots real?”

Question 1.3 would appear to be easy. Here, learners would have to simplify the left-hand side, bring the 6 to the left-hand side, then factorise the new expression on the left-hand side and then look for the region where the product of the two factors on the left are less than 0. This involves understanding the concept of quadratic inequality and then using the procedures learnt to solve the inequality.

In question 1.4, the roots of a quadratic equation are given. Here, learners have to analyse the roots in this form and then work out the value(s) of $k$ which give real roots. Learners would only know the procedure to work out this question if they knew how to analyse the roots in the manner displayed and then to work with the expression under the $\sqrt{\text{-}}$ symbol. Once again, this question required knowledge of the concepts of roots and real numbers.

Quadratics was introduced into question 3.1 where learners had to work with a quadratic sequence. This question was set as follows:

Given the quadratic sequence: $-1; -7; -11; p; \ldots$

“3.1.1 Write down the value of $p$.

3.1.2 Determine the $n^{th}$ term of the sequence.

3.1.3 The first difference between two consecutive terms of the
sequence is 96. Calculate the values of these two terms.”

The solution required in 3.1.2 is a quadratic expression and the memorandum shows four ways of working out the $n^{th}$ term of this sequence. Once learners get the value for $p$, then they would just have to follow certain procedures to get the $n^{th}$ term of
this sequence. Question 3.1.3 makes use of the answer arrived at in 3.1.2. Thus, this question is at a higher cognitive level and would require knowledge of the various concepts involved, rather than the procedure. If learners are not familiar with concepts such as “first difference” and “consecutive terms” then they would not be in a position to answer 3.1.3. Although the question involves obtaining then solving a quadratic equation, learners would have to do quite a bit of work before they can calculate the values requested.

In question 6.1, it is given that \( g(x) = 4x^2 - 6 \). To find the \( x \)-intercepts, learners have to equate \( g(x) \) to 0 and then solve for \( x \). This question involves procedural knowledge. In question 6.3.1 they had to write down the length of QKT in terms of \( x \) where \( x \) is the \( x \)-coordinate of K. The solution to 6.3.1 is indicated below:

\[
"QT = f(x) - g(x) \\
= 2\sqrt{x} - (4x^2 - 6) \\
= 2\sqrt{x} - 4x^2 + 6"
\]

Now \( QT \) is a combination of a quadratic and square root expression.

In 6.3.2, learners had to calculate the maximum length of \( QT \). First they had to find the derivative of the expression for \( QT \) and then equate to 0. The expression for the derivative \( x^{-\frac{1}{2}} - 8x \) also involves roots. If we let \( x^{-\frac{1}{2}} = k \), then the expression becomes \( \frac{1}{k} - 8k^2 \) which although not being truly quadratic, could be seen as being “quadratic in nature” (the highest power of \( k \) is 2). The solution to 6.3.2 is found by substituting the single \( x \)-value obtained after equating the derivative of \( QT \) to 0, into the above expression for \( QT \).

In question 9 they were given the function:

\[
"f(x) = (x + 2)(x^2 - 6x + 9) \\
= x^3 - 4x^2 - 3x + 18"
\]

In question 9.1, they had to calculate the coordinates of the stationary points \( f \). This is done by first finding the derivative of \( f \) and then equating to 0. Now we have:
\[ f'(x) = 3x^2 - 8x - 3 \]
\[ \Rightarrow 3x^2 - 8x - 3 = 0 \]

We notice that the derivative is a quadratic expression. When we equate this expression to 0, it becomes the solution to a simple quadratic equation. However, we do not only work with the \( x \)-values. We have to substitute these values into \( f \) to obtain the corresponding \( y \)-values and consequently the two turning points of the graph. Question 9.1 involves finding the derivative of the cubic function, which is procedural in nature, and then solving the resulting quadratic equation for \( x \) (since the expression is equal to 0 for turning points). This is also procedural knowledge. However, integrated into this question is the knowledge of concepts such as derivatives, why the derivative is 0 at the turning point and obtaining the \( y \)-value by substituting the \( x \)-values into the original function \( f \).

In question 10, a box and its net are shown. Question 10.1 and 10.2 builds up the formula for the volume \( V = h(50-h)(40-2h) \). Now question 10.3 is an interesting question:

“For which value of \( h \) will the volume of the box be a maximum?”

Here, the solution is obtained by first multiplying out on the right-hand side.

\[
V = h(50-h)(40-2h) \\
= h(2000-140h + 2h^2) \\
= 2h^3 - 140h^2 + 2000h
\]

Then to find the value of the height for which the volume is a maximum, learners should know that the expression is differentiated and equated to 0 which is “procedural knowledge”. The act of solving the resulting quadratic equation is also procedural in nature.

\[
V' = 6h^2 - 280h + 2000h = 0 \\
\Rightarrow 6h^2 - 280h + 2000h = 0 \\
\Rightarrow h = \frac{280 \pm \sqrt{(-280)^2 - 4(6)(2000)}}{2(6)} \\
\Rightarrow h = 8,8; \quad h \neq 37,86
\]
This question probably caught learners by surprise as it did not factorise. So, they would have to use the quadratic formula to solve for \( h \). Once they have the two values for \( h \) they would have to choose one value. Now learners should be in a position to discard 37.86 cm as a solution since 40 cm – 2(37.86) cm = – 35.72 cm and the length cannot be negative. This part, where the correct value for the height has to be chosen, depends on learners’ understanding of the concepts of volume and length in this context.

A summary of the questions involving Quadratics in paper 1 is shown in table 5.

**Table 5 Summary of questions involving Quadratics in Mathematics Paper 1**

<table>
<thead>
<tr>
<th>Question</th>
<th>The nature of the question</th>
<th>Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1.1</td>
<td>Quadratic equation in factorised form</td>
<td>2</td>
</tr>
<tr>
<td>1.1.2</td>
<td>Solving a quadratic equation through use of the formula</td>
<td>4</td>
</tr>
<tr>
<td>1.2</td>
<td>Solving two equations, one linear and one quadratic simultaneously</td>
<td>6</td>
</tr>
<tr>
<td>1.3</td>
<td>Solving a quadratic inequality</td>
<td>4</td>
</tr>
<tr>
<td>1.4</td>
<td>Analysis of roots of a quadratic equation</td>
<td>2</td>
</tr>
<tr>
<td>3.1.2</td>
<td>The ( n )th term of a quadratic sequence</td>
<td>4</td>
</tr>
<tr>
<td>3.1.3</td>
<td>Given the first difference between two consecutive terms of the quadratic sequence, calculate the value of these terms</td>
<td>4</td>
</tr>
<tr>
<td>6.1</td>
<td>Calculating the positive ( x )-coordinate of a quadratic function</td>
<td>2</td>
</tr>
<tr>
<td>6.3.1</td>
<td>Obtaining the expression for ( QKT ) in terms of ( x )</td>
<td>3</td>
</tr>
<tr>
<td>6.3.2</td>
<td>Determining the maximum length of QT</td>
<td>6</td>
</tr>
<tr>
<td>9.1</td>
<td>Calculating the coordinates of the turning point of a cubic function; the derivative, which is quadratic, is equated to 0</td>
<td>6</td>
</tr>
<tr>
<td>10.3</td>
<td>Optimising volume; the expression for volume is differentiated and equated to 0. The resulting quadratic equation can only be solved by using the quadratic formula.</td>
<td>5</td>
</tr>
</tbody>
</table>

**TOTAL** 48

A quick glance at table 5 shows that questions involving Quadratics comprised 48 out of 150 marks. This equates to 32% of the paper.
Questions involving Quadratics also appeared in Mathematics paper 2 (DBE, 2014b). In question 3.2, learners had to read the given stimulus, in conjunction with the diagram, identify the centre and radius and write it in the appropriate form.

The answer, which is \((x - 5)^2 + (y - 4)^2 = 25\), was quadratic in nature. This was a familiar question and involved mostly procedural knowledge as the form of the answer was given.

Question 3.3 involved the use of the question 3.2. Here, learners had to calculate the coordinates of the smaller of the two \(x\)-intercepts of the circle. This was also an easy question. Learners had to know that on the \(x\)-axis, \(y = 0\) and then the working out of the solution becomes procedural.

Thus, \(y = 0\) is substituted in the equation.
\[(x - 5)^2 + (0 - 4)^2 = 25\]
\[\Rightarrow x^2 - 10x + 25 + 16 = 25\]
\[\Rightarrow x^2 - 10x + 16 = 0\]
\[\Rightarrow (x - 8)(x - 2) = 0\]
\[\Rightarrow x = 8 \text{ or } 2\]

The smaller one is chosen so the coordinates of A is (2;0).

In question 3.8, the equation of the circle through the points K, L and M had to be calculated. Learners had to prove that ML was the diameter and that the midpoint of ML was the centre of the circle. These ensured that the radius and coordinates of the centre of the circle could be calculated and then substituted into the given form of the equation. The final answer was \((x-12,5)^2 + (y-6,5)^2 = 62,5\). This required quite a bit of analyses and interpretation of the given situation. This question was of a high cognitive demand and would be underpinned by conceptual rather than procedural knowledge.

The identity in 6.1 simplifies to \(\cos^2x - \sin^2x\) and this is equal to \(1 - 2\sin^2x\) or \(2\cos^2x - 1\) and these are quadratic in nature. We know that all three of these are equal to the right-hand side, \(\cos2x\). This question tended to be more procedural in nature as the structure of the question dictated the approach to be followed. Once they simplified the left-hand side, they could go to the information sheet for further assistance.

In question 6.3, it was given that \(\sin76^\circ = x\) and \(\cos76^\circ = y\), and learners had to do some working to show \(x^2 - y^2 = \sin62^\circ\). This involves the use of double or compound angle identities. The left-hand side \(x^2 - y^2\), is clearly quadratic in nature. The national memorandum showed three approaches to obtain this proof, meaning that a procedural approach would not have been suitable for this question. This question depended on understanding the concepts of compound and double angles and their relevance in answering this question.

In question 7.2, learners had to show that \(\sin x + 1 = \cos2x\) can be written as \((2\sin x + 1)(\sin x) = 0\). The left-hand side is in factorised form and is used in
question 7.2 to determine the general solution. If one has to expand the left-hand side then we get \(2 \sin^2 x + \sin x = 0\). This is a quadratic equation in \(\sin x\). Since the left-hand side was already factorised, the question would be classified as being “procedural” in nature.

In the DBE exemplar paper for grade 12 in 2014 (DBE, 2014 c), the following question was set to capture the understanding of the general solution of trigonometric equations.

“Determine the general solution of \(\cos 2\theta + 4 \sin^2 \theta - 5 \sin \theta - 4 = 0\).”

This question requires learners to replace \(\cos 2\theta\) by \(1 - 2 \sin^2 \theta\) and then the equation becomes a quadratic equation in \(\sin \theta\). In simplified form, this quadratic equation is \(2 \sin^2 \theta + 5 \sin \theta - 3 = 0\) which is easily solved through factorisation. Although this question could be regarded as straightforward, it would be leaning towards conceptual understanding as learners would not be able to work out the solution if they did not know the appropriate expansion for \(\cos 2\theta\). Further, it would require learners to understand the concepts of “general solution” and the writing of the answer(s) in a certain form; an acute or obtuse angle in combination with \(\text{k} \cdot 360^\circ\).

In question 8.3.2 of the November 2014 paper learners had to calculate the length of AB, which was a tangent to the circle.
Question 8.3.1 was given as support or background for learners to answer this question.

In 8.3.1 learners were given the statements that $\hat{ABC} = 90^\circ$ and $AB = x$ and had to provide reasons for these statements. They had to use this information in 8.3.2. $\Delta ABC$ is a right-angled triangle and the theorem of Pythagoras will be used. This question is worked out as follows:

"$AB^2 + BC^2 = AC^2$ (theorem of Pythagoras) \\
$\Rightarrow x^2 + (x + 7)^2 = 13^2$ \\
$\Rightarrow x^2 + x^2 + 14x + 49 = 169$ \\
$\Rightarrow x^2 + 7x - 60 = 0$ \\
$\Rightarrow (x - 5)(x + 12) = 0$ \\
$\Rightarrow x = 5; x \neq -12$"

Once again there was a very simple quadratic equation to solve. The background information in 8.3.1 made it easier for learners and would be more procedural in nature. Only in the selection of the appropriate $x$-value would concepts play a role, that is, length can never be negative.

A summary of the questions involving Quadratics in the 2014 Mathematics paper 2 is shown in table 6.

Table 6 Summary of questions involving Quadratics in Mathematics Paper 2

<table>
<thead>
<tr>
<th>Question</th>
<th>The nature of the question</th>
<th>Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2</td>
<td>Equation of a circle in a given form</td>
<td>1</td>
</tr>
<tr>
<td>3.3</td>
<td>Calculation of smaller $x$-intercept of circle</td>
<td>3</td>
</tr>
<tr>
<td>3.8</td>
<td>Equation of a circle in a given form</td>
<td>5</td>
</tr>
<tr>
<td>7.2</td>
<td>General solution of trigonometric equation</td>
<td>2</td>
</tr>
<tr>
<td>6.1</td>
<td>Proving an identity (involving squaring)</td>
<td>5</td>
</tr>
<tr>
<td>6.3</td>
<td>Prove an identity; the left hand side is quadratic</td>
<td>4</td>
</tr>
<tr>
<td>8.3.2</td>
<td>Calculate the length of tangent AB using the theorem of Pythagoras</td>
<td>4</td>
</tr>
<tr>
<td>TOTAL</td>
<td></td>
<td>24</td>
</tr>
</tbody>
</table>

A glance at table 6 shows that questions, of a quadratic nature or involving quadratics, comprised 24 out of 150 marks. This equates to 16% of the paper.
DISCUSSION

A detailed scrutiny and analyses of the 2014 grade 12 mathematics examination papers shows that there is a strong emphasis on questions involving quadratic equations and functions. These type of questions comprised 32% of paper 1 and 16% of paper 2. In total, 24% of the total marks for mathematics, that is, 72 marks came from questions involving quadratic equations and functions. It should be noted that the analyses earlier in tables 1 to 4 only makes provision for the location of quadratic equations and functions in Mathematics paper 1. It is not possible to ascertain, in advance, what type of questions involving Quadratics would feature in paper 2. However, one could probably expect such questions in Analytical Geometry (working with circles), Trigonometry (trigonometric expressions and equations) and Geometry (questions featuring the theorem of Pythagoras). This would mean that teachers need to draw their learners’ attention to such occurrences during the teaching of the appropriate paper 2 topics. If learners are given a proper foundation in quadratic equations and functions (and being able to recognise these in different contexts), then there is no reason why learners should not do well in these sections and overall in mathematics. However, it would mean that teachers should prepare well and make learners aware of how the various sections are linked and connected.

Implications for teaching and learning

There is a tendency for mathematics teachers to treat the various topics in the curriculum as separate, without regard for linkages and connectedness to work done previously in such topics. The research by Li (2011) points to the important role by teachers of mathematics, especially with regard to their content knowledge and preparedness for their classrooms. This was in sharp contrast to the study by Makgakga (2014) who found that the teachers’ approaches contributed towards learners’ errors and misconceptions.

The author of this study believes that teachers’ preparedness, content knowledge and sound pedagogical approaches are key factors when teaching mathematics. This would ensure that children would learn mathematics in a more structured and coordinated manner. This study on Quadratics has shown how working “smartly” in one section of mathematics can impact positively on other sections as well. In this regard, the following recommendations are made, with respect to the teaching and learning of Mathematics, in general, and Quadratics, in particular:

- Teacher preparedness is the key for successful mathematics lessons. In this regard, the injudicious use of textbooks should be discouraged. Teachers should be able to interpret content in textbooks and adapt such
content to suit one’s own learners. This calls for teachers to look at the curriculum on a wider level so that the necessary connections can be made during the teaching. This paper on Quadratics has shown how important it is for learners to be made aware of such connections.

- For any mathematics lesson, it is important to take note of learners’ prior knowledge and abilities which may impact or influence such learners. Teachers can build on what has been previously learnt. For example, in a lesson on trigonometric equations involving a trinomial such as the one from the exemplar paper which simplifies to \(2\sin^2 \theta + 5\sin \theta - 3 = 0\), teachers could ask learners about the type of equation that must be solved. Learners are likely to say that it is a trigonometric equation; but teachers should ask learners what would happen if \(\sin \theta\) were replaced by \(k\). They would get the equation \(2k^2 + 5k - 3 = 0\) which is a quadratic equation.

- In the FET, learners are likely to work with linear, quadratic and cubic functions. Teachers may ask learners for similarities and differences, when introducing these functions. For example, when determining the turning points of a cubic function, learners have to differentiate the function and equate to 0. Now the derivative of a cubic function \(f\) results in a new function \(f'\) which is a quadratic function. So to find the turning points of \(f\) the derivative \(f'\) is equated to 0; this involves solving a quadratic equation. In the diagram that follows the graphs of \(f(x) = x^3 - 4x^2 - 3x + 18\); its derivative \(f'(x) = 3x^2 - 8x - 3\) and its second derivative \(f''(x) = 6x - 8\) are drawn on the same system of axes. Learners may be asked to use the first and second derivatives to come up with properties of the cubic function \(f\).
• Learners tend to do well in quadratic equations when it forms part of paper 1. However, it is important that learners are made aware of where quadratic equations are likely to feature in paper 2 (see discussion). These types of examples should be discussed in class so that learners are better prepared to work through these questions with confidence.

• Another area where quadratics is very likely to feature is in Physics. For example, in kinematics, the formula for distance \( s = ut + \frac{1}{2}at^2 \) is quadratic in nature. Learners who have been taught Quadratics in a meaningful and connected manner should have no problem with this section of Physics.

CONCLUSION

Quadratics (or quadratic equations and functions) is a critical part of the FET mathematics curriculum in South Africa. This study has shown that the assessment of this important section of the work, especially at grade 12 should have a direct bearing on its teaching and the way it is learned. Its relevance and importance transcends Mathematics paper 1 and paper 2 and beyond. Should teachers take note of and
implement the recommendations as stated in the previous section of this paper, there is no reason why learner performance in these content areas, and Mathematics as a whole, should not improve. However, a lot will depend on teachers’ knowledge and pedagogical approaches.

REFERENCES


Problem solving is an integral part of Mathematics teaching and learning. This paper probes the views of primary and high school teachers on problem solving. It would seem that teachers are aware of the importance of problem solving and the need to pay more attention to this part of their Mathematics teaching. Also, the results of problem solving contests for both sets of teachers suggests that the teachers, themselves, are not competent in problem solving, thus, leading to a lack of confidence. It is clear that teachers need a great deal of support on this crucial aspect of their work and some suggestions, in this regard, are given at the end of the paper.

INTRODUCTION

One of the aims of the National Curriculum Statement Grades R-12, in South Africa, is to produce learners who are able to “identify and solve problems and make decisions using critical and creative thinking” (DBE, 2011a).

With problem solving being an integral part of Mathematics, it is clear that Mathematics is one of the key subjects in the South African curriculum which is able to achieve this aim. The term “problem solving” refers to mathematics tasks that have the potential to provide intellectual challenges for enhancing children’s mathematical understanding and development (NCTM, 2010)

In the South African context, four cognitive levels of assessment are prescribed. These are Knowledge (level 1), Routine Procedures (level 2), Complex Procedures (level 3) and Problem Solving (level 4) (DBE, 2011a: 53). The table below shows the percentage allocation of the “Problem Solving” cognitive level to assessment at various phases.

Table 1: School phases and problem solving

<table>
<thead>
<tr>
<th>Phase</th>
<th>Problem Solving</th>
</tr>
</thead>
<tbody>
<tr>
<td>FET</td>
<td>15%</td>
</tr>
<tr>
<td>Senior Phase</td>
<td>10%</td>
</tr>
<tr>
<td>Intermediate Phase</td>
<td>10%</td>
</tr>
<tr>
<td>Foundation Phase</td>
<td>Integrated into class activities; rubrics given on how to assess problem solving</td>
</tr>
</tbody>
</table>
Table 1 show that learners across the phases should be involved in “problem solving”. In this regard, the National Curriculum Statement Grades R-12, commonly known as the Curriculum and Assessment Policy Statement or CAPS, gives examples of the types of problem solving questions which could be done in the mathematics classroom.

**Examples of Problem Solving type questions**

The CAPS document gives the following examples of problem solving type questions:

*Grade 4:* “The sum of 3 consecutive whole numbers is 27. Find the numbers”

*Grade 5:* “Heidi divided a certain number by 16. She found an answer of 246 with a remainder of 4. What is the number?”

*Grade 6:* “Busi has a bag containing six coloured balls: 1 blue; 2 red balls and 3 yellow balls. She puts her hand in the bag and draws a ball. What is the chance that she will draw a red ball? Write the answer in simplest fraction form

*Grade 7:* “The sum of three consecutive numbers is 87. Find the numbers.”

*Grade 8:* “Mary travels a distance (in km) in 6 hours if she travels at an average speed of 20 km/h on her bicycle. What should be her average speed if she wants to cover the same distance in 5 hours”

*Grade 9:* “The combined age of a father and son is 84 years. In 6 years time the father will be twice as old as the son was 3 years ago. How old are they now?

*Grade 10 -12:* “Suppose a piece of wire could be tied around the earth at the equator. Imagine that this wire is then lengthened by exactly one metre and held so it is still around the earth at the equator. Would a mouse be able to crawl between the wire and earth? Why or why not”

(DBE, 2011a ,b, c)

One observes that these questions increase with difficulty. For example in grade 4, the sum of the three consecutive whole numbers is 27 while in grade 7 the word “whole” is left out and the sum is higher. Teachers are encouraged to work through these types of examples in the mathematics classroom in an effort to develop the problem solving skills of their learners. These questions would not be out of place in mathematics
competitions such as the “South African Mathematics Challenge (SAMF)” for grade 4 – 7 learners and the “South African Mathematics Olympiad (SAMO)” for grade 8 – 12 learners. So if teachers place some emphasis on problem solving as is suggested by the CAPS document, then they are also, indirectly, preparing their learners for problem solving competitions and contests.

**Some related literature on problem solving**

There have been several studies on problem solving and some recent South African ones are highlighted here.

Peters (2011) investigated mathematics teachers’ beliefs and knowledge in a grade 7 mathematics class. She also investigated how teacher beliefs and teacher knowledge was used to incorporate word problem-solving and problem solving strategies in a given task. Some of the themes emerging from this study were:

- Learners’ attitudes towards solving word problems varies;
- Learners find solving word problems difficult;
- Teacher experience influences teaching and learner performance;
- Teaching strategies used in the classroom are not assisting learners;
- Learners have a backlog in problem solving strategies.

Govender (2012) investigated teacher preparation for integrating “problem-solving” type questions into Mathematical Literacy lessons. He found that teachers had identified a wide range of knowledge and skills when working with “problem-solving” type questions. This included reading and understanding questions, noting the numeric data in such questions and determining the operations to be used.

Sepeng and Webb (2012) explored whether discussion as a teaching strategy in mathematics classrooms could improve learners’ problem solving abilities and to make sense of real “word problems”. The main finding of this study revealed that where the discussion strategy was successfully implemented, there was a statistically significant improvement in the learners’ competence in solving word problems.

A common feature from these studies is the role played by the teacher. In this regard, the experience of the teacher, the teachers’ knowledge and the use of appropriate teaching strategies are key factors when it comes to problem solving in the mathematics classroom.

However, there are difficulties which are associated with problem solving in the classroom and some of these are:
Teachers are not usually comfortable with problem solving activities;
Learners may become insecure;
There is not enough time for problem solving and it takes too long to teach;
There may be challenges when working with below average mathematics learners;
It requires a lot of preparation on the part of the teacher.

(Difficulties of teaching problem solving, 2010)

Despite the difficulties associated with the teaching of problem solving, there are several benefits of problem solving. Some of these are listed here:

- Learners are able to develop mathematically using their current knowledge;
- It is an interesting and enjoyable way to learn mathematics;
- It is a way to learn new mathematics with greater understanding;
- It produces positive attitudes towards mathematics;
- It teaches thinking, flexibility and creativity;
- Children are able to learn general problem solving skills;
- It encourages cooperative skills.

(Benefits of problem solving, 2010)

The benefits associated with the teaching of problem solving make it imperative that mathematics teachers take this part of their work very seriously. There are three critical components to effective mathematics teaching. These are:

- Teaching for conceptual understanding;
- Developing children’s procedural literacy;
- Promoting strategic competence through meaningful problem-solving investigations.

(Shellard & Moyer, 2002)

The third component “Promoting strategic competence through meaningful problem-solving investigations” provides the basis on which this study was conducted.

Research question

This research focuses on a group of primary school and high school teachers of mathematics and their views on problem solving and its incorporation in their lessons. Thus, the following research question was developed for this study.
“What views do mathematics teachers have on problem solving and how do these views influence their incorporation of problem solving activities in their mathematics classrooms?”

To answer the research question the following subsidiary questions were formulated in the context of the research question:

- What do mathematics teachers know about problem solving?
- Are teachers aware of the similarity and/or difference between “problem solving” and “solving word problems”?
- Do teachers see the need for including problem solving activities in their mathematics classrooms?
- Is it important for learners to be involved in problem solving activities?
- Are learners encouraged to participate in problem solving competitions such as Mathematics Challenges and Competitions?
- Can teachers themselves do problem solving?

Sample

18 teachers participated in this research activity, six high school teachers and 12 primary school teachers. They participated voluntarily in this activity and came from a wide range of schools.

Operational strategy

Due to the differences in the type of problem solving activities which high school and primary school learners should be involved in, separate sessions were held for the teachers. However, each session had a similar format. These were:

- At the beginning the teachers were given the first part of a questionnaire to complete. This part elicited responses on their understanding of “problem solving”, the similarities and/or differences between “problem solving” and “solving word problems” and the incorporation of problem solving activities in their classroom.
- This was followed by a presentation on “problem solving”, which is shown later in this paper.
- Immediately after this presentation, they completed the second part of the questionnaire. The second part of the questionnaire sought their responses on why it is important for learners to do problem
solving; whether learners get enough support or training on problem solving; participation in Mathematics Challenges/Olympiads/Competitions and to give any other comment(s) on problem solving.

- The session ended with teachers participating in a problem solving contest, made up of questions appropriate to the grade(s) they were teaching.

THEORETICAL FRAMEWORK

Since this research involved problem solving, a suitable framework for this research would be Problem-Based Learning (PBL). PBL fosters both inquiry- and knowledge-based approaches to problem solving. As an inquiry-based approach, its focus is on helping teachers work through authentic and complex problems (Bereiter and Scardamalia, 2006). As a knowledge-based approach, it lets prospective or inexperienced teachers create a rich foundation for solving similar or more serious problems in the classroom, thus preparing them for their future learning (Schwartz, Bransford, and Sears, 2005). Elements of both the inquiry-based and knowledge-based approaches of PBL featured in this research.

In this regard, the research probed the views of both primary and high school mathematics teachers on problem solving and how they were able to incorporate problem solving activities in their classrooms. Some problem solving strategies were discussed and used in certain problems, thus, covering the inquiry-based part of PBL. This research also involved teachers participating in a competition to ascertain whether they were able to work with mathematics questions from Mathematics challenges and Olympiads and to, thus, create a “rich foundation” for solving similar problems and to probably enable them to teach their learners how to solve such problems, ensuring that the knowledge-based part of PBL was covered as well.

Since the views and experiences of the teachers with regard to problem solving were the objects of this research, the research was mainly qualitative in nature (Hatch, 2002). Quantitative data (the marks obtained in the problem solving contest) was also collected in this study. As this study involved both the description and an interpretation of the processes involved, it would be located in the interpretive research paradigm (Bassey, 1999).

RESULTS: THE DATA

In the writing up of the results of this research, with reference to the questionnaires and problem solving competition results, the data for the primary school teachers are
recorded separately from that of the high school teachers. Although the responses to the questionnaire represent the exact words of the teachers, some editing has been done to correct spelling and grammatical mistakes in the responses.

**Questionnaire (part 1)**

In response to the question on what they understood by problem solving in mathematics, the primary school teachers provided the following responses:

“Finding solutions to a given problem”; “To solve a problem by moving from the known to the unknown or from simple to difficult”; “To systematically and logically try to find the answer to a given problem”; “Solve problems by building puzzles and using different materials and shapes”; “Read and understand a problem and then find ways to solve it”; “Learners using various strategies to solve mathematics problems”; “To take a complicated mathematics problem, break it down or simplify it to learners so they can grasp the concepts involved and solve the problem”

The high school teachers responded as follows:

“It covers any type of problem in mathematics and also develops learners to think constructively and be able to deal with real life situations”; “A way of solving or getting a solution for certain abstract questions in mathematics”; “It is when we use given information to investigate/solve things we don’t know yet”; “Solving a problem with no clear solution or recipe”; “Ways of tackling problems”; “Develops thinking ability and communication skills”

All the primary school teachers responded in the affirmative that there was a difference between “problem solving” and “solving word problems”. Some of the reasons given were:

“You use problem solving in solving word problems but not the other way round”; “Problem solving gives you the operations while when you solve word problems, you look for keywords to help you solve the problem”; “Solving word problems refers to answering a problem in a story while problem solving refers to finding a solution”; “Problem solving can include many different types of problems while word problems require the child to understand what they are reading to solve the problems and involves Mathematics and English integrated learning”; “Problem solving involves steps and different methods while solving word problems requires a given question to be solved in two or three steps”; “With problem solving you have to find a method while with word problems the method is already implied”; “Problem solving involves solving any mathematical problem while solving word problems means to answer a mathematical question that is asked using words and sentences”; “Problem solving involves thinking outside the box and experimenting with
methods to get an answer while with word problems you have to take the information out of the sum to come to a conclusion and deliver an answer.”

Three of the high school teachers responded in the affirmative, with only one giving a reason for this choice, saying that:

“Solving word problems is part of problem solving.”

Only one of the other three teachers gave a reason for responding in the negative, stating that:

“Problem solving and solving word problems are the same. They are just written or interpreted in words and need to be formulated into equations.”

All primary school teachers in the sample responded in the affirmative that problem solving is an integral part of mathematics teaching and learning, citing the following reasons:

“If you are not taught to think creatively when faced with problems in maths, I believe it will be reflected in the solutions you choose with respect to the problems you face in life”; “Nearly all mathematics can be used to obtain the necessary information”; “It helps learners to discover, explore and find ways to solve problems in their daily lives”; “Learners thinking ability is tested and developed”; “Because through life, one can always need to solve problems; finding a strategy that works for you allows you to experience success”; “Helps learners to see how they come to their answers; easier to identify mistakes”; “Children learn to expand their thinking and find different solutions to solve problems”; “Maths needs problem solving as every mathematical sum or question is a problem to be solved” “It is important for learners to follow processes to help with logical thinking and developing creative processes”; “It is because with any sum whether it is geometry or just any of maths, they would hope to find a solution to that sum; they would have to solve that problem.”

The high school teachers also responded in the affirmative and provided the following reasons:

“It deals with real life situations and teachers should be able to facilitate learners to analyse and interpret problems and then find solutions”; “It engages both the learners and educators”; “It helps learners solve mathematical problems in real life situations”; “If you have a mathematical problem you must decide which knowledge and methods you are going to use but you first have to analyse the problem”; “It should be but in reality it is not being done”; “It involves real life situations in context and involves the analysis and interpretation of mathematical problems.”
All the primary school teachers indicated that they incorporated problem solving activities in their teaching. One also mentioned that separate sessions are also held for learners. Their reasons are captured below:

“These activities should be included in class to encourage problem solving thinking and challenge learners but should be “moderate” so as not to “scare off” the weaker learners from mathematics. There are separate sessions aimed at the top achievers to strengthen their skills”; “It helps learners develop an understanding of problem solving”; “It is part of the activities of my lesson”; “It is important to show the children that maths can be included into almost anything. Therefore, at times, where possible, I incorporate problem solving into my classroom teaching”; “I incorporate problem solving activities in the classroom by ask learners to solve problems: they cannot give one word answers and need to show their steps”; “I use concrete examples with learners by explaining and solving problems with objects and give separate worksheets where learners solve problems using methods they are familiar with”; “Brainteasers are used before the maths lessons as a warm up activity. For example, how many triangles are there in the diagram below?”

Once again, the high school teachers responded in the affirmative about incorporating problem solving activities in their classrooms.

“I incorporate problem solving activities because I want the learners to relate these things to the real world”; “I incorporate problem solving using real-life situations”; “I prefer to integrate problem solving activities. Mathematics learners must be detectives who analyse the problem, look at the evidence and then decide on a strategy”; “It forms part of my lesson”; “I incorporate it only if there is time; otherwise it is done separately.”

Some trends emerging from the first part of the questionnaire

The responses of both sets of teachers were analysed with a view to detecting trends and patterns of coherence.
All teachers in the sample had some ideas of problem solving in the classroom. They were able to describe some mathematical principles such as “moving from the known to the unknown” or the steps involved in problem solving. Other views focused on developing learners’ ability to think and to deal with real-life situations. The views of two teachers are now highlighted as these may be more significant in the context of this research. A primary school teacher spoke about “learners using various strategies to solve problems” and a high school teacher stating that “it is when we use given information to investigate/solve things we don’t know yet.”

The teachers stated there were differences between “problem solving” and “solving word problems”. It is noteworthy that many of them knew that problem solving involved more than just solving word problems. This is in keeping with the view that when researchers use the term problem solving, they are referring to mathematical tasks that can challenge learners and help in their mathematical development. This is unlike some word problems or story sums which are very basic and do not sufficiently challenge learners (NCTM, 2010).

All teachers agreed that problem solving was an integral part of mathematics. They stated that they incorporated problem solving activities in their lessons, providing some very compelling reasons for doing so. Only one teacher was able to provide a problem solving example which could be incorporated into a mathematics lesson.

Problem solving presentation

Some of the main areas of the presentation were:

- Despite the various changes to the Mathematics curriculum in South Africa since 1994, problem solving has been a key feature of all the curricula.
- The role of the teacher in developing and implementing problem solving in the mathematics classroom.
- Key conventions of mathematics and teaching principles which should underpin our teaching.
- A review of some grade appropriate problems which feature in Mathematics Challenges/Olympiads/Challenges. This review included some strategies in solving these problems.
In response to why they thought it was important for learners to do problem solving, the primary school teachers responded as follows:

“To develop creative ways of thinking and a better and more logical approach to problems both in Mathematics and the real world”; “It develops their way of thinking”; “Learners can find answers in their daily lives”; “It can help them become good thinkers”; Learners can expand their thinking and use the problem solving to build on their prior knowledge and make it as part of their living”; “To develop the necessary skills for their daily lives; to improve their logical thinking and creative abilities”; “Because in everyday life there are going to be problems which they will have to solve and they should be prepared for.”

For the same question, the high school teachers stated:

“So they can remember why a certain rule for problem solving is given; to introduce rules”; “Problem solving challenges learners and forces them to think about different methods”; “If learners are successful in one problem; it will give them self-confidence and they will try to solve more difficult problems”; “It enables them to think logically and to solve problems using different strategies”; It develops thinking and the ability to communicate or explain solutions.”

All the primary school teachers reported their learners do not get enough support or training in problem solving. Some of their responses are recorded below:

“I would say yes; but the answer is probably not as I know there is still more I could do to support them”; “I think if the teacher is well-equipped then the teacher could help learners become good thinkers”; “More support is required at this level”; “Not where assessment is concerned; teachers need to incorporate problem solving during their everyday lesson planning in order to give the support that learners need”; “I feel that teachers are pressed for time in regard to the CAPS curriculum. Learners do the work to keep up and not to understand and apply. They are getting with the pace as time passes.”

There were similar responses by the high school teachers who also said that their learners do not get enough support or training in problem solving. Only two teachers explained further. These are captured below:

“I don’t think learners get enough support as they should”; “Teachers struggle to teach the “recipes” and homework is not a favourite activity for learners.”

All the primary school teachers agreed that participation in Mathematics Challenges/Competitions/Olympiads would help improve learner problem solving.
“These are not regular class-based questions so it challenges them on a different level and exposes them to new experiences at the same time”; “We don’t always have the time or knowledge to help them”; “The more they practice the more experience they get”; “It will expand learners knowledge and it will encourage them to put in extra efforts in their studies”; They would need to practice with different problems and their thinking and methods will expand”; “Learners will be challenged to think logically and with time pressure. A sense of achievement will inspire them to participate more; encouragement is the key recipe”; “They would need to practise and to prepare for these maths challenges; they would be involved in problem solving and they will improve.”

The high school teachers also agreed that participation in Mathematics Challenges/Competitions/Olympiads would help improve learners’ problem solving. They provided the following reasons:

“It will give them exposure and techniques of solving some maths problems”; “It can give them a different insight to problem solving”; “There are previous Olympiad papers that can be used to let learners practice”; “It exposes them to different ways of solving problems and also makes them feel part of a bigger group of mathematics teachers”; “Being able to learn from other learners and learn more methods and strategies to deal with maths.”

Space was provided on the questionnaire to capture other comments from the teachers about problem solving. The primary school teachers provided the following additional comments:

“I think we need to do more as teachers to encourage creative thinking and problem solving in our children as ultimately they are going to be the leaders of our country in the future and there are many challenges that they could be faced with”; “Our lessons will be more interesting if we incorporate problem solving”; “The presentation was an eye-opener to me”; “We need to be more “hands-on” in class”; “Teachers need to use problem solving in everyday lesson planning and encourage learners to try different methods”; “Materials should be available for learners and there should be worksheets which teachers can use. This would make learning more fun and interesting.”

The high school teachers provided these additional comments:

“I think the educators need to be trained or given workshops in problem solving”; “I enjoyed the introduction of using problem solving to answer the questions of why we multiply first and then add”; “Learners must realise that all mathematics were created to simplify problems and find solutions; not the opposite”; “Problem solving helps learners to investigate mathematics”; “Problem solving should be taught to educators so that they know how to teach it to their learners.”
Some trends emerging from the second part of the questionnaire

The second part of the questionnaire was answered after the problem solving presentation. Once again, the responses of both sets of teachers were analysed with a view to detecting trends and patterns of coherence.

- All the teachers agreed that it was important for learners to do problem solving in the classroom. Most of the teachers stated that problem solving assisted in the development of learners thinking ability with some also mentioning logical and creative reasoning. It was also about building self-confidence in learners.
- There was consensus among the teachers that their learners do not get enough support or training on problem solving. Various reasons were given. Some of the reasons given were “lack of time”, teacher preparedness and learner apathy.
- They knew about Mathematics Competitions and the importance of their own learners participating in such competitions. However, it would appear that they were not sure about being able to help their learners prepare for such competitions.
- The teachers provided further information on the need to include problem solving activities in their classrooms. One interesting comment is shown below:

  “I think we need to do more as teachers to encourage creative thinking and problem solving in our children as ultimately they are going to be the leaders of our country in the future and there are many challenges that they could be faced with.”

- Two high school teachers emphasised the need for teachers to be trained in problem solving. Their responses are captured here again:

  “I think the educators need to be trained or given workshops in problem solving”; “Problem solving should be taught to educators so that they know how to teach it to their learners”.

Problem solving contest for teachers

Each group of teachers then participated in a “problem solving” contest. The questions for the primary school teachers were obtained from the South African Mathematics Challenge (for grades 4 – 7) papers of recent years. Questions from previous years
South African Mathematics Olympiad (SAMO) papers for grades 8 – 12 were included in the problem solving contest for high school teachers.

Each paper consisted of 15 questions. The mark allocation was similar to what is used in the South African Mathematics Team (SAMT) Competition. Questions 1 – 10 were six marks each and questions 11 – 15 were eight marks each, making each paper out of 100. If no answer was given then one mark was scored. A wrong answer received no marks. The marks obtained by the teachers are shown in the next two tables.

**Table 2: Marks of the primary school teachers**

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Mark</th>
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<tbody>
<tr>
<td>A</td>
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<td>B</td>
<td>32</td>
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<td>C</td>
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<td>I</td>
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<td>J</td>
<td>47</td>
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<td>K</td>
<td>48</td>
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<tr>
<td>L</td>
<td>58</td>
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**Table 3: Marks of the high school teachers**

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<th>Teacher</th>
<th>Mark</th>
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<tbody>
<tr>
<td>M</td>
<td>66</td>
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<td>60</td>
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<tr>
<td>O</td>
<td>59</td>
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<tr>
<td>P</td>
<td>38</td>
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<tr>
<td>Q</td>
<td>36</td>
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<tr>
<td>R</td>
<td>13</td>
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</tbody>
</table>

The marks obtained by the teachers give an overall picture of their results in the problem solving contest. It would appear that the primary school teachers struggled. If there was a second round paper and the qualifying mark was say, 50 %, then only one teacher (out of 12) would have made it to the second round. This could probably point to the fact that these teachers are not really experienced at working with problem solving questions and triangulates with their earlier pronouncements of not doing enough problem solving in their own classrooms. Using the same criteria, three of the six high
school teachers would have made it to the second round. Once again, the lack of experience in problem solving activities would have probably accounted the marks obtained. It is also useful to note what types of questions the teachers had difficulty with. Tables 4 and 5 give a question-by-question analysis of each paper.

**Table 4: Primary school paper (question-by-question analysis)**

<table>
<thead>
<tr>
<th>Question</th>
<th>Number wrong</th>
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<tbody>
<tr>
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<td>4</td>
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<td>14</td>
<td>10</td>
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<td>15</td>
<td>11</td>
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**Table 5: High school paper (question-by-question analysis)**

<table>
<thead>
<tr>
<th>Question</th>
<th>Number wrong</th>
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<tbody>
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<td>1</td>
<td>1</td>
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<td>2</td>
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<td>14</td>
<td>4</td>
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<td>15</td>
<td>5</td>
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</table>
Using a yard-stick that teachers should give correct responses to at least 50% of the questions then the following questions challenged teachers:

Primary school teachers: Questions 2; 3; 4; 5; 9; 12; 14; 15

High school teachers: Questions 6; 12; 13; 14; 15

It would appear that the high school teachers did slightly better than the primary school teachers. The questions which the primary school teachers had difficulty with is shown in annexure A and the questions which the high school teachers had difficulty with is shown in annexure B. The teachers had difficulty with a wide range of questions. These were not confined to any specific topic or content area. Should their learners come to them for assistance with these types of questions, they would not be able to provide this assistance. This is likely to continue unless teachers start doing more problem solving questions themselves or get support or training to implement problem solving in their classrooms.

FINDINGS

The findings of this research are written with the research question and subsidiary questions in mind.

- It would appear that the teachers were knowledgeable about problem solving and its importance of including problem solving activities in their classrooms. They were aware that there were differences between “problem solving” and “solving word problems” but were not able to clearly explain these differences.

- They believed that it was important for learners to be involved in problem solving activities in order to impact positively on their mathematical ability and to develop their mathematical thinking. However, they conceded that they do not spend enough time on problem solving in their classrooms. Although various reasons were given, these reasons appeared to be “excuses”. This is rather disappointing as it would seem that learners do not get much needed practice in problem solving, especially for external examinations.

- Although they were aware of Mathematics contests such as the South African Mathematics Challenge and the South African Mathematics Olympiad and its importance in exposing their learners to such contests, some of them conceded that these contests were “difficult” and they would not be able help their learners to prepare for such contests. One
can infer that their learners probably do not participate in such contests as the teachers themselves, are not confident about helping them. However, only two of them indicated that there should be workshops or training on problem solving for teachers.

- This possible lack of confidence in the teachers came to the fore in the problem solving contests. They did not fare well in the problem solving contests. Only one primary school teacher and three high school teachers would have qualified to write the second round of the contest (if one existed). It is probably due to a lack of experience or lack of exposure to problem solving that the teachers fared poorly. One cannot expect teachers who are not confident about problem solving to expose their own learners to problem solving. However, one cannot leave it at that. More should be done about supporting teachers in problem activities at all school levels. There are initiatives by the South African Mathematics Foundation, various universities and non-governmental organisations which target teachers and learners with respect to problem solving and participation in Mathematics Competitions and contest. These initiatives should be supported by the various provincial departments of education and education districts if there is to be “change” in our mathematics classrooms.

CONCLUSION

Problem solving should be an integral part of mathematics teaching and learning. Although one can distinguish between problem solving in the classroom and mathematics competitions such as Mathematics Challenges/Olympiads, the kinds of problem solving activities that are implemented in the classroom could also help prepare learners for these competitions. The sad reality is that problem solving is not actively pursued by our teachers in their classrooms due to various factors, one of which is a lack of confidence. As evident from this research, it would appear that the much vaunted Problem-Based Learning or PBL may be but a dream in the classrooms of the sampled teachers and probably many other mathematics classrooms.

If we are to change the way mathematics is taught in our classrooms, then it is important that mathematics teachers, at all levels, get more training and support to implement and integrate problem solving in their lessons.

REFERENCES


Annexure A: Primary school questions

<table>
<thead>
<tr>
<th>Question 2 (6 marks)</th>
<th>Question 3 (6 marks)</th>
</tr>
</thead>
<tbody>
<tr>
<td>If an object travels at 5 metres per second, how many metres will it travel in one hour?</td>
<td>What is the average (mean) of all multiples of 10 from 10 to 190 inclusive?</td>
</tr>
<tr>
<td>A. 30 m</td>
<td>A. 90</td>
</tr>
<tr>
<td>B. 300 m</td>
<td>B. 95</td>
</tr>
<tr>
<td>C. 720 m</td>
<td>C. 100</td>
</tr>
<tr>
<td>D. 1800 m</td>
<td>D. 105</td>
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<tr>
<td>E. 18 000 m</td>
<td>E. 110</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Question 4 (6 marks)</th>
<th>Question 5 (6 marks)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A cubical block of metal weighs 6 kg. How much will another cube made of the same material weigh if its sides are twice as long?</td>
<td>In a class of 78 students, 41 are taking English and 22 are taking Afrikaans. Of the students taking English or Afrikaans, 9 do both languages. How many students do neither English nor Afrikaans?</td>
</tr>
<tr>
<td>A. 48 kg</td>
<td>A. 6</td>
</tr>
<tr>
<td>B. 32 kg</td>
<td>B. 15</td>
</tr>
<tr>
<td>C. 24 kg</td>
<td>C. 24</td>
</tr>
<tr>
<td>D. 12 kg</td>
<td>D. 33</td>
</tr>
<tr>
<td>E. 18 kg</td>
<td>E. 54</td>
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</table>

<table>
<thead>
<tr>
<th>Question 9 (6 marks)</th>
<th>Question 12 (8 marks)</th>
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<tbody>
<tr>
<td>A teacher writes five numbers on the board. The five numbers have an average of 30. She then erases one of the numbers. The average of the four remaining numbers is 28. What number did she erase?</td>
<td>A reservoir is $\frac{5}{8}$ full. If 135 litres of water is added, the reservoir is $\frac{8}{11}$ full. What is the capacity (in litres) of the reservoir when full?</td>
</tr>
<tr>
<td>A. 32</td>
<td>A. 16</td>
</tr>
<tr>
<td>B. 10</td>
<td>B. 88</td>
</tr>
<tr>
<td>C. 8</td>
<td>C. 729</td>
</tr>
<tr>
<td></td>
<td>D. 1 320</td>
</tr>
<tr>
<td></td>
<td>E. 3 520</td>
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</tbody>
</table>
Question 14 (8 marks)
The three digits of a three-digit number add up to 25. How many such three-digit numbers are there?

A. 2  
B. 4  
C. 6  
D. 8  
E. 10

Question 15 (8 marks)
Calculate
\[
\left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \left(1 - \frac{1}{4}\right) \times \ldots \times \left(1 - \frac{1}{2013}\right) \times \left(1 - \frac{1}{2014}\right)
\]

A. 2013  
B. 2014  
C. \frac{1}{1007}  
D. \frac{1}{24}  
E. \frac{24}{2014}

Annexure B High school questions

Question 6 (6 marks)
If \(n > 0\) which of the following must be true?

(i) \(n^2 > 1\)  
(ii) \(n - n^2 < 0\)  
(iii) \(2n - 1 > 0\)

A. (i) only  
B. (ii) only  
C. (iii) only  
D. (i) and (ii) only  
E. None of the above

Question 12 (8 marks)
A full can of paint has a mass of 1000g. If half the paint is used up, the mass of the can and the remaining paint is 700g. What is the mass of the empty paint can?

A. 600 g  
B. 500 g  
C. 350 g  
D. 400 g  
E. 300 g

Question 13 (8 marks)
A triangle with perimeter 7 units has integer side lengths. What is the maximum possible area (in square units) of such a triangle?

A. \(\frac{\sqrt{35}}{4}\)

Question 14 (8 marks)
A rectangular piece of paper measures 17 cm by 8 cm. It is folded so that a right angle is formed between the two segments of the original bottom edge, as shown. What is the area of the new figure in cm²?
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>B.</td>
<td>( \frac{3\sqrt{7}}{4} )</td>
</tr>
<tr>
<td>C.</td>
<td>( \frac{49}{4} )</td>
</tr>
<tr>
<td>D.</td>
<td>( \frac{1}{7} )</td>
</tr>
<tr>
<td>E.</td>
<td>( \frac{22}{7} )</td>
</tr>
</tbody>
</table>

**Question 15 (8 marks)**

The figure shows a square inscribed in a circle of diameter 2 cm. Find the area shaded, in cm\(^2\).

A. 1 cm\(^2\)
B. \((\pi - 2)\) cm\(^2\)
C. \(\frac{1}{2}\) cm\(^2\)
D. 2 cm\(^2\)
E. \(\frac{\pi}{2}\) cm\(^2\)
In this study, pre-service mathematics teachers (PMTs) were introduced to some complex ideas in mathematics (fractals, infinite sequences, limiting value of an infinite geometric sequence connected to the concept of infinity). In so doing, the PMTs investigated the relationship between mathematics and the Sierpinski gasket within the context of patterns and sequences. Results of the study indicate that many students were not able to establish a correct algebraic expression to represent the area of the \( n^{th} \) stage Sierpinski gasket, but the majority of those who could were able to intuitively connect it to the concept of infinity, and consequently were able to justify that the area of the Sierpinski gasket at stage infinity tended to 0 (or 0). In retrospect, we believe that this study makes a contribution to possible ways in which we can deepen PMTs mathematical thinking through pattern generalization and further interrogation thereof.

INTRODUCTION

Yeo & Yeap (2009) describe a mathematical investigation as a process that invokes mathematical thinking via four core processes, namely specialization, conjecturing, justification and generalisation. Similarly, the South African Curriculum and Assessment Policy Statement (CAPS): Mathematics: Senior Phase (Grades 7-9) (DBE, 2011b) asserts that an investigation promotes critical and creative thinking and can be used to enable learners to discover patterns and general trends. The aforementioned emphasis of the South African National Curriculum means that mathematics teachers themselves need to be appropriately educated to enlighten and engage their learners in core mathematical processes in meaningful ways. As pre-service mathematics teachers (PMTs) would continuously feed into the pool of mathematics teachers, who are expected to implement CAPS, it is prudent that PMTs themselves experience such processes in meaningful ways supported by appropriate contexts and environments.

Such experience gained by pre-service teachers will hopefully enable them to effectively select, design and implement mathematical investigation tasks more meaningfully in their mathematics classrooms both during their teaching practice sessions and when they assume duties at schools as qualified teachers.

Hence, this study provided an opportunity for a group of senior phase pre-service mathematics teachers to experience the core processes of mathematical thinking
through actually doing a pencil and paper task based Sierpinski triangle investigation. In the main, the purpose of this study is to analyse PMTs ability to investigate a Sierpinski gasket and discern an algebraic expression that can represent the area of a stage \( n \) Sierpinski Gasket primarily through the processes of specialization, conjecturing, justification and generalization, and also ascertain the extent to which they can apply the limit notion to establish the area of a Sierpinski Gasket at Stage infinity.

In this paper we begin with statement of the main research questions, followed respectively by a literature review and theoretical consideration, research methodology, results and discussions, and end with some concluding remarks.

**RESEARCH QUESTIONS**

1. Can pre-service teachers find the area in each successive stage of the Sierpinski gasket and generalize the pattern for a stage \( n \) gasket?

2. To what extent can pre-service mathematics teachers make connections with the concept of infinity when determining and/or justifying the area of a Sierpinski gasket at stage infinity.

**LITERATURE REVIEW AND THEORETICAL CONSIDERATIONS**

The emphasis and fostering of mathematical investigations could be attributed to the initiatives of the Nuffield Project, which was embarked upon in 1967, and the ideas put forth in the Plowden Report (1967). In the main, these reports express that the necessary space and environment should be created in our classrooms to enable learners to engage with investigative tasks that provide an opportunity for them explore and discover during the process of learning. In a similar vein, Sangster (2012) citing Garrard (1986) says:

> Mathematical investigations require additional skills of enquiry such as assembling information, drawing upon previous knowledge, coming up with strategies and employing them, interpreting and recording findings, seeking patterns and relationships, extracting general points and considering whether they can be transferred to new situations. (p.45)

Currently, as per CAPS for Mathematics, schools in South Africa are being encouraged to focus on doing mathematical investigations (DBE, 2011a & DBE 2011b).

As the process of specializing and generalizing within the context of pattern generalization underpins the investigation reported in this paper, it will, hereunder, be
discussed within the broader context of generalization. According to Govender (2013),
generalization is the reflection upon a number of cases, and the discovery of either a
pattern, or stable relationship, or common property (attribute) across these, which is
subsequently proclaimed to hold true for cases beyond the considered cases. Focusing
on the process of generalization, Clements and Sarama (2009), in concurrence, state
that patterning involves the study of both ordered and unordered data and the
consequent establishment of mathematical regularities and structures in them. Drawing
from this Riviera (2010.147) says:

When students perform a pattern generalization, it basically involves mutually
coordinating their perceptual and symbolic inferential abilities so that they are
able to construct and justify a plausible and algebraically useful structure that
could be conveyed in the form of a direct formula. (p.147)

The development of pattern generalizations embraces processes of specializing,
abstracting, generalizing and testing (compare: Canadas & Castro 2005; James, 1992;
Polya, 1967; Reid 2002). These processes can be enumerated as follows:

- **Specializing**: Students examine particular cases and become familiar with the
details of each case;

- **Seeing generality**: As the students move through particular cases they begin
to see some regularity across the cases. This awareness of the abstracted
regularity or underlying sameness becomes more and more prominent as the
students pass from one particular case to the next, and consequently boosts
their sense of confidence in their observed degree of sameness across the
special (or particular) cases.

- **Expressing the generality**: As soon as the student is quite confident with the
underlying sameness that he or she has increasingly seen across the particular
cases, then he or she begins to articulate the sameness in his or her own words
and also comes to grips with concepts that underpin it.

- **Checking and Convincing (empirically)**: By empirical testing students try
to see if their generalizations also hold for new particular cases as well.

Specializing and generalizing are mutually dependent processes and it is in this sense
that Mason (1999) says: “specializing can produce fodder for generalization and
generalizations must be checked to see that they do specialize back to the particular
cases which spawned them”. (p.33). In a similar vein Badger, Sangwin, Hawkes, Burn;
Mason, and Pope (2013) reiterate: “…it is through ‘seeing the general through the
particular’ that we encounter generalization and it is through ‘seeing the particular in
the general’ that we test intuitions and contact structural relationships that may lead to
justifications or conjectures” (p.48).
According to Zazkis & Liljedhal (2002), a patterning approach creates the opportunity for students to verbalize their generalizations and to record them symbolically. This resonates with Mason’s (1996) view that “expressing generality” is a gateway to algebra. However, a teaching experiment focusing on development of patterns in algebra, revealed that the major problem for students was not in “seeing a pattern” but in perceiving an “algebraically useful pattern” (Lee, 1996, p.95).

**RESEARCH METHODOLOGY**

In this study both quantitative and qualitative methodology were employed. The curriculum material from Bennett (2002) was adapted for use in this study, and is accordingly shown in Figure 1. The purposive sample consisted of 53 Senior Phase (grades 7-9) pre-service mathematics educators doing their Mathematics Education method module (Mathematics Education 401) in their final (fourth) year of study at a resident South African university. The nature of the task posited to the pre-service teachers during one of the method lectures, which was about an hour long, was investigative in nature and pertained initially to the construction of Stages 0-4 of a Sierpenski gasket using pencil and paper, and thereafter the establishment of the area of each of the Sierpinski stages 1-4. This activity, which was facilitated by the main author, was further extended by asking PMTs to generalize the area pattern to a nth stage Sierpinski gasket, and then using generalized area sequence to determine and/or justify the area of a Sierpinski gasket at Stage infinity.

Data was collected from the completed task-based worksheet only, and analysed using both descriptive statistics supported by analytical/inductive method governed by an interpretive paradigm (see Govender, 2013, p.215). The theoretical considerations on generalization (both as a process and product) and infinite geometric series including provided a useful framework for coding, interpreting and analysing PMTs responses to the Sierpinski task based activities.

Figure 1: Mathematical Investigation Task - Sierpinski Triangle
Sketch

Start with a coloured triangle, a Stage 0 Sierpinski triangle. To create a Stage 1 triangle, connect the midpoints of the sides to form four smaller triangles; colour the three outer triangles and make the inner one white. Do this again to each of the shaded triangles in the Stage 1 triangle to get the Stage 2 triangle. Continue in this way.

Investigate

1. Think about what happens to the area (the shaded part) in each successive stage of the Sierpinski triangle. Suppose the Stage 0 triangle has area 1 square unit.

<table>
<thead>
<tr>
<th>Stage number</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>n</th>
<th>...</th>
<th>50</th>
</tr>
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<tbody>
<tr>
<td>Area</td>
<td>1</td>
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</tbody>
</table>

1.1 Complete the table with the areas of Stage 1, 2, 3 and 4 gaskets.

1.2 As you increase the number of stages, what’s happening to the area of the gasket (the shaded part)?

1.3 Enter an expression in the table for the area of a stage n gasket.

1.4 Calculate the area of a stage 50 gasket.

1.5 (a) What would the area of a Sierpinski gasket be at stage infinity?

(b) Provide an explanation to justify your response provided in 1.5(a).
Start with a coloured triangle, a Stage 0 Sierpinski triangle. To create a Stage 1 triangle, connect the midpoints of the sides to form four smaller triangles; colour the three outer triangles and make the inner one white. Do this again to each of the shaded triangles in the Stage 1 triangle to get the Stage 2 triangle. Continue in this way.

**Investigate**

1. Think about what happens to the area (the shaded part) in each successive stage of the Sierpinski triangle. Suppose the Stage 0 triangle has area 1 square unit.

<table>
<thead>
<tr>
<th>Stage number</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>…</th>
<th>n</th>
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<th>50</th>
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<tr>
<td>Area</td>
<td>1</td>
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<td></td>
<td></td>
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</tbody>
</table>

1. Complete the table with the areas of Stage 1, 2, 3 and 4 gaskets.
2. As you increase the number of stages, what’s happening to the area of the gasket (the shaded part)?
3. Enter an expression in the table for the area of a stage \( n \) gasket.
4. Calculate the area of a stage 50 gasket.
5. (a) What would the area of a Sierpinski gasket be at stage infinity?

(b) Provide an explanation to justify your response provided in 1.5(a).
RESULTS AND DISCUSSION
The analysis and findings will be presented in the context of each of the critical research questions.

Research Question 1:

Can pre-service teachers find the area in each successive stage of the Sierpinski gasket and generalize the pattern for a stage n gasket?

Initially, all 53 students, were able to follow the guidelines in the task-based worksheet and construct a Stages 0 to 4 Sierpinski gaskets pencil and paper, as illustrated by one of the PMT’s constructions in Figure 2. In sub-activity 1.1, PMTs were required to complete the table (as shown in Figure 1) with areas of Stage 1, 2, 3 and 4 gaskets, with the assumption that the Stage 0 triangle has area 1 square. In order to analyse as to whether PMTs could find the areas of Stage 1, 2, 3, 4 gaskets, their responses for each stage area calculation was classified into either one of three categories, namely, did not answer; incorrect and correct.

Figure 2: Sierpinski Gasket- Stages 0-4

Figure 3, illustrates stage-wise the number of PMTs that did not respond or responded but did so either correctly or incorrectly when asked to complete the table in Figure 1 with the areas of Stage 1, 2, 3, 4 gaskets.
Figure 3: Stages 1-4 area calculations by PMTs

For Stage 1 area, the analysis show that 42% (25) of candidates did correctly state that the area (referring to the shaded area) of the Stage 1 gasket will be \( \frac{3}{4} \) units (i.e. square units) as can be visually seen in the Stage 1 Sierpinski gasket in Figure 3.

However, the analysis for Stage 1 area shows that 1.9% of the PMTs did not record a response and 50.9% of the PMTs provided an incorrect response (see for example Figures 4, 5). This suggests that 52.8% of PMTs did not conceptually comprehend their construction move (namely, connect the midpoints of the sides of a Stage 0 Sierpinski gasket to form four smaller triangles; colour the three outer triangles, and make the inner one white) on a Stage 0 Sierpinski gasket and its related impact on the resultant area of a Stage 1 Sierpinski gasket. It seems that the visual artefact had virtually not helped these 52.8% of students to precisely recognise and see what was happening to the area of the stage 1 gasket even though it was decreasing. The following are what some of these students wrote in their worksheet:

![Figure 4: Tony’s response](image)
In addition, the analysis show that number of PMTs who stated an incorrect area for a Stage 2 gasket increased to 35 as compared to 27 for the Stage 1 gasket. This effectively represents 15% increase relative to the number (53) of PMTs in the sample. It seems that as the stage number increased and the corresponding number of shaded triangles increase, PMTs struggled to conjecture (or see) what is happening to the sum of the areas of shaded triangles. This suggests that they were not able to discern a pattern as yet at stage 2. The latter is illustrated in the following two examples as shown in Figures 6 and 7:

The PMTs who did not provide a response for Stage 1, also did not provide an area response for stage 2, stage 3 and stage 4. Furthermore, the number of students who provided an incorrect response for stage 3 area increased to 62.1% as compared to 60.3% which prevailed for Stage 2 gasket (see Figure 3. Although, at stage 4 the percentage of incorrect responses moved down to 58.6%, this gain was eradicated by
three more students (an increase of 5.2%) not providing a response for stage 4 area (See Figure 3). A downward trend in the correct area response rate from Stage 1 to Stage 4 (see Figure 3) was also prevalent across the sample. In fact the correct response rate dropped from 47.2% in Stage 1 to 32.1% in Stage 2 to 30.2% in Stage 3 and finally 28.35 in Stage 4.

However, amongst the group that got the areas of stage 1-4 incorrect there were 10 PMTs who presented answers as shown in Figure 5, 3 PMTs who presented answers as presented as shown in Figure 7, and 2 PMTs presented answers as shown in Figure 9. These 15 PMTs is referred to as the ‘ONE THIRD GROUP’ in this paper for discussion purposes.

In order to ascertain as to whether PMTs could observe (or reason) that the shaded area of the of Sierpinski gasket decreases as the number of stages decreases, the following question was posited to all PMTs: *As you increase the number of stages, what’s happening to the area of the gasket (the shaded part)?*

Figure 10, broadly described the nature of PMTs responses in terms of the following three categories: did not answer, incorrect, and correct. Even though the number of PMTs who were able to provide the correct area for each of the stages dropped from 25
for Stage 1 to 15 for Stage 4, the number of PMTs who were able to correctly discern that the area of the gasket becomes smaller and smaller as one moves from Stage 1 to Stage 4 has been relatively high, namely 38 out 53 (i.e. 71.7%). This increase is largely due to the fact that there were about 15 students (as illustrated in Figures 5, 8 and 9) who generated an incorrect geometric pattern using a common ratio of \( \frac{1}{3} \) instead of \( \frac{3}{4} \). This suggests that these PMTs have not necessarily been able to see that the shaded area is being reduced by a scale ratio of \( \frac{3}{4} \) as they moved from the construction of one stage to the next.

![PMTs responses](chart)

Figure 10: PMTs responses as to what happens to area of gasket as the number of stages increase

Nevertheless, their pattern portrayed a similar kind of decreasing area trend as for those who correctly used the common ratio of \( \frac{3}{4} \) and hence they were able to make the same generalization, namely that the area decreases as the you increase the number of stages.

Furthermore, it seems that the fourteen students, who provided incorrect answers as shown in Figure 10, did not suggest that area decreases as one moves from one stage to the next. They could not naturally see that the shaded area (i.e. the area of the gasket) is becoming smaller as the number of white triangles (holes that manifest in a gasket) was increasing as they were practically constructing the Sierpinski gasket. Moreover, not having obtained the precise area value for each specific stage may have disadvantaged these PMTs from seeing a decrease in the area as one moves from one stage to the next. The reason why many students were not able to calculate or discern the precise unit area for each special case (i.e. each Sierpinski stage), could possibly be attributed to poor content knowledge associated with the concepts of division and ratio as well as mathematics operations with fractions.
The next task activity, namely task 1.3, required the PMTs enter an expression in the table for the area of a stage $n$ gasket. Figure 11 shows how the students responded to the task. The analysis shows that only 18.9% (that is 10 out of 53 students) were finally able to provide an expression to represent the area of a stage $n$ gasket. This could largely be attributed to the fact that many PMTs were not able to ascertain the correct areas for each of the special cases (i.e. Stages 1-4). For example, at Stage 2 the cumulative percent of PMTs who either did not answer or responded incorrectly was 67.9% (compare data in Figure 2), and this cumulative percent trend manifested itself in Stage 3 at 69.8% and finally at Stage 4 at 71.7%. Not having these correct area values at each stage, would have meant that no regularity would have necessarily prevailed across the area values for the respective stages 1-4. This in itself, would have deprived PMTs from seeing any regularity prevailing across the specific cases, and thus generalize the area pattern of a Sierpinski gasket, and thereby construct the expression, $\left(\frac{3}{4}\right)^n$, which could be used to determine the area of a stage $n$ gasket.

![Percent Chart]

Figure 11: PMTs responses to Task 1.3

However, there was a group of 15 PMTs (28.3% of the sample), as can be seen from Figure 3 who were able to discern the unit area of a stage 4 gasket, and in retrospect also the respective unit areas for stages 1, 2 and 3 (see examples in Figure 12 and Figure 13).

However, only 10 out of these 15 students were able to correctly provide an expression for the area of a stage $n$ gasket. The following are what some of these ten students wrote in their worksheets:
As evident in Figure 12 and Figure 13, it seems that if students do have the correct terms in a given table (or sequence) then it is highly likely that they cannot abstract the regularity and hence generalize the pattern correctly using algebraic expressions (or formulae). This suggests that failure to specialize can lead to failure to generalize or hamper the process of successful generalization.

The remaining the five students, who obtained the area of stages 1-4 correctly did not provide a correct expression for the area of a stage \( n \) gasket. Figure 14 and 15, illustrates two such responses. The responses of the five aforementioned PMTs do suggest that even though PMTs could establish the areas of each stage (i.e. specialize), they were not able to obtain general term (or expression) that could characterise the generalization.
Figure 15: Logan’s response
The responses of the five aforementioned PMTs do suggest that even though PMTs could establish the areas of each stage (i.e. specialize), they were not able to obtain general term (or expression) that could characterise the generalization.
Now Reverting to the ‘ONE THIRD GROUP’, who determined the area of the stages using the $\frac{1}{3}$ reduction factor, it was found that 12 out 15 of them did generate an expression for the area of stage $n$ that was consistent the sequence of areas generated for each successive stages. Figure 9 and Figure 16 illustrates such kind of cases:

<table>
<thead>
<tr>
<th>Stage number</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{27}{64}$</td>
<td>$\frac{91}{256}$</td>
<td>$\frac{3}{4}\left(\frac{2}{3}\right)^n$</td>
<td>$\uparrow$</td>
</tr>
</tbody>
</table>

Figure 16: Christian’s response
This again suggests that if students are able to specialize in a consistent way, and generate a sequence of terms (even though are incorrect) with a consistent pattern, they can abstract a general pattern in terms of an algebraic expression. This again supports Mason’s assertion that specialization supports the process of generalization.

Research Question 2:
To what extent pre-service make connections with the concept of infinity in determining the area of a Sierpinski gasket at stage infinity.
In an attempt to provide answers to the aforementioned research question, the researcher put to the PMTs the following questions in the task based activity worksheet.

1.5 What would the area of a Sierpinski gasket be at stage infinity?

The first purpose of asking the question 1.5 was to see if students could intuitively see that the area of a Sierpinski gasket will approach 0 at stage infinity, and secondly whether they could connect, invoke and reflect on their work on convergent geometric
series that they would have probably done in Mathematics at grade 12 level to try to determine the *area of a Sierpinski gasket be at stage infinity*. In particular, this question explores students ability to recognize that the areas of successive stages of a Sierpinski gasket can be considered as an infinite geometric series, and that this area sequence converges to 0 because its common(or constant) ratio \(r, \frac{3}{4}\), is less than 1 (i.e. \(-1 < r < 1\)). In retrospect, this question invokes PMTs understanding of the conception saying: the limit of \(\left(\frac{3}{4}\right)^n\) as \(n\) increases without bounds is zero, or 

\[
\lim_{n \to \infty} \left(\frac{3}{4}\right)^n = 0.
\]

<table>
<thead>
<tr>
<th>Stage – Area</th>
<th>Frequency</th>
<th>Percent</th>
<th>Cumulative Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Did not answer</td>
<td>20</td>
<td>37.7</td>
<td>37.7</td>
</tr>
<tr>
<td>Incorrect</td>
<td>20</td>
<td>37.7</td>
<td>75.4</td>
</tr>
<tr>
<td>Correct</td>
<td>13</td>
<td>24.6</td>
<td>100</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: PMTs responses on the area of a Sierpinski gasket Stage infinity.

Table 1 provides an overview of PMTs responses, which were initially categorized as follows: did not answer, incorrect, and correct. As evident in Table 1, 20 out of 53 students (37.7%) did not attempt to answer this question. This could be attributed to the fact that these students did not know the concept of infinity and what it means in the context of the area of the Sierpinski gasket at stage infinity, or they purely did not comprehend what can be done to answer this question or they had incorrect or incomplete data as discussed earlier in this report, which prevented them from making any further cognitive moves. A further 20 of 53 students (37.7%) attempted the question but provided incorrect responses.

Thirteen out of the 15 PMTs, who had their areas correctly presented for stages 1-4 and the correct expression for the area of a stage \(n\) gasket, did establish that area of a Sierpinski gasket be at stage infinity will be 0 (or approach 0). Majority of these 13 PMTs seemed to have displayed an intuitive understanding of the concept of infinity and/or limit concept. The following are what some of the students (see Figures 17-20) wrote in their worksheets as they intuitively deduced that the area of a Sierpinski gasket at Stage infinity would be zero.
The two students from the ‘All Stage Group’, actually did not realize that they were supposed to calculate the area of a Sierpinski gasket at stage infinity only and not the sum of the areas across an infinite number of stages of a Sierpinski gasket. Consequently, these two students used the Sum to Infinity formula, 

\[ S_\infty = \frac{a}{1-r}, \]

equation for an infinite geometric series defined by 

\[ a + ar + ar^2 + \cdots + ar^{n-1} + \cdots, \]

to calculate the sum of the areas across an infinite set of Sierpinski gasket stages (see Figures 17 and 18).
Figure 21 for an example of student response). This could be attributed to PMTs not being conversant with the mathematical language used and hence not comprehending the requirements of the task.

![Equation Image]

Figure 21: Betty’s response

The same kind of mistake was exhibited by 7 students from the ‘One Third Group’, as illustrated in a sample of two such cases via Figures 21 and 23.

![Equation Image]

Figure 22: Renny’s response

![Equation Image]

Figure 23: Shannon’s response

**Concluding remarks**

This study took place in a pencil and paper context, where a Sierpinski Gasket was drawn starting from stage 0 and culminating stage 4. Results show that more than 50% of the PMTs were not able to correctly establish that the resultant area of their Stage 1 Sierpinski gasket. The situation worsened with more than 60% of the PMTs not being able to discern the resultant area for each of the stages 2-4 of the Sierpinski gasket. This to an extent suggests that the visual artefact did not help PMTs to determine the resultant area for each of stages 1 to 4, and/or that they did not comprehend the effect of their geometrical construction moves on the available gasket area for stages 1-4 respectively. As these PMTs were not able to specialize correctly across some/all of stages 1-4, they were not able to see a regularity permeating their area values for stages 1-4, and hence could not articulate a generality in terms of an algebraic expression, i.e. they were just not able to provide an algebraic expression to represent the area of a
This demonstrates that one’s failure to specialize correctly prohibits one’s ability to generalize.

Conversely, all PMTs (15) who were able to establish the correct areas of stages 1-4, succeeded in seeing the geometric pattern permeating the sequence of areas associated with the successive stages (0-4) of the Sierpinski gasket, and hence were able to discern the required algebraic expression. This resonates with Mason’s (1999) assertion, which states that specialization is the fodder for generalization.

The majority (13) of the 15 PMTs who were able to perceive an algebraically useful (or correct) expression for the area of stage $n$ gasket, were able to intuitively discern that the area of the Sierpinski gasket will approach zero (or will be zero) at stage infinity. This suggests that patterning activities should not necessarily end up with the establishment of an algebraic expression (or algebraic formulae), but rather and engagement of learners with more complex ideas (such as the concept of infinity) that could stretch learner’s thinking.

Acknowledgement

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REFERENCES


In this paper I share a proposed study aimed at investigating teacher reflection on the teaching of a particular topic in the context of lesson study. Lesson study is becoming an increasingly popular form of professional development around the world and more recently it is being investigated within the South African context (see for example Pillay, 2014b). In this paper, I provide some general context and literature review as to why an investigation into teacher reflections on lesson study is important within both the South African context and in relation to my role as Foundation Phase Mathematics subject adviser at Dr. Kenneth Kaunda district in the North West Province of South Africa. The topic of my focus was prompted by the observations made while fulfilling my professional responsibilities. From my numerous school visits, it is apparent that many learners have problems when it comes to multidigit subtraction. Hence, the purpose of my proposed study is to investigate teacher reflections on the teaching of multidigit subtraction to Grade 3 learners. It is envisaged that this inquiry will be carried out through the use of a lesson study involving foundation phase teachers. The interpretive qualitative research paradigm supports this study and a case study approach will be used to conduct this research. Vygotsky’s socio-cultural perspective of teaching and learning will guide both data collection and analysis. The findings of this study should point to possible solutions to the challenges encountered in the teaching of multidigit subtraction in my area and I also plan to incorporate these into my future work with teachers.

INTRODUCTION

The South African education system is faced with a serious challenge in which learners struggle with the basic concepts of Mathematics. For example, in the 2003 Trends in Mathematics and Science Study (TIMSS) international study of science and mathematics teaching in various countries around the world, South African learners did not perform well at all, being placed at the bottom. Furthermore, the inequality of performance between South African learners in these studies is increasing (Reddy, 2006). According to Reddy (2006), this could be attributed to a multitude of factors including teacher development.

Selter (2002) argues that researchers do not know much about children’s mathematics with respect to an array of topics. This applies, for example, to addition and subtraction with three-digit numbers. According to Selter (2002), there are few research reports concerning multidigit subtraction, other than for instance, the report by Fuson et al. (1997). Graven, Venkat, Westaway and Tshesane (2013) point to gaps in mathematics
teacher content knowledge and pedagogic content knowledge (Carnoy et al., 2008) with foundation phase addition and subtraction of multidigit numbers within the South African context. Similarly, in my work as Subject Adviser responsible for Foundation Phase Mathematics I have noted that learners struggle with the subtraction of multidigit numbers. This observation was made during my numerous visits to various Grade 3 mathematics classrooms. Hence there is a need to investigate possible solutions to the challenges encountered in mathematics especially in our previously disadvantaged poor performing schools.

Local and international research does however point to the problem of the prevalence of rote/ritualised methods as opposed to developing conceptual number sense for teaching addition and subtraction of multidigit numbers (Graven et al. 2013; Selter, 2002; Fosnot & Dolk, 2001). Furthermore, research also points to a range of strategies that can be used to support learners in developing number sense and proficiency in multidigit computations.

While Fosnot and Dolk (2001) agree that ‘algorithms - a structured series of procedures that can be used across problems, regardless of the numbers – do have an important place in mathematics’ (p. 124), they emphasize that this should only come after the students have a ‘deep understanding of number relationships and operations and have developed a repertoire of computation strategies’ (p. 124).

I discovered in my area that the problem with algorithmic methods is that the learners use them without understanding. I experienced the above-stated problem as both a teacher and as a district subject adviser. For example, during my numerous schools visits in Maquassi Hills, I have observed that some learners do the following when subtracting:

\[
\begin{align*}
92 \\
- 76 \\
\end{align*}
\]

Graven et al. (2013) provide similar anecdotal experiences of learners who were attending in Eastern Cape and Gauteng primary schools. In addition, two-digit subtraction problems that require borrowing pose a serious challenge to learners. In essence, the solution provided by the learner as illustrated above suggests that the learner subtracted the units and the tens as per the algorithm but erroneously adhered to the principle of ‘always subtract the smaller from the bigger’. Thus, in the above case, the learner lacks two key strands of mathematical proficiency, namely; procedural fluency and conceptual understanding (Kilpatrick, Swafford and Findell 2011). In my view, these student errors show that students focus on trying to remember the steps of
algorithm but do not make sense of numbers (Clements & Sarama, 2014). According to what is stated above, it might be wise if we do not teach algorithm (Clements & Sarama, 2014) to foundation phase learners if we want learners to grasp and develop number sense. The preceding statement by Clements & Sarama, 2014 coheres with the South African Curriculum and Assessment Policy Statement (CAPS) for Mathematics in the Foundation Phase for Grade 3 which states that learners should “understand subtraction and use subtraction vocabulary by the end of the year” (DBE, 2011, 322). Thus, the Curriculum and Assessment Policy Statement for Foundation Phase does not promote or suggest the use of algorithms. Perhaps, this explains why the departmentally issued Grade 3 Numeracy workbooks do not include the vertical algorithm method but instead suggest variety of methods.

The 2013 ANA marking memorandum for Grade 3 shows the following options as ‘valid method for multidigit subtraction at this level:

\[
\begin{align*}
9.1 & \\
795 - 213 & = 700 + 90 + 5 - 200 - 10 - 3 \\
& = 700 - 200 + 90 - 10 + 5 - 3 \checkmark \\
& = 500 + 80 + 2 \\
& = 582 \checkmark
\end{align*}
\]

or

\[
\begin{align*}
5 - 3 & = 2 \\
90 - 10 & = 80 \checkmark \\
700 - 200 & = 500 \\
795 - 213 & = 582 \checkmark
\end{align*}
\]

or

\[
\begin{align*}
795 - 200 \rightarrow 595 - 10 \rightarrow 585 - 3 \rightarrow 582
\end{align*}
\]

However since the algorithm method is suggested in the CAPS for Mathematics in the Intermediate Phase (DBE, 2011b) and many teachers themselves use this method for their own calculations, these methods are often found (erroneously executed) in grade 3 classrooms in my area. Additionally, the first two methods indicated in the memorandum above do not work well when, for example, a larger unit is subtracted from a smaller unit as in the case of 23 – 19 where many learners answer 20–10 = 10 and 9-3 = 6 so 23 – 9 = 16.
A scan of literature shows that there is little local research currently available that addresses, from a classroom teaching perspective, challenges with teaching multidigit subtraction and how to address the widely noted ‘common errors’ such as Gap in research on local teacher informed reflection on practice. Departmentally coordinated in-service teacher development in SA has been widely criticised for being predominantly ‘lecture style’ and largely ineffective since it does not sufficiently take into account classroom conditions and existing teaching practices (Chisholm et al.; 2000; Graven, 2004; Pausigere & Graven, 2013). I perceive teacher development that is separated from classroom practice as lacking a key learning opportunity and a possible reason for the widespread failure of much teacher development in SA.

Indeed, in their Integrated Strategic Planning Framework for Teacher Education and Development South Africa 2011 – 2025, the Department of Basic Education and Department of Higher Education Training (2011) acknowledge that there is a need for a new approach to teacher development. Here they emphasise the need for localised teacher support through the development of subject based professional learning communities (PLCs). In my professional capacity, as already referred to in the preceding paragraphs, I initiated after school Maths Clubs in the area with six Foundation Phase Grade 3 teachers. This was done in an attempt to address the subtraction of multidigit in foundation phase and various other mathematics challenges. To this end, in 2012 I began collaborating with the South African Numeracy Chair Project (SANCP) which is based at Rhodes University and in particular began working with their notion of after school mathematics clubs (Graven, 2011). The initiation of the maths clubs was motivated by my first interaction with members of the SANCP and the clubs were structured in line with the design suggestions of Graven & Stott (2012) and Stott & Graven (2013). The clubs have typically 25 Grade 3 learners from four different schools. Teachers chose to take learners from different schools in order to see various ways of working. Through my collaboration with SANCP I am planning to conduct an inquiry or research that will enable me to improve my support of teachers in my current district role and also to shed light on the opportunities of lesson study as a form of professional development for clusters of teachers. I would like to explore teacher reflection that foregrounds strengthening local teaching practice of the particular topic of multidigit subtraction through joint planning, implementation and reflection on classroom teaching. In this respect a lesson study approach (Fernandez, 2002) will maintain this exploration because within this approach there is a strong emphasis on reflection on practice.

Having developed strong relationships with these teachers through these clubs provides me with an opportunity to develop a professional learning community (PLC) with these teachers focused on lesson study. When visiting with these teachers in their after school maths clubs they have often shared with me several challenges they have noticed with
learner computations from the different schools. Additionally, they shared ways they have addressed these with individual or small groups of learners in these clubs. A challenge for them, however, is how to draw on these insights in order to address the challenges in their Grade 3 school classrooms with about 45 learners per class.

This study will be guided by an assumption that language is central in the learning and teaching of mathematics. As such I will draw on a Vygotsky’s socio-cultural perspective of teaching and learning to frame my study and data analysis. In this respect and given the above, I hope to offer lesson study focused on multidigit subtraction as an opportunity for drawing together these six teachers insights and reflections on designing a lesson for the teaching of multidigit subtraction and reflection on that lesson for improved teacher understanding. However Likando (2014) notes that adaptations need to be made when using lesson study in culturally different contexts such as the USA and I would expect some adaptations within the South African context. According to Lewis and colleagues (Lewis, et al., 2006) lesson study was also used successfully in the UK to improve the teachers’ pedagogy. In my view, lesson study is a useful opportunity for both improving teacher practice in the classroom as well as stimulating teacher reflection and thus provides a rich empirical field enabling me to gather data on teacher reflection on a particular topic of interest (in this case multidigit subtraction). Thus this collaboratively planned, taught, observed, reflection process of lesson study will provide a useful opportunity for my study on teacher reflection and insights into multidigit (three digit) subtractions.

I intend to adapt lesson study in relation to teacher participation. As such the number of lesson cycles and who will teach the lesson will be jointly planned and will be openly negotiated. Additionally, should they all wish to be involved in the lesson, all teachers may participate in a team teaching lesson approach. In my role as subject adviser I will arrange for permission for teachers to jointly participate/observe in this lesson that will take place in one Grade 3 classroom. All schools are within a 5km radius of each other enabling this possibility.

In this respect the research and lesson design study aims to:

* Develop reflective insights on how to address the problem of multidigit subtraction in Maquassi Hills Grade 3 classrooms through a focus on designing and analyzing a lesson focused on the teaching of this.

* Research teacher reflective insights on the teaching of this topic and the enablers and constraints of participation in this lesson study process.
Thus addressing the central question for this study is: What are teachers’ reflections and insights/strategies on the teaching of multidigit subtraction (as revealed through participation in lesson study lesson on the teaching of it to Grade 3 classrooms)?

The unit of analysis will be teacher reflection as communicated in the lesson study sessions. This data will be primarily obtained from transcriptions derived from video recordings of sessions. Additionally teacher reflection data on this topic will be derived from post lesson study individual semi-structured interviews with each of the participating teachers.

**RESEARCH DESIGN AND METHODOLOGY**

This collaborative ‘lesson study’ will form the empirical field of my research. Given the choice of lesson study as the empirical field and teacher reflection and insights on the teaching of multidigit subtraction as the focus of my research it is appropriate to use an interpretive qualitative research paradigm. It will underpin this study with a multiple case study approach. The multiple cases will be the 6 teachers participating in the lesson study professional development cycle.

The participants will be invited from the opportunity sample of teachers that I have been working with in 2014. As indicated above I have developed good working relationships with six Grade 3 teachers through our mathematics club work in 2014. I have thus selected (to invite) these teachers because I am already working with them in foundation phase and their work and their demonstrated interest in strengthening their teaching provides me with an opportunity. I intend to be a participant observer within the lesson study. My role as a participant observer would be to observe the event as it would be “naturally occurring” (Macmillan & Schumacher, 1997) at the “field site” (Neuman, 2011). I would also write notes to “describe what people did and said” (Kelly, 2006). While a participating member of the lesson study even while I will additionally assume a co-coordinating role. As their mathematics subject adviser I will be particularly aware of possible power relations that might influence the process and their participation. I will endeavour to establish a partnership with them in the investigation. The schools and teachers also provide a convenient sample of teachers as their schools are situated nearer to my place. Therefore, it will be easier for me to visit these schools.

My methods of data collection will primarily involve:

* Video recording of all lesson study sessions - Since teacher talk (reflection, strategies and insights) is the unit of analysis for this study, video recordings will be fully transcribed, it is envisaged that video recording will be used to capture all lesson study sessions. Member checks will be used to support accuracy.
* Post lesson study semi-structured one-to-one interviews with all participating teachers. These will be audio recorded and fully transcribed. Since “interviewing people is a natural way of interacting with people than making them fill out questionnaires or do a test” (Kelly 2006: 297) one-on-one interviews will be used in this study. A semi-structured interview schedule will be designed following the lesson study process before the interviews are conducted. The interview questions on the interview schedule will be in line with the research questions developed for this study. In essence, they will be given an opportunity to explain how they believe the invented strategies can be used to develop an understanding of multi-digit subtraction.

* Researcher/participant journal – I would record interactions (Phelps, 2005) with the participants at my field site, as well as the ‘informal’ dialogue between me and the teachers that may occur outside of the recorded lesson study sessions.

* The constructed lesson plans of the planning sessions. - The documentary analysis of the lesson plans developed in the lesson study process will provide rich data on strategies and insights developed by the teachers.

Thematic analysis of the teacher insights and reflections present in lesson study session transcripts will be conducted. I will draw on Lee’s (2005, p.703) framework of reflection to assist in this analysis. This framework provides three levels of teacher reflection:

Level 1 – Recall: When the teacher provides simple description of their initial teaching experience, from what their own past experience as teacher or learner of the particular mathematical topic. Further express why they use a certain approach to teach the topic.

Level 2 – Rationalization: When the teacher describes how they start to question their belief about teaching a topic in a particular way. Thus seeks to look for their belief about teaching a topic in a particular way. Thus seeks to look for alternative ways that relates to change their past experience. The teacher develops a relational meaning of mathematical teaching and learning experiences.

Level 3 – Reflectivity: The teacher shows an awareness of various perspectives and is intent to change/improve, and is able to see the influence of their cooperating teachers on their learners’ values/behaviour/achievement.

Informed consent, respect for the rights of research participants, confidentiality and anonymity are some of the ethical considerations that I will attend to prior to conducting the investigation. For example, I will write letters to the North West Department of
Education, the Education District Director of Maquassi Hills and the school principals to request permission to conduct my research investigation.

Similarly, during the recruitment process I will provide potential participants with full details concerning the purposes of and the manner in which the study will be conducted. I will also assure them that their participation will be both anonymous and confidential. In this regard, I will indicate that their names and those of their schools will not be used anywhere in the documents that will form part of the study. I shall strive to use codes and/or pseudonyms for this purpose (anonymity).

REFERENCES


This paper reports on a preliminary study of a research project that seeks to explore secondary school mathematics teachers’ knowledge for teaching quadratic functions. At this stage of the study, the focus is on exploring the nature of the teachers’ subject matter knowledge of quadratic functions. Quantitative and qualitative data were obtained from five practising teachers by means of content knowledge assessment followed by an interview. Analysis of the instruments provided insights into teachers’ subject matter knowledge in quadratic functions. The data indicated that the teachers have good procedural knowledge. However, there were lapses in certain aspects of the teachers’ subject matter knowledge of quadratic functions showing therefore the need to address certain aspects of teachers’ subject matter knowledge in teacher education and in in-service training.

BACKGROUND

This paper reports on a part of a preliminary investigation of a research project that seeks to explore secondary school mathematics teachers’ knowledge in teaching quadratic functions. The focus of this paper is on secondary school teachers’ subject matter knowledge. The teachers involved in this study are qualified and practising mathematics teachers differing in their qualifications and years of experience. The sample is drawn from the South African school system where a teacher is deemed qualified if s/he possesses a three-year diploma in Education or a four-year Bachelor of Education degree or a Bachelor degree in the relevant discipline followed by a Post Graduate Certificate in Education. The teachers under discussion have witnessed several curriculum reform initiatives as in many other countries of the world. Creating and carrying out mathematics lessons as envisioned in the reform documents makes more demands on teachers than before. The literature is full of evidence which points that teacher’s ability to adapt to reforms and curriculum changes depends on the nature of the subject matter knowledge which the teacher possesses.

In response to strengthening the professional knowledge of the teaching force in the face of the curriculum reforms as alluded in above, various in-service programs has been put in place including the Accelerated Certificate in Education program (ACE) for teachers possessing a three-year diploma. However, the effectiveness and quality of these programmes has been questioned by individual(s) and the Council on Higher Education (CHE). In their study, Likwanbe & Christiansen (2008) has found that the
ACE programme was ineffective in deepening the conceptual knowledge of mathematics teachers, especially teachers from historically disadvantaged communities. In its Report on the National Review of Academic and Professional Programmes in Education, the CHE had this to say:

“... HEIs end up paying insufficient attention to the ACE Mathematics, and indeed to other ACEs, because the ACE is perceived as the lowest of their priorities. ... The absence of a sustained plan that addresses the continuum of learning that is required, and in particular that addresses poor subject specialisation knowledge, is perhaps the greatest weakness of the ACE programmes”.


Within this period of curriculum changes and the various in-service and teacher support programmes provided, there has been a poor showing of learners in national and international examinations in mathematics for more than a decade. At the primary school level (Intermediate Phase) and at the General Education and Training (GET) level, learners have performed dismally in the ANA, TIMMS, and SACMEQ examinations (Department of Basic Education, 2012; DBE, 2013; Taylor & Shindler 2008; van der Berg, Burger, Burger, de Vos, du Rand, Gustafson, et al. 2011; Spaull, 2011; TIMMS, 2011; TIMMS, 2002). At the Further Education and Training (FET) band mathematics remained the only school subject where learners have performed below fifty percent in the Matriculation examinations (Matric) except for the last three years. The seemingly improvement in mathematics performance for the past three years was traced to a practice whereby school principals in a bid to improve their pass percentage direct students away from mathematics towards Mathematical Literacy – a form of Mathematics that is easily passed by learners. (Department of Basic Education, 2011b; Taylor, 2011). This practice narrows the learner options for further study.

As a result of poor learner performance in mathematics, some individuals and have questioned teachers’ competency in teaching mathematics (Taylor 2009b, Spaull, 2011). In its reaction to students’ poor performance in mathematics, the South Africa Democratic Teachers’ Union (SADTU) – a teachers’ union with the largest followership, proposes that the Department of Basic Education adopt the disaggregation model, in response to educational reforms. This model involves the study of each province/region, and school separately with the view to finding a workable intervention strategy. It requires an audit of the factors that impact on quality teaching and learning, which includes parental involvement, teacher professional development, school leadership and management, socio-economic status and available resources.

The background of my study as highlighted above informed my purpose which is to shed light into the nature of some secondary school mathematics teachers’ subject
matter knowledge in quadratic functions. The intention is not to find faults with teachers’ knowledge; rather, my interest is to study what the teacher knows in quadratic functions, with a view of providing opportunities for deepening teachers’ knowledge and their practice.

THEORETICAL FRAMEWORK

The complex nature of the domain of teachers’ knowledge of mathematics makes it very difficult to categorise and conceptualise. In this paper, teachers’ subject matter knowledge shall be defined as: teachers’ knowledge of the key facts, concepts and principles in mathematics and the ways in which they are connected and organised, as well as alternative ways of approaching a concept and their ability to use this knowledge to judge a given solution as right or wrong. This notion of teachers’ subject matter knowledge as used in this study draws from the work of Shulman (1986, 1987), Grossman (1990), Fennema and Frank (1992) and Ball et al (1990).

Shulman (1986) identified various components of the knowledge base necessary for teaching. Specifically, he proposed a theoretical framework consisting of three categories of content knowledge: subject matter content knowledge, pedagogical content knowledge, and curricular content knowledge. Subject matter content knowledge was characterized as more than the information a teacher knows. Shulman indicated that teachers needed to know about both the content of their subject and the structure of their subject, that is, how the content fits together. They needed to know “what” is true, “why” it is true, and why it is worth knowing in the first place (Shulman, 1986).

Grossman (1990) reorganized the seven categories defined by Shulman into four main categories: subject-matter knowledge, general pedagogical knowledge, pedagogical content knowledge, and knowledge of context. The subject matter content knowledge was composed of the same components Shulman and his colleagues defined: knowledge of content, syntactic structure of a discipline, and substantive structures (Shulman, 1987). Grossman placed Shulman’s third component of curriculum knowledge into the pedagogical content knowledge category (labelled as curricular knowledge). Aligned with the later work of Shulman and his colleagues, beliefs became a part of the knowledge base for teaching. Grossman’s component related to belief was listed as a part of pedagogical content knowledge and comprised “knowledge and beliefs about the purposes for teaching a subject at different grade levels” (Grossman, 1990, p. 8).

Fennema and Franke (1992) suggested a model of mathematics teachers’ knowledge that consisted of three components and a separate factor of teachers’ beliefs about
mathematics. In this model the category called ‘the content of mathematics’, resembled Shulman’s subject matter content knowledge. The content of mathematics includes teacher knowledge of the concepts, procedures, and problem-solving processes within the domain in which they teach, as well as in related content domains. It includes knowledge of the concepts that is behind the procedures, how these concepts are related to one another, and how these concepts and procedures are used in various types of problem solving. Crucial also to teacher knowledge of content is the manner in which the knowledge is organized, indicating teacher knowledge of the relationships between mathematical ideas. Essentially the teachers’ knowledge in this model comprises of the knowledge of the content of mathematics, pedagogical knowledge and knowledge of students’ cognitions. These three components combined with teachers’ beliefs to “create a unique set of knowledge which drives classroom behaviour” (Fennema & Franke, 1992).

Ball (1990) looked at teachers’ knowledge by asking “what kind” of subject matter knowledge was necessary for teaching rather than “how much” a teacher should know about various aspects of mathematics. Her analysis included four dimensions of understanding: knowledge of the substance of mathematics, knowledge about the nature and discourse of mathematics, knowledge of mathematics in culture and society, and the capacity for pedagogical reasoning. Ball’s definition of knowledge about mathematics is closely related to Shulman’s definition of substantive knowledge while her definition of knowledge of mathematics is related to Shulman’s definition of syntactic knowledge. Knowledge of mathematics in turn can be divided into both conceptual and procedural knowledge (Hiebert & Lefevre 1986). Procedural knowledge refers to computational skills and knowledge of procedures for identifying mathematical components, algorithms and definitions. Conceptual knowledge refers to the ability to show understanding of mathematical concepts by being able to interpret and apply them correctly to a variety of situations. In conceptual knowledge mathematical relationships is as important as the separate bits of information.

I have unified the subcomponents of teachers’ subject matter sited in the literature above and shall use same as the conceptual frame which guides this study. The diagrammatic illustration is given in Figure 1 below.
As shown in the conceptual framework, the teacher’s subject matter knowledge is a content knowledge which consists of facts, concepts, ideas, and principles of mathematics. The content knowledge can be viewed as syntactic (knowledge of mathematics) or substantive (knowledge about mathematics). Knowledge about mathematics can further be classified as procedural (instrumental understanding) or conceptual (relational understanding). All the subcomponents of the teacher subject matter are interwoven hence subject matter knowledge shall be seen as a domain whose components are integrated with one another.

**Researches in teachers’ Subject Matter Knowledge**

Ball, (199a) investigated the understanding of division by both elementary and secondary pre-service teachers’. She found that both had significant difficulties with the *meaning* of division by fractions. Most of them could do the calculations, but their explanations were rule-bound, with a reliance on memorising rather than conceptual
understanding. Hill, Rowan, & Ball, (2005) have shown a statistically significant and positive relationship between teacher knowledge measured in terms of the mathematical work that teachers do in the classroom (SMKT), and student achievement. Their study was conducted with educators and learners from 115 elementary schools during the 2000 – 2001 and 2003 – 2004 school years in the USA.

Capraro, Capraro, Parker, Kulm and Raulerson (2005) indicate that mathematically competent pre-service educators exhibit progressive pedagogical content knowledge as they are exposed to mathematics pedagogy during their mathematics method course.

Ma (1999) studied the mathematics content knowledge and pedagogical content knowledge of USA elementary school teachers and compared it with their counterparts from China. Her study indicated that the USA mathematics teachers were unable to clearly represent division by fractions and did not explain its meaning correctly. Their knowledge was relatively instrumental, unconnected and devoid of conceptual grounding. On the contrary, Chinese educators were able to generate multiple representations and also use variety of models of division by fractions which were pedagogically effective. Ma called the strong conceptual grounding in mathematics ‘profound understanding of fundamental mathematics’ (PIFM). PUFM according to Ma represents an understanding in a terrain that is deep, broad, and thorough.

Most of the researches on SMK have been focused on primary school teachers and pre-service teachers whereas the nature of the subject matter knowledge of practising secondary school mathematics teacher is relatively under-researched. The work of Ball and her colleagues are principally located among primary school educators in the United States. Adler and her colleagues in the Quantum project in South Africa and in the UK has focused on deepening pre-service or in-service teachers’ subject matter knowledge in such a way that it is useful in teaching. Not much has been said about practising teachers who are not attending in-service program. Studies on how practising teachers carry out the work of teaching mathematics in specific topics is scanty. The nature of teachers’ subject matter knowledge in all topics in mathematics and particularly at the High School level needs to be investigated. This study will fill this gap by investigating the nature of practising teachers’ subject matter knowledge in quadratic functions.

**METHODOLOGY**

As I mentioned earlier, the study investigates the nature of mathematics teachers’ subject matter knowledge in quadratic functions. The mixed methods approach will be used in this study because using qualitative and quantitative methods together yields a better understanding of the phenomena than using a single method. My aim
of using a mixed methods approach in this study was for one research method to complement the other, and not for the purpose of triangulation. Said differently, the different research methods will address different aspects of the phenomenon, and convergence is not necessarily expected. Findings from the separate components are then fitted together like a jigsaw puzzle. Essentially, the study is descriptive.

Participants
At this stage of the study the sample constitutes of five Grade 11 secondary school mathematics teachers (the main study is made up of twenty-three practising mathematics teachers. The main study is on process) who agreed to participate in the study. The sample is therefore a convenient sample. The teachers are qualified (having at least three years of college preparation or a bachelor of education degree, a degree in the subject area followed by a post graduate certificate in education (PGCE) mathematics). The teachers therefore vary in their experience in teaching mathematics and academic background.

Data collection
Data were collected through two subject matter knowledge tests and semi-structured interview. Subject matter knowledge test 1 constitutes of 10 items which cover the knowledge components indicative of knowing quadratic functions as indicated in the South Africa curriculum. The subject matter knowledge test 2 was used to probe into teachers’ knowledge of the meaning behind their algorithms and skills. Subject matter knowledge 2 and the interview were used for qualitative description of aspects teachers’ subject matter knowledge, which may not be achieved by quantitative means. The subject matter knowledge tests 1 and 2 were validated by two experienced Heads of Department (mathematics), a Mathematics specialist (Subject adviser) in the Department of Education and a lecturer in Mathematics Education in a university. The interview was used to clarify the responses given by teachers in the subject matter knowledge tests; and teachers’ understanding of some concepts in quadratic functions.

A sample of three questions in the Subject Matter Knowledge Test 1 (SMKT1) is presented below.

<table>
<thead>
<tr>
<th>Question 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>The graphs of $m(x) = \left( \frac{1}{2}x + 1 \right)^2 + 6$ and $j(x) = x^2 - 5x$ are shown below.</td>
</tr>
<tr>
<td>$m$ and $j$ intersect at points A and B.</td>
</tr>
</tbody>
</table>
meaningful decision has been reached (Webb, Nemer, & Ing, 2006). These researchers reported that the use of discussion as a tool to increase reasoning has gained emphasis in classrooms worldwide, consistent with earlier reports (Yore, Bisanz & Hand, 2003). Discussion, however, requires scaffolding and structure in order to support learning (Norris & Phillips, 2003).

Wood (2006) found variation in students’ ways of seeing and reasoning, and these were assigned in the first place to the particular differences established in classrooms early in the year pertaining *when* and *how to contribute* to mathematical discussions and *what to do as a listener*, consistent with findings reported by a number of other researchers (e.g., Dekker & Elshout-Mohr, 2004; Ding, Li, Piccolo, & Kulm, 2007; Gillies & Boyle, 2006; Webb et al., 2006). Moreover, participation obligations put boundaries around the opportunities for students to share their ideas and to engage in mathematical practices (Ding et al., 2007; Webb et al., 2006).

Issues of interest to mathematics educators, such as, for example, knowing, can be examined from the perspective of participants in interaction, rather than as underlying cognitive processes which can be used to explain what people do and say (Edwards, 1997). As Edwards & Potter (1992) acknowledge, this is not to say that people explicitly talk about these things. As Sacks showed, these patterns of interaction arise through the social actions of the participants; actions which bring about the on-going organisation of their talk (see Sacks, 1987). For discursive psychology, the social action through which interaction is organised takes precedence over other aspects of interaction, so that the psychological structures and functions of language became shaped by language’s primary social functions (Edwards, 1997).

Talk is about more than its surface content. Every utterance, for example, also constructs the identity and reflects the interests of the speaker, who may present themselves as, loud or polite, knowledgeable or uncertain, biased or neutral. Each utterance, therefore, reflects the partiality or interest of the speaker (Antaki, 1994). Amongst empirical studies of foreign language attainment, a focus on recycling in local classroom communities can be seen in the work of Rampton (1999) on how foreign language teaching is recycled in peer group interactions and participation among adolescents as substantial resources in performance-based identity work.

For learners, discussion, debate and critique are all learned strategies. Sfard and Kieran (2001, p.70) emphasise that "the art of communicating has to be taught". As such, learners should be afforded appropriate time and intelligent space for exploring ideas and making connections (Stein, Grover, & Henningsen, 1996) between classroom mathematics and out-of-school mathematical knowledge and a sustained press for explanation, meaning, and understanding (Fraivillig, Murphy, & Fuson, 1999). Such a
(x + 1)(x - 6) = 0
x = -1 or x = 6

(i) Do you think that the function \( g(x) = -x^2 + 5x + 6 \) and
\[ f(x) = x^2 - 5x - 6 \]
are the same?

(ii) How would you justify that your solution? Apart from substituting the values \( x = -1 \) or \( x = 6 \) in the two expressions.

Question 2

Given that \( f(x) = 2x^2 - 4x - 6 \) and \( g(x) = -2x - 5 \). \( f \) and \( g \) intersect at \( P \) and \( Q \). \( A \) is a point on \( g \) while \( B \) is a point on \( f \). \( AB \) is always parallel to the y-axis. Is it true that the largest distance between the functions \( f \) and \( g \) is at the x - coordinate of the midpoint between \( P \) and \( Q \)? Justify your answer.

Question 3

Given the function \( f(x) = x^2 \). Since the graph of \( f \) is a curve hence any two points in the graph of \( f \) should not be joined with straight a line as in 3(i). Any two points in \( f \) is joined with a curve as in 3(ii).
Show that the distance between any two points in this function should be joined with a curve.
**Data analysis**

Data which were collected from the subject matter knowledge test 1 was analysed quantitatively by using descriptive statistics. The subject matter knowledge test 2 and the interview were analysed qualitatively by first presenting any difficulty found followed by the illustration of each (Cohen, 2007). Teachers’ preferences in solving questions were presented and discussed also.

**RESULTS AND DISCUSSION**

**Result of the quantitative data.**

In the tables that follow, column 1 shows performance in questions 1 to 7 (questions that are not word problems) and column 2 shows teachers scores in questions 8 to 10 (questions that are word problems).

The performance of the teachers in the Subject Matter Knowledge Test 1 is presented in Table 1a and Table 1b below.

Table 1a. Teachers’ scores (in percentage) in SMKT 1
<table>
<thead>
<tr>
<th>Teacher</th>
<th>Score (Questions 1-7)</th>
<th>Score (Questions 8 – 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ms Jennifer</td>
<td>78</td>
<td>33</td>
</tr>
<tr>
<td>Mr Maine</td>
<td>90</td>
<td>0 (did not attempt)</td>
</tr>
<tr>
<td>Mr Paul</td>
<td>64</td>
<td>30</td>
</tr>
<tr>
<td>Mr Seema</td>
<td>95</td>
<td>100</td>
</tr>
<tr>
<td>Ms Mary</td>
<td>85</td>
<td>55</td>
</tr>
</tbody>
</table>

Table 1b. Mean and standard deviation of teachers’ performance (in percentage) in SMKT 1

<table>
<thead>
<tr>
<th></th>
<th>Performance in questions 1 – 7</th>
<th>Performance in questions 8 – 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>82.4</td>
<td>43.6</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>10.4</td>
<td>33.2</td>
</tr>
</tbody>
</table>

Table 1 shows the performance of the teachers in the subject matter knowledge tests. It indicates that four teachers performed very well in questions 1 – 7 (questions that are not word problems) apart from Mr Paul who scored 64 percent. Table 1 also seems to suggest that the teachers have relatively low subject matter knowledge in word problems leading to quadratic functions. Teachers’ difficulties in word problems could be one of the reasons why students have difficulties in solving word problems (e.g., Mohammed et. al., 2012; Sara, 2009). This difficulty has implication on the areas of emphasis in quadratic functions during teacher preparation or in-service programs.

Preferences. The answering pattern in teachers’ responses in SMKT 1 revealed that teachers preferred to proffer algebraic solutions to questions even when the question demanded a geometric approach. In question 1(iii) of SMKT 1 teachers were asked to use the graph of \( m \) to determine the value(s) of \( k \) for which the equation \(- \frac{1}{4} x^2 - x + k = 0\), has real roots.

Three teachers determined the value of \( k \) by using the discriminant (\( \Delta \)): \( \Delta \geq 0 \). Two teachers tried to use the graph of \( m \) to solve the question but they interpreted it wrongly. This result seems to suggest of teachers’ difficulty in moving from one representational form to the other.
Results of the qualitative data

Qualitative analysis of the teachers’ responses in the subject matter knowledge test and interview reveal the following in the teachers’ subject matter knowledge:

- Difficulty in providing meaning behind algorithms.
- Limited conception of the quadratic function concept.
- Over dependency on textbook in proving what is true or wrong.

1. Difficulty in providing meaning behind algorithms.

*Teachers always draw the graph of quadratic functions (parabola) with curve but they had difficulty in explaining why two points in a parabola should be joined with a curve and not a line.* In question 3 of (SMKT 2) the explanation given by the teachers on how they can show that the points in parabola should be joined with curve are shown below:

(i) “I have to use the concept of gradient between two points on the curve. I will draw tangents and calculate gradient between various points. I will show them that the gradients vary therefore we cannot connect points with a ruler since it is non-linear function”.

(ii) “They must know that the parabola has a line of symmetry and should be drawn using free hand and not a ruler. They must know that it is a graph of the second degree so it must have one turning point which is a curve”.

None of the two explanations is capable of convincing the learner to joints with a curve. The first response above only proved that the two points did not form a straight line. It did not prove that the two points formed a curve. The second response has nothing to offer in addressing the question. By selecting any two points on the $x$-axis (for instance 0 and 1) and plotting smaller intervals example 0; 0.1; 0.2; 0.3; to 1, the teacher could have been able to show that any two points in $f$ is a curve as shown below:
However, the teachers were not able to give a relevant explanation though they know the shape of the parabola.

Teachers’ response to question 5 in the SMKT1 (not included in the sample above) was another situation which seems to show teachers’ knowledge of procedures without a deep understanding behind the procedure. In question 5, teachers’ were asked the following questions: Given that \( f(x) = 2x^2 - 6x - 8 \). (i) Complete the square in the expression \( 2x^2 - 6x - 8 \). (ii) Determine the minimum value of \( f \) and the value of \( x \) for which this occur.

Each of the five teachers was able to complete the square in the given expression. They also determined the minimum value of \( f \) and the value of \( x \) for which it occurred. However, none of the teachers read the minimum value of \( f \) and the corresponding value of \( x \) from the expression they got by completing the square. Rather, three teachers used their knowledge of calculus whereby they solved \( f'(x) = 0 \) to get the \( x \) value and thereafter the minimum value of \( f \). Two teachers determined the value of \( x \) by using the formula \( x \) (at minimum point) = \( -b/2a \) and there after they calculated the minimum value of \( f \). Teachers not reading the minimum value of \( f \) and the corresponding value of \( x \) from the expression of completing the square seems to suggest that the teachers lack understanding of the entailment of the expression they got from completing the square. Or could it be that the teachers wanted to show case their higher knowledge of mathematics and hence they didn’t want to read the answer from the expression they obtained from completing the square? Unfortunately, we did not verify this from the interview.

2. Limited conception of the quadratic function concept.

Quadratic functions can be represented symbolically (algebraically) as an equation but that does not mean that a quadratic function is the same as quadratic equation. In fact there are other representations of quadratic functions which include tabular,
diagrammatic, as set of ordered pairs etc. However, teachers perceived quadratic functions to mean the same thing as quadratic functions. Other representations for instance the tabular representation (“table of values”) were conceived not as representations but only as a tool or as one of the processes that must be followed in drawing graphs. When the teachers were asked if there are any differences or similarities between quadratic equation and quadratic function, the following are responses of the teachers:

Respondent 1. “I have not thought about their differences before”.

Respondent 2. “With quadratic equations basically we are looking at the roots of a function. When you are talking of equations we concentrate in may be on the roots. Like if you are to draw a graph for example the x-intercepts of that particular equation or functions in fact these things are related. So when you talk of function, a function would have input values and output values. Right. So for a function you have got so many input values given out many output values. Right. And then when you talk of an equation you concentrate on specific kind of input and output. The intercept of that graph. Basically we are saying that the intercepts are not like for example if the intercepts do not exist it would mean that the roots will not be a real root. When you try to solve the equation you get non real roots”.

To respondent 2, the relationship is that quadratic equation is just a means of finding the roots of the quadratic functions. This teacher seems to have ignored that the equation of the quadratic function is just a representation of the quadratic functions. It can be used to find not only the x - intercept(s) of graph (the zeros of the function) but can be used also to find the value of the range for every value in the domain.

3. Over dependency on textbook in proving what is true or wrong

Generalizations and testing conjectures is a very important aspect in knowing quadratic functions in particular and mathematics in general. But teachers depend on textbooks to verify if a conjecture is true. In question 2 of (SMKT 2) teachers were asked if a conjecture that was presented in the question was true. Instead of testing the conjecture by solving some question of that nature and comparing the answer with the conjecture, a teacher’s response was that he will check out the answer in the textbook as if the conjecture and the answer to the conjecture were in a textbook.

CONCLUSIONS

This study is part of a research project that focuses on secondary mathematics teachers’ subject matter knowledge in teaching quadratic functions. Our objective is in this paper is to explore nature of the teachers’ subject matter knowledge. This study has
highlighted some empirical evidence for discussing teachers’ subject matter knowledge. On the basis of this study we concluded that some teachers possess low level of subject matter knowledge in word problems leading to quadratic functions. We also concluded that teachers have difficulties in providing meaning behind algorithms; limited conception of the quadratic function concept; and over dependence on textbook in proving what is right or wrong. These findings imply that emphasis be given to be given to word problems and application in teacher preparation programs. Multiple representations and the inter relatedness of the different representations should also be emphasised. Teachers should be trained to analyse problem situations so as to differentiate a correct statement from a wrong one. Teachers should be provided with opportunities to explore and discover different methods and strategies to solve mathematics problems.

REFERENCES


van der Berg, S., Burger, C., Burger, R., de Vos, M., du Rand, G., Gustafsson, M., Shepherd,
This paper is concerned with the field of continuous professional development of mathematics teachers, specifically focusing on classroom teaching analysis. In particular we describe and analyse one teacher’s single lesson in a classroom and her interaction with her learners. We viewed this lesson through three specific lenses: communication, classroom management and the mathematical content of the lesson. The findings show that the dominant teaching style used was the expository form. But as the analysis shows, the use of this form is not simple, but complicated. When Brendefur & Frykholm (2000)’s four communication types were used to analyse communication between the teacher and the learners in the classroom, we found a predominance of two kinds of communication, namely uni-directional and contributive. The mathematical content of the lesson was clearly outlined and enacted, however, we also found that the learners were mainly passive in during the lesson and that the lesson was driven by the teacher.

INTRODUCTION

Recent lines of research have alluded to the growing quest for high quality continuous professional development (CPD) where mathematics teachers work together in communities of practice (CoP) with the aim improving their practice – key to this notion is that these learning opportunities through the CoP should be located in the practice of teaching (Hung & Chen, 2001). Lave and Wenger (1991) using the coinage - legitimate peripheral participation, argue that learning is achieved through increasing participation in CoP - essentially what happens is that teachers, start by being mere observers with minimum interaction in the activities of the community, that is being on the periphery of the CoP. However, with the passage of time, the teachers become full participants in the activities of the CoP thereby making contributions that not only influence the individual but also CoP and its future (Brown et al., 1989; Hung & Chen, 2001; 2007; Roschelle & Clancey, 1992). In this research, the Local Evidence Driven Improvement of Mathematics Teaching and Learning Initiative (LEDIMTALI) project as a CoP, provides the ideal learning environment where mathematics teachers, both novice and experienced, from a designated group of schools come together, analyse videos of themselves and their colleagues teaching in their own everyday practice situations – that is, work practices situated in their own contexts. Video recordings that teachers use in the CPD provide powerful authentic learning contexts where the teachers’ learning is anchored in their own classroom practices (Borko et al., 2008;
Putnam & Borko, 2000; Van Es & Sherin, 2008; Sherin & Van Es, 2002, 2005). Because we are highlighting video in this research, we are aware of other alternative data sources such as learners’ written work, learners’ written tests, and curricular materials that that can be used in CPDs – these are not discussed in this study.

Analysing teaching episodes among in-service teachers using video as tool

This paper describes use of video in a CPD programme whose aims, among others, are to improve the teachers’ pedagogical content knowledge (PCK) and the mathematical content knowledge (MCK), and improve classroom practice, thus ultimately fostering quality lifelong acquisition of mathematical skills. In the past decade the video has become an important tool for in-servicing teachers, thus providing professional, intellectual, and a research platform where novice, veteran teachers, researchers in mathematics teacher education at universities interact in CoP placing “emphasis on helping teachers learn what to do in the classroom” (Sherin, 2004, p.1). In so far as enhancing classroom practice is concerned, video can be used to initiate discourses on how practising teachers can present problematic mathematical concepts during lessons, in other cases illustrate key methodological episodes of practice (Beck et al., 2002). The objective of this study was to deepen understanding on the episodes of engagements between teachers and learners as they interact during lessons. By an episodic moment of engagement we are referring to a time based phase of interaction between the teacher and a learner or learners which highlight a particular aspect of mathematics education. These episodes will explicitly be pointed out in the paper. In the past decade, as corroborated by recent lines of research (Borko et al., 2008; Brophy, 2004; Seidel et al., 2011; Tripp & Rich, 2012), the use of videos in teacher education has become increasingly prominent primarily because of their “unique capability to capture the richness and complexity of elusive classroom practice” (Zhang et al., 2011, p. 454). The quest to understand classroom interaction between teachers and learners is something that has been part of mathematics teaching and learning research – through classroom observations between peers, education authorities visiting schools, and teacher educators training pre-service teachers (Beck et al., 2002). Our attempt in this paper is to show how, through video analysis; we can capture episodic moments of engagements between the teacher and the learners which inform research in mathematics teacher education.

The Study

The purpose of this study is to characterise the episodes of engagements of teaching and learning through the descriptions of video recorded lessons of one mathematics teacher participating in a CPD through a CoP. Our aim is to show a particular methodology of description of these episodes. To be clear, the study will focus on the...
following episodes of engagements during lessons: classroom management, communication, and mathematics content. This study has both theoretical and practical implications to highlighting tool of description in CPD, in that it advances our understanding of the nature of the selected episodes of engagement of teaching and learning. In addition, the findings of the study will provide the much needed valuable information for teacher educators, researchers, and teachers of mathematics of the use of videos to support CPD activities. Therefore, the research question for this study is: How can one best describe episodes of teaching and learning engagements, in particular classroom management, communication, and mathematics content?

THEORETICAL FRAMEWORK

Situated cognition

The CPD of mathematics teachers as espoused by the objectives of LEDIMTALI is grounded in the situated cognition theory of learning. Proponents of situated cognition argue that in addition to the acquisition of knowledge, learning is a continuously occurring phenomenon situated in a context (Greeno, 2003; Greeno & Moore, 1993; Kirshner & Whitson, 1997). Borko et al. (2008) posits that in situated cognition “knowing and learning are constructed through participation in the discourse and practices of a particular community, and are situated in particular physical and social contexts” (p. 418). In sum, Little (2002) argues that situated cognition can foster strong CPDs that can enhance the teachers’ pedagogical content knowledge (PCK) and mathematics content knowledge (MCK) leading to an improvement in practice.

METHOD

Participants

One mathematics teacher participating in the LEDIMTALI project for CPD gave her consent for her lesson to be video recorded for the purposes of sharing classroom practices within a CoP. The content and the timing of the lesson which was video recorded was left to the discretion of the teacher. For the purposes of this study, the teacher whose video recorded lesson is analysed is referred to with pseudo name “Sarah”.

DATA COLLECTION

The data source for this paper was the transcription and analysis of the video recorded lesson of Sarah, a high school teacher, teaching in a government school. The video recording forms part of the data recorded in the LEDIMTALI project. The field workers who recorded the videos focussed on the teacher. We accept that the data is skewed and
that we are not the authors of the data. Our role is in the selection of aspects of the video recording which suited out research intentions. As such we accept a certain bias in our findings. We began the data collection process by viewing the video several times, this was followed by creating a list of observed episodes of engagements of the lesson and the teacher’s practices. A pre-determined framework, based on the work of Brendefur & Frykholm (2000), Frederiksen et al. (2008), and Star and Strickland (2008), was used to identify and to characterise the three episodes of teaching engagements from the video transcriptions namely, classroom management, mathematical content, and communication illustrated in Table 1. The characteristics were coded in the data as follows: For mathematical content we looked for instances when the teacher or the learners referred or used mathematics. Though this may seem odd as this was a mathematics lesson, it will be shown that there are numerous other activities which occur in the classroom, not all of which can be classified as mathematical content. Hence it was necessary for us to single out these instances. So we listened out for the use of mathematical terms and operations. And we looked out for mathematical signs and symbols on the blackboard. For the classroom management instances we listened out for indications that the teacher was managing what happens in the classroom, from straightforward instructions such as “take out your books” to oblique ones such as, “learner X I am watching you”. We also watched when she made use of a learner or learners to participate in the management of the class, for example when they handed out paper for the learners to work on. As far as communication types are concerned, we identified four; we listened out for shifts and also identified the type(s) of communication that was predominantly used by the teacher during her lesson.

**Table 1: Episodes of teaching and learning engagements**

<table>
<thead>
<tr>
<th>Episodes of teaching and learning engagements</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classroom management</td>
<td>Includes the ways the teacher deals with disruptive events, pace changes, procedures for calling on learners or handling homework, and the teacher’s physical presence (e.g. patterns of moving around the classroom, strategies for maintaining visibility, tone and volume of voice).</td>
</tr>
<tr>
<td>Mathematical content</td>
<td>Includes representations of the mathematics conceptions, examples used, sequencing of the problems, nature of the problems posed, cognitive level of the questions.</td>
</tr>
<tr>
<td>Communication</td>
<td>Refers to learner-learner as well as teacher-learner talk, includes questions posed, answers or suggestions offered or word choice. Four groups of communication emerged and we used categories adopted from Star and Strickland (2008)’s framework: uni-directional – teachers dominate...</td>
</tr>
</tbody>
</table>
discussions through lecturing, posing closed questions, that is questions where there were definite answers that the teacher expected, and limiting learners’ participation in classroom participation; contributive communication – interactions among learners, and between learners and teachers is limited to assistance or sharing, that is when the learners “complete” sentences begun by the teacher. These sentences usually refer to a mathematical operation. Key to this type of communication is the “this is how it should be done”; Reflective communication – In addition to sharing ideas teachers and learners use their experiences or knowledge gained to deepen their understanding of the discourses of the mathematical conceptions. We recognise this when a learner reflects back to the teacher something that has been learnt. Instructive communication – this type of communication will allow learners’ mathematical thought processes to be visible to the teachers, thus giving teachers the opportunities to understand the conceptual misconceptions their learners have – this in-turn will shape and inform the subsequent instructions designed by the teachers.

Source: Adapted from – Star and Strickland (2008, p.113)

It is important to note that we regard these categories of episodes of learning and teaching engagements as illustrations of the different types of substantial classroom features that in-service teachers grapple with during practice. However, it is also apropos to point out that we are not claiming that this framework is optimal, though we contend that it is useful and sufficient for the objectives of this study.

RESULTS

In analysing the video lesson of Sarah, the researchers were interested in observing episodes of interaction between teacher and learners which can be described as communication, classroom management and mathematical content. Of course, these and other elements are not appearing exclusively, but come about as a result of the interaction between the teacher and student in the classroom.
Classroom management

In the previous lesson Sarah had taught the lesson on factorising the quadratic trinomial $x^2 + bx + c$ where $b < 0$ and $c > 0$. Sarah sets the tone for the current lesson, which is focused on the factorisation of the quadratic trinomial $x^2 + bx + c$ where $b < 0$ and $c < 0$ by announcing the goal:

We basically continue where we stopped yesterday, ok. Remember what we learnt yesterday (reads off the board). We learned to factorise quadratic trinomials with a positive coefficient of 1 for $x^2$, a negative coefficient for $x$, and a positive constant term. … Today we say (writes) we have two negative numbers $x^2 - x - 12$ and a positive coefficient of 1 for $x^2$.

This action functions on at least two levels: it alerts the learners to what is to come and in so doing it alerts them to the action that is required of them. Sarah does not need to spell it out to them as this is not new. Later, as she organises herself for the lesson, she glances at the class and reminds them of their responsibility:

Your books should have been out long ago, right?

Then, before she is drawn to her tasks at hand, she informs them, by referring to the board, what changes they are going to make to yesterday’s lesson and instructs them to start writing down the outline of the lesson from the day before which is still on the board. In that time (maybe the first minute), she has demonstrated her confidence with the class, since she is able to leave them to get on with the task while she is involved in another task (setting herself up), she has reminded them of her working style and their role in the lesson, namely that they will prepare themselves for what is to come while she engages in a different activity of preparation. All this while moving around the front of the room, she is alert and calls out individual names when she spots potential infringements of her instruction.

When she and a learner start to hand out the white paper (“white board”) we notice that the learners sit in rows of three (groups), while some have larger groups (those on the corners). We are informed through the shift of the video focus that the teacher may use some form of group work in her class, since the learners are grouped. The standard seating of the classroom desks is that they all face forward. She refers to their groups and checks that the group has the resources she wants them to have and she emphasises:

Ok on your white board you are going to show me the answers.
When she signals that she is ready to begin the lesson she declares:

Ok right, date today the 24th (makes the change), and here, we factorise two positive terms and one negative term.

This action functions both as an organising method to prepare the learners for the focus they will need for the lesson, but at the same time it brings the class to a central focus point. The focus on the lesson to come and the focus on what the teacher wants from the class which the call also brings function integrally: both functions are served at the same time. Furthermore, involving the class in reading what is written and changing the signs of the quadratic trinomial written on the board in anticipation of the lesson, all function in this double way. Her reference to the positives and negative refer to a>0, c>0 and b<0. There are other more ways in which Sarah manages the class: she draws out individual answers and gets learners to raise their hands to indicate their choice (see excerpt 1):

Excerpt 1

Teacher: Ok Alia I am taking that risk to ask you. Ok. Right. Let me quickly ask you this small question. Who still does not understand that? (Sarah counts 6 hands and then addresses them). If I say factorise the last number (points to 12). Let’s leave out the negative sign. Then I mean which two numbers you multiply with each other to get to that number. Let us take a simple number. Come we if we say, let’s say here we have minus six (circles 6) and we only look at 6 which numbers do you multiply with each other to get to 6?

Learners: 1x6, 2x3. (Then she asks for a show of hands) (Learners are requested to raise their hands if they want to respond to the question)

Teacher: 2x3. What? Again? Now you must just swing it around 3x2. (asks another learner for the last pair reversed). We have 1x6, 2x3, 3x2, the answer is here. (Class shouts out the answer in the meantime). Chesline hesitates. Answers. (Teachers nods)

Teacher: 6x1. Ok right, so those numbers you multiply to get to the last number. Has everyone got...who still does not understand that? (Silence). Everyone got it now.

There are ways in which Sarah’s management breaks down, mostly when the learners get excited and clamour to shout the answer to a (closed) question:
Teacher: Ok right, now try again: what do we say we are going to factorise, what do we mean?

Learners: (shout): Factorise!

Teacher: ah ah no you don’t shout out to me.

But more times during the lesson she enlists the whole class to join in chorus with answers. A big part of her classroom management style is also to keep everyone involved in the lesson, engaged in the whole class activity which she leads from the front. Thus throughout the lesson she is on the lookout for learners who may be losing focus. This may be indicated by those who are no longer participating in the lesson (they may look away, or talk to a friend, or be distracted by something else) – we see her calling on specific learners to respond to questions which members of the class shout out anyway, for example Alia and Chesline (see excerpt 1).

What is instructive, and is easily lost unless one is looking for it, is that Sarah has to go through quite a lot of preparation which would, in a better endowed school, be taken for granted as a given: the piece of chalk she has hidden on top of the blackboard (presumably so that it is not removed between the last lesson and this one), she uses a piece of paper to clean her board; she leaves previous work on the board (which may indicate her particular teaching style or could reflect the shortage of chalk); she hands out A4 white pages and khokis on which the learners will work as if they are white boards. These are all pointers to a classroom type situation which is akin to schools in working class areas with poor resources. Although some may argue that a blackboard, white paper and khokis are a luxury.

Communication

The dominant form of communication of the four categories is contributive communication, in which the learners complete the mathematical statements which the teacher starts. This is usually done by the teacher to check that sufficient numbers of learners are following her process. There is, however, an opportunity right at the start of the lesson for the teacher to engage the whole class when she checks if the class has learnt the mathematics of the previous lesson

Excerpt 2

Teacher: Right, come we see, come let us be sure if we know ... (Moves away from the board, closer to the class). Who is going to explain to me what factorisation is? (After a pause): Who is going to explain if we say we are going to factorise what we mean? (Another pause) Who wants to tell me? No one? (Nervous laughter).
Learner: Miss must explain that again … (One student near the front speaks).
Teacher: Must I explain that again?

Learners: Yes (chorus)
This is a form of instructive communication, in which the learners make visible their mathematical understanding. We see that the teacher abandons her attempt to get them to inform her about what they remember about the process of factorisation of a quadratic trinomial fairly quickly, after she asks them the question, when they hesitate to answer (in part because of the presence of the strangers in the classroom, perhaps) she says: “I am not going to explain the whole thing; what I am going to explain” – this compels the learners to take initiatives in the learning process. But the use of contributive communication ties in with a teacher led, teacher controlled lesson in which the learners play a participatory role, but as support, not joint leaders, or initiators of new ideas, as for example, the reform mathematics movement, that is the notion that the learners should be involved in constructing their own knowledge in mathematics rather than being told by the teacher what content to learn. It has been argued in the section under classroom management above that this form of communication is consistent with an expository didactic form of pedagogy. In an environment, both within the classroom and without, in which control is key, the teacher places a strong emphasis on control in the classroom, dictating the pace, the changes which occur in a lesson, the responses to her questions, and so on.

There are examples where learners attempt to respond to the teacher’s questions individually but she glosses over them or takes them on board, but stays focussed on a whole class response. Again this is as much a form of classroom management as it is a type of teacher centred pedagogy (see excerpt 1).

Mathematical content

With the mathematical content aspect we were interested to observe any instances of conceptual development or discussion of concepts as may be made visible in the discussion or questions the learners asked. Within the expository teaching framework, that is, traditional teacher dominated didactic teaching, conceptual development “from within”, as in constructivist learning, is often lagging; the teacher brings the concepts from “without” the experience of the learners in this format. For example, we notice this when she tells the learners how to recognise the different types of quadratic trinomials and as a result how to factorise them.
The mathematical content of the lesson is a key focus of the paper, largely because classroom management and communication are geared to aid the mathematics in the lesson. The teacher’s reminder of the previous lesson (as already indicated), contextualises the lesson for the learners. While she organises herself and organises them, that is the focus to which they return when she is ready. Looking back to the previous lesson onto which she will tie this lesson, she tests whether the learners are set up for the lesson: do they remember what concepts were learnt in the previous lesson.

**Excerpt 3** (with some overlap with excerpt 2)

Learner: Miss must explain that again. (One learner near the front says).

Teacher: Must I explain that again?

Learners: Yes (chorus).

Teacher: I am not going to explain the whole thing; what I am going to explain…

Learner: Please show us an example.

Teacher: An example, right. (Writes on blackboard and continues to talk to the class). If we have 12, what do we do with that?

Learner and other voices: factors.

Another learner: factorise.

The use of the very personal: who can explain “for me”, although a colloquialism, underlines the uni-directional communication that underlies much of the classroom activity. The learners are invited to exhibit their knowledge to the teacher, almost as if it is for the teacher’s benefit (thus personalising the task). But this seemingly innocuous form also underlies the direction of control in the classroom, that is, from the teacher to the learners. This is clear at this early stage of the lesson. An alternative option would have been to ask the learners to explain “to the class” the meaning of factorisation.

With her eye perhaps on the lesson to come, she moves straight into a revision of the previous lesson - a good teacher will cover ground even if it has been covered, in the event that some of her learners may not have kept pace with the class, and, particularly, they need the concept of factorisation in order to make sense of the lesson to come.

The teacher checks if her class understands the concept and shows her preparedness to re-teach it a second or third time (see excerpt 1). She accepts their word that they
understand after her explanation: at this point there is no written or individual assessment to confirm that they do in fact understand.

It is when we analyse the various interchanges between the teacher and the learners, that we find her actually changing the intent of the question midway.

Excerpt 4:

Teacher: Ok right, now try again: what do we say we are going to factorise, what do we mean?

Learners: factorise (shouting).

Teacher: ah ah no you don’t shout out to me.

Learners: factors of 12.

Teacher: (goes to the blackboard). Right, that still does not tell me nicely what we mean. If we say factorisation what do we mean by factorisation.

Learner: factorise.

Teacher: You are right (to the student) but what do you do when you say you are going to factorise.

Learner: You must see what goes into 12 Miss.

Teacher: Ok right (clearly not happy with the responses). (More answers are shouted out).

Learners: How many numbers go into 12 Miss?

Teacher: Ok right, the description I am looking for is, those numbers (points at the factors of 12 on the board), especially in terms of what we are busy with, you must see which two numbers you must multiply to get 12. Does that sound familiar now?

Learners: Yes Miss (chorus), that is what we said Miss.

Teacher: Yes, but you say the word factorise, still. I want you use the word factorise when you are telling me what you are going to do.

With her eye on what she wants to do with the factors of 12, she is looking for factor pairs, but nowhere gives the learners a clear idea what it is she actually wants. Instead she waves away proper answers of learners and then produces her pairs as if that has been the answer to the question all along. What is striking about this kind of teacher-
learner relationship is that no-one contradicts her strongly enough that she has to deal with it. The learners echo their consent when she asks for it, even though there were a few grumblings from those who felt they had given the correct answers anyway (see excerpt 4).

This form of silent consent (from the rest who may also have felt aggrieved) is consistent with the kind of classroom we are dealing with: the learners are essentially disempowered and depend on the teacher to put her stamp on everything which occurs.

Part of this exercise is to get the learners to respond to the question she posed at the start in the way she describes it to them here: she links finding factors to the process “factorisation” and says very explicitly: “when I ask you the question I asked you then this is what you must say”. In other words she is inducting them into the language of mathematics in a very direct way: this is how you must respond to the question about factorisation. Her style is very direct.

When the teacher introduces an example of the new type this is sign change from the previous lesson, which considered the trinomial: \(x^2 - x + 12\). Now we look at: \(x^2 - x - 12\). When she removes the 1 in front of the x, she refers to the 1 as imaginary.

**Excerpt 5:**

Teacher: Ok we must take an example (Cleans one board. There is still work unrelated to the lesson on other boards). Yesterday we saw there were two positive numbers and one negative number (repeat) today we say (writes) we have two negative numbers \(x^2 - x - 12\). … One x means actually what? (Erases the 1). Just x. 1 is imaginary (points a finger to her head).

Perhaps she meant to say invisible, which would be more acceptable (actually it is a convention: the 1 is assumed). It is incorrect to use imaginary as in the extension of the number system that word has a specific meaning as part of a complex number, something which the learners will learn about in a higher grade. Almost immediately after, she adds the 1 again, without explaining why, and again no one asks or challenges her on it.
She clearly points out the contrast between the previous quadratic trinomial type and the new one, and then pauses as she figures out what she has written before and how it should change.

**Excerpt 6:**

Teacher: Right now we have to completely change this piece: continues to reads her notes aloud …write the two numbers in brackets which give the middle term when you add them. … So up to that …up to that… (Re-reads aloud to herself what she wrote before) (And then aloud to the class as she erases the “add” Not when you add but when you… (Looks expectantly at the class)

Learner: Subtract! (shouts)

Teacher: There you are! (she says triumphantly, smiles). Who shouted that?

Learner: Tyrone. (someone says). (Momentary respite as she looks around, smiles, continues).

Teacher: Ok but when you subtract minus (to herself) minus minus minus minus (thinking). Minus is the process, the mathematical term we are going to use when you subtract.

Learner: Is it not different Miss? (Shouts).

The procedure that gets taught is that, because the difference between the factors in the factor pair has to equal the numerical value of the middle term (“ignoring the sign”), that “subtraction” is the method. The difficulty of teaching the method so that learners memorises it is that the learners have to remember to insert the negative and positive signs inside the brackets in such a way that the correct middle term results when the brackets are removed. This method is usually consistent with the method of teaching addition and subtraction of integers, in which the sign are ignored initially and the correct sign inserted afterwards. The method may work well but it has the drawback that it presents as a recipe which learners have to remember, with little fall back on the principles involved.
CONCLUSION

In this paper we analysed the practice of a teacher, focusing on classroom management, communication between teacher and learners and the mathematics content of the lesson. Our analysis of the lesson shows that the findings show that the dominant teaching style used was the expository form. Clearly there was little evidence of learner centredness. But as the analysis shows, the use of this form is not straightforward: there were many instances in which the learners intervened and offered directions for the lesson.

Brendefur & Frykholm (2000)’s four communication types, uni-directional, contributive, reflective and instructive was used to analyse communication between the teacher and the learners in the classroom. We found a pre-dominance of the first two kinds of communication. We also found forms of communication which did not fit easily into the categories, such as communication to assist with classroom management.

The mathematical content of the lesson was outlined throughout the lesson, interspersed with classroom management activities, including keeping discipline and instructing the class on the progress and pacing of the lesson. We found that the teacher engaged the class as a whole as the dominant means and as such she attempted to control and contain the class. In addition, the findings show that the learners were mainly passive in their involvement, as a consequence of the teaching style of the teacher; the lesson was driven mainly by the teacher. We also found that the teacher hesitated at times while delivering the mathematical content, and at times made wrong mathematical statements.

RECOMMENDATIONS

Teaching involves a myriad of activities. We have only focused on communication, classroom management and the subject content through analysing a video recorded lesson using the framework. Follow up interviews with the participants would assist the overall picture of what happens in a class. The responses of the learners to a researcher may throw more light on the process and their experience of being in a class such as this. Video analysis of teachers teaching should be used in a developmental form as part of CPD, both to assist teachers in their teaching and the showcase good practice.

REFERENCES


This paper reports on a study following the classroom practices of two high school teachers whilst teaching inequalities to grade 10 learners in KwaZulu Natal of South Africa. The study was conducted to establish the nature of instructional support that can generate in learners the kinds of mental representations that will enable them to think about critical differences when engaging in symbol manipulation activities involving inequalities. This was a qualitative study which explored APOS and Variation theory to compare and establish the best practices in teaching inequalities through observation of two teachers’ presentations of inequalities to grade 10 learners. Semi-structured interviews were conducted with the teachers on the choice of activities, classroom discussions, and exercises used to enforce mental representations of the concept. Data was analyzed using APOS (actions, process, objects and schema) theory together with the variation theory in investigating the level at which learners operated. The paper concludes by suggesting that learners should be allowed discernment of the concept, so as to be aware of certain features which are critical to the intended way of seeing this concept. It was also found that the essential features of the concept through varying the non-essential features should be highlighted. Lastly it was established that the learners who demonstrate, argue and explain their solutions to others operate on either the Object stage of APOS and demonstrate ability to fuse, connect and relate the concept learnt to other concepts.

INTRODUCTION

The concern on mathematics teacher knowledge has been on the teacher education and research in South Africa since the publication of the Presidential Education Initiative (PEI) report (Brodie, 2004:65). In this report it was found that teachers needed a relevant conceptual knowledge of mathematics in order to help learners to understand and internalise mathematical content. Ball & Bass (2000) in their interpretations of pedagogical knowledge, identify knowledge of teaching strategies and multiple representations through use of (i) appropriate activities, (ii) real life examples and analogies, (iii) different instructional strategies, (iv) different representations like graphics, tables, and formulas in instruction. This study embarked to shed some light on the instructional design in enriching mental constructions in the learning of inequalities where two teachers reflected on their choices of instruction regarding the topic.

(Carter, 2011) explains instructional design as a process by which instruction is improved through the analysis of learning needs and systematic development of
learning material. According to him, for a teacher instructional design should include creation of effective meaningful lessons, helping learners to make sense of information, and cutting through extraneous information in a lesson. He further articulates that the choice of instruction to be used in a lesson depends on the teacher’s knowledge of the concept, but should be guided by preconceptions, misconceptions and the difficulties that learners could experience in learning the concept. Ernest (1991) asserts that Mathematics Education understood in its simplest and most concrete sense concerns the activity or practice of teaching mathematics. He further asserts that learning is inseparable from teaching and that the process involves the exercise of the mind and intellect in thought, enquiry, and reasoning. There is however a growing research support for designing classroom instruction that focuses on developing deep knowledge about mathematics procedures (Burke, Erickson, Lott & Obert, 2001). Meanwhile when instruction is focused only on skillful execution, learners develop automated procedural knowledge that is not strongly connected to any conceptual knowledge network (Star, 2000). This instruction usually results in procedures that are not intelligently or systematically executed.

When Blanco and Garrote (2007) conducted a study with first year college learners in Spain, their findings indicated that not only many middle and high school learners possess misconceptions and difficulties about inequalities, but first year college learners also possess these misconceptions in solving and interpreting inequalities and equations. Much research has been conducted on equality but little has been done on inequalities (Verikios & Farmaki, 2010). Mental representations for inequalities should be fostered through an instructional choice of activities meant to evoke learners to make mental constructions critical to enable the learner to fit the concept into bigger picture. With regard to learning inequalities in mathematics with understanding, the teacher has to use different teaching strategies like the art of questioning to establish learners’ internal networks. This has nothing to do with what was taught the previous day, but rather mental representations that each learner forms as part of a network of representations. The manner in which new knowledge fit with what the learners already know, and how those representations can be connected to existing knowledge networks, determines the degree of understanding indicated by the number and strength of connections.

LITERATURE REVIEW

Equations were often found to serve as a prototype in the algorithms for solving linear inequalities (Tsamir & Bazzini, 2004). This is an obscure view since mathematical inequalities are one of the crucial mathematical topics requiring student understanding of various other mathematical topics such as analytical geometry, trigonometry, and algebra. They also play a critical role in developing conceptual understanding of
equality and equations because inequalities have been considered complementary to learners’ equality understanding (Tsamir & Almog, 2001). Also, many researchers, (Kieran, 2004; Tsamir & Bazzini, 2004; Vaiyavutjamai & Clements, 2006; Ellerton & Clements, 2011) have noted that often learners operate on inequalities as if they were equations. Vaiyavutjamai & Clements (2006: 131) echoed this when their findings on learners’ solution of inequalities revealed that the most common incorrect answer was the answer that would be obtained by regarding the inequality as an equation. In this case an inequality would be solved by treating it as an equation, except that the symbol used in the original statement of the inequality would be repeated throughout the problem. They also found that most of the time the learners assumed that there was one answer to an inequality problem. Ellerton & Clements (2011:399) noted that “most of the mathematics teachers treated inequalities as equations that happened to have inequality signs rather than equal signs.”

Inequalities are taught in secondary school as a subordinate subject (in relationship with equations), dealt with in a purely algorithmic manner, taught in a manner to avoid the difficulties inherent in the concept of function in a sequence of routine procedures which are not easy for learners to understand. Also interpretations and control algebraic transformations are performed without taking care of the constraints deriving from the fact that the \( \leq \) or \( \geq \) sign does not behave like the = sign. In order to think about mathematical ideas there is a need to represent them internally in a way that allows the mind to operate on them. As relationships are constructed between internal representations of ideas, they produce networks which could be structured like vertical hierarchies or webs. With regard to learning mathematics with understanding: a mathematical idea or procedure or fact is understood if it is part of an internal network, the mathematics is understood if its mental representation is part of a network of representations. This forms the genetic decomposition of a concept which refers to the set of mental constructs which the learners should construct in order to understand a given mathematical concept. The genetic decomposition in this study was composed in terms of mental constructions (actions, processes, objects, schemas) and mechanisms (contrast, separation, generalisation and fusion) learners might employ when learning inequalities.

THEORETICAL FRAMEWORK

This study is based on both the variation theories, (Leung, 2012) and APOS (Dubinsky & McDonald, 2001). Leung (2012) is of the opinion that teaching and learning of mathematics is about providing learners opportunities to experience mathematics and to create (new) mathematical experiences. To this end Leung (2010) describes mathematical experience as the discernment of invariant pattern concerning numbers and/or shapes and the re-production or re-presentation of that pattern. He further argues
that there needs to be a connection between the learner and the ‘object of learning’ in mathematics. According to him, the object of learning refers to the mathematical concept being learnt and it can be distinguished from or connected to other concepts. He therefore proposed the variation theory where he defined variation as what changes, what stays constant and what the underlying rule is, in any phenomenon.

Variation is defined by its critical features that must be discerned in order to constitute the meaning aimed for in a lesson (Marton & Tsui, 2004). The teacher has to decide on an instructional design that accommodates a pattern of variation as a useful tool in structuring teaching to help the learners to construct relevant mental constructs for the concept to be learnt or object of learning. The object of learning refers to algebraic inequalities in this study. Marton (2009) proposed four kinds of awareness brought about by different patterns of variation. These include:

- **Contrast** which presupposes that for one to know what a concept is, a learner has to discern and know what the concept is not, (Leung, 2012). Teachers could use both examples and non-examples to stress how the concept differs from related ones. For example it is very important for learners to know that inequalities are not necessarily equal signs and should be treated differently from them.

- **Separation** which assumes that all concepts have a multitude of features, each of which gives rise to different understandings of the concept. Ling (2012) therefore suggests that it is necessary for teachers to consider learning as a function of how learners’ attention is selectively drawn to the critical aspects of the concept.

- **Generalisation** according to Chik, Leung and & Marton (2010) refers to the verification and conjecture making activity that checks out the validity of a separation.

- **Fusion** is the simultaneous discernment of all the critical features of a concept and a relationship between them which allows a learner to make connections gained in past and present interactions. For more elaborations on the variation theory see Mhlolo (2013, p11).

According to Asiala, Brown, De Vries, Dubinsky, Mathews & Thomas (2004) proposed a specific framework for APOS Theory based on research and curriculum development, in undergraduate mathematics education. The framework consists of the following three components: theoretical analysis, instructional treatment, and observations and assessment of student learning. The theoretical analysis helps to predict the mental structures that are required to learn the concept in the form of a proposed genetic decomposition. This then informs the instructional design and implementation suitable for each mathematical concept. These are then used for collection and analysis of data.
APOS Theory proposes that an individual has to have appropriate mental structures to make sense of a given mathematical concept. The mental structures refer to the likely actions, processes, objects and schema for learning the concept. For elaboration on these mental structures see Jojo, Maharaj, & Brijlall (2013), p 648.

While the APOS Theory and its application to teaching practice are based on an assumption on mathematical knowledge and a hypothesis on learning, variation is concerned mainly with important features on the object of learning that must be discerned in order to afford relevant mental constructions for a mathematical concept. It is for this reason that this paper analyses the data collected on the bases of these theories (see fig 1).

This was a qualitative study which explored APOS and Variation theory to compare and establish the best practices in teaching inequalities through observation of two teachers’ presentations of inequalities to grade 10 learners in different schools. Semi-structured interviews were conducted with the teachers on the choice of activities, classroom discussions, and exercises used to enforce mental representations of the concept. The observation and interviews with the teachers were triangulated with data collected from the learners’ responses in exercises whilst some learners explained and presented their work on the chalkboard justifying on how they worked out the problems.

Figure 1: Variation and APOS combined
The questions asked followed a guide designed to elicit the learners’ understanding of the inequalities based on the tasks given. The study was conducted to establish the nature of instructional support that can generate in learners the kinds of mental representations that will enable them to think about critical differences when engaging in symbol manipulation activity involving inequalities. This was a case study in which two teachers’ roles and functions in the abstract theories or principles were used in teaching inequalities. In particular, the study sought to respond to the research questions: (i) what are learners’ conceptions of inequalities?; (ii) what are the possible sources of learners’ incorrect solutions? and (iii) How can the teaching of inequalities be approached?

DISCUSSION AND FINDINGS

Discussions on findings in this paper are captured by illustrating the form of techniques used by the teachers to lay out the genetic decomposition assumed for inequalities in each of their lessons. This will be followed by a discussion analysis according to the APOS and variation theory on the level of understanding displayed by the learners in response to the approach used as instruction.

A structured set of mental constructs which might describe how the concept can develop in the mind of an individual is called the genetic decomposition of that particular concept, Dubinsky (1991). The genetic decomposition was composed in terms of mental constructions (actions, processes, objects, schemas) and mechanisms (contrast, separation, generalisation and fusion) learners might employ when learning inequalities. The genetic decomposition guides the teacher and material developer to provide the learner with activities which will enhance his/her progress in developing his understanding of the concept through the different levels of actions-processes-object-schema. For example it was noticed that:

Action-A transformation is first conceived as an action, when it is a reaction to stimuli which an individual perceives as external and still needs to be guided by instruction. For example:
Teacher 1:

An inequality is like an equation, but instead of an equal sign (=) it has one of these signs. For example, Let us try to solve the problem, ‘$2x - 5 \geq 7$’.

This was followed by verbal reading of the questions similar to the first laid out problem. This example and two others like this one which had positive coefficients of the unknown were calculated on the chalkboard.

Do we do the same with the next example? When do we deviate from this routine procedure?

These are some of the questions he asked from the class before engaging the class with examples where the coefficient of the unknown was negative. The teacher then issued worksheets for learners to do the following problems in class: (i) $x - 6 < 2$  (ii) $2x + 3 < 5$  (iii) $3x - 2 \geq 1$. This is to provide a glimpse of the tasks learners confronted in the worksheets.

Volunteers from the class were then sought to attempt the problems on the chalkboard. The teacher would probe questions to each volunteer in an aid to guide or indicate and iron out misconceptions that the learners had. The learner also had to explain how he/she arrived at each solution on the chalkboard. In the course of the explanations, the learner sometimes paused and relooked at the response displayed, changed it and resumed with a convincing explanation and argument to the class. This happened after the learner had applied mental structures to make sense of a concept. Thus the learner repeats and reflects on the solution process, the action. According to APOS, this entailed interiorisation of the action to a process. I argue that when the learner defends his presentation of the response, he also becomes aware of a process as a totality, realizes that transformations can act on that totality and can actually construct transformations in his imagination. To this, we say the individual has encapsulated the process into a cognitive object. After discussion and an interactive discussion that ensued between the learners amongst themselves and the teacher, classwork was given in the form of exercises to be done in class. These exercises were of the same level and nature as the examples discussed.

DISCUSSION

There was one concern in Teacher 1’s presentation that he did not expose the learners to non-examples. Discernment could not occur since the learners only knew what inequalities were and, not what they were not. The teacher only presented the learners with examples and left out the non-examples. They could not separate the critical features of working with inequalities. There was therefore no guarantee that when
learners deal with inequalities in future that they won’t treat them as equations. The reflection was good and the opportunity for learners to present their work to the class was strengthening their mental representations of the concept. Nonetheless, the learners actions and processes could not be linked together to form a coherent network. This would help the learner to decide, when presented with a particular mathematical situation, whether the inequality schema applies. For example, learners with an inequality schema would easily handle problems involving finding maxima and minima at a later stage. Also although the given tasks were sufficient to enable the learners to make mental constructions relevant to the concept of inequalities, the process was incomplete. It is only after learners can experience, contrast and separate features of the procedures followed when dealing with inequalities that learners can be able to generalise the effect of $<$; $>$; $\leq$; and $\geq$ in a given expression.

Teacher 2:

Where and how do I start with the teaching of inequalities?

Teacher 2’s approach was investigative since he presented examples that sought to engage learners to use what they already knew to understand inequalities. He created a scenario on which learners had to construct meaning. He first cited examples that would help learners to make a distinct understanding on when to use, ‘or equal to’, strictly greater ($>$) or less than ($<$). During interviews this teacher opted to draw a concept map to illustrate his understanding of the all the sub-topics which he thought were important for the learners to know regarding this concept. He called this ‘a draft skeleton’ and that the concept map served as a yardstick against which he would measure a complete understanding of the concept by his learners. He pointed out that for the grade 10 class, he starts introducing learners to the single variable problems using user friendly scenarios that need to be mathematically represented (see Fig 2). This is in line to Shulman’s idea on classroom practice as he stated that ‘teaching necessarily begins with a teacher’s understanding of what is to be learned and how it is taught’ (Shulman, 1987:7). Shulman believed that the teacher can transform their conceptual understanding to pedagogical representations and actions. These include explanation, talking, enacting or representing the idea such that the unknowing can come to know, those without understanding can comprehend and discern and unskilled can become adept (Shulman, 1987:7).

This teacher contextualised the problem using an authentic task as an instructional strategy to make sense of some of the exceptional transformation rules used in solving inequalities. He also highlighted the properties underlying valid equation-solving transformations which are not the same as those underlying valid inequality-solving transformations. In contrast he warned his learners that multiplying both sides by the same number, which produces equivalent equations, can lead to pitfalls for inequalities.
During the time of discussions Teacher 2 engaged his learners in cooperative group discussions. Each group was issued with a worksheet card containing two problems. After 10 minutes of group discussions, group representatives presented work for the whole class on how the problem was attempted on the chalkboard. Questions from other groups regarding the presentation were addressed by the learner’s group members. In this way the whole group took ownership of the response presented by their representative.

**Figure 2**: Concept map used by teacher 2

Examples included:

(i) The doctor instructed my grandfather to take no more than 3 pills per day  
(ii) My instructor advised me to run at least 10km per day to prepare for the marathon (iii) I ate at most two meals today.

He then emphasised the use of terminology in the above examples, words like, ‘at most’, ‘no more than’, ‘at least’ and ‘no less than’ as words deviating from ‘equality. Also, that such problems, could not be represented as equations. During interviews it was evident that Teacher 1 approached his teaching in the following manner:

It would depend on the situation, but what I simply do is to consult as many textbooks relevant to the topics as I can find. I then pick examples from all of them and arrange them in order of easy to difficult ones.
This clearly indicated that this teacher was teaching the concept through examples. The non-examples were not in his agenda at all. Meanwhile Teacher 2 said:

*I design a concept map so that I can recall all the other concepts to which the topic at hand is linked, I then think about some misconceptions that I have seen in past with my former learners. I then prepare and open up discussions around the topic such that learners would be aware and understand the concept, not only that they must see where it fit in the bigger picture.*

On further probing, he further elaborated that

*The bigger picture refers to how the learners can use the concept, connect it to a network of what they know already, and use it correctly whenever it is necessary for it to be used.*

This clearly indicates that the second teacher is likely to lead his learners to a deeper understanding of the concept. Ball and Bass (2000:8) agreed with this when they argued that knowing content is crucial to being inventive in creating worthwhile opportunities for learners that take learners’ experiences, interests and understanding into account. This second teacher indicates advances into leading his learners to schema understanding of the concept. In exposing them to bigger picture of the concept and how it connects with the other concepts. Also this happens after the learners already know and can separate critical features of the concept and can identify the relationship between them which allows a learner to make connections gained in past and present interactions.

*Learners’ conception of inequalities*

Clearly learners’ understanding of inequalities in the two classrooms, were different since the first group only worked with examples presented to them by the teacher. There was very scarce chance of interiorisation of the concept or being able to vary what it is and what it is not. Meanwhile the second group stood a greater chance of processing the object of learning to encapsulation. They were exposed to various ways in which they could discern critical features of working with inequalities. They also enjoyed a platform where they argued for their responses and could connect the concept to internal and external networks.

*Possible sources of incorrect solutions*

The physical entities where learners simply use a set of rules in working with inequalities without interiorising the meaning of their actions may lead learners to incorrect solutions. This often is the case when learners are exposed to examples only
of that particular concept. Also teachers should probe follow up questions to help learners make meaning of exercises presented to them.

Teaching inequalities

What was clear from the findings was that both teachers had content knowledge that pertained to the concept of inequalities. The first teacher though was unable to demonstrate this since he only exposed his learners to examples only, leaving out the bigger picture of how the concept connected with other concepts. Learners in his class could only operate in the process stage of APOS and could not generalise on the critical features of this concept. The instruction designed was not sufficient to allow learners to make mental constructions relevant to inequalities and to be able to lead them to construction of their own knowledge regarding them. The strength of activities designed for teaching a particular concept is directly proportional to the depth of the content knowledge the teacher possesses. In instructional design, it is this kind of knowledge that is restructured to make it accessible to learners and hence affording the learners to make mental constructions relevant to each concept.

The second teacher demonstrated being able to modify tasks during introduction of his lessons to make learning easier. He also opened a platform for fruitful discussions to ensue on probing the ‘why’ questions during the learners’ presentations. This allowed the learners to engage deeply with reasoning on how they performed certain operation with inequalities. Learners were exposed to discernment of the concept. They were aware of certain features which are critical to the intended way of seeing inequalities. Also the teacher highlighted the essential features of the concept through varying the non-essential features. He also gave the learners a chance to demonstrate argue and explain their solutions to others. He saw the learners as constructors of meaning and assisted them to actively try things out. The learners then experienced the construction of multiple perspectives of mathematical concepts related to find components of the concept that are interconnected with each other. His instructional support involved leading the learners to: (i) extend the original problem by varying the conditions, changing the results and generalize; (ii) use multiple methods of solving a problem by varying the different processes of solving a problem and associating different methods of solving a problem and (iii) use multiple applications of a method by applying the same method to a group of similar problems.

CONCLUSION

The findings in this study recommend that the kind of practice as employed by teacher 2, would generate in learners mental representations suitable enough to enable learners to think about critical differences and where inequalities feature in the bigger picture.
Learners need to be enabled to conceptualisation beyond the action stage where their solutions are based on rules and algorithms. No adequate thinking is involved where a rule is practised repeatedly until it becomes routine. Learners should be assisted to construct meaning beyond the manipulation of entities externally from memory or a clearly given instruction, but should realize that transformations regarding the concept explicitly in their imagination so as to encapsulate the process into a cognitive object.

The paper concludes by suggesting that learners should be allowed discernment of the concept, so as to be aware of certain features which are critical to the intended way of seeing this concept. It was also found that the essential features of the concept through varying the non-essential features should be highlighted. Lastly it was established that the learners who demonstrate, argue and explain their solutions to others operate on either the Object stage of APOS and demonstrate ability to fuse, connect and relate the concept learnt to other concepts.

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A SYSTEMATIC REVIEW OF TEACHING APPROACHES THAT LEAD TO THE DEVELOPMENT OF MATHEMATICAL MODELLING COMPETENCIES IN HIGH SCHOOLS

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Teaching approaches fostering the development of competencies associated with mathematical modelling were systematically reviewed by considering one set of conference proceedings. A weight of evidence procedure was applied to the findings emanating from studies relevant to the review question. It was found that a guided holistic approach to the teaching of mathematical modelling contribute towards the development of a limited number of mathematical modelling competencies.

INTRODUCTION

Currently much attention is accorded to the notion of evidenced-based development and design in education. This resulted from assertion in political and public circles that the return on investment in educational research is extremely low. To address these criticisms efforts have been made in recent times to synthesise research on educational phenomena in order to provide ‘best evidence’ for practice, policy and research. The generic term given to this kind of research is a systematic review of which one description is that “Systematic reviews are one form of research synthesis which contribute to evidence-based policy and practice by identifying the accumulated research evidence on a topic or question, critically appraising it for its methodological quality and findings, and determining the consistent and variable messages that are generated by this body of work.” (Davies, 2004: 22. Of importance is that in systematic reviews the research findings are appraised regardless of whether they support or refute whatever the research pursuit regarding the phenomenon is. This paper reports on a sub-study of an in-progress larger systematic review concerned with teaching approaches for mathematical modelling and the development of competencies for mathematical modelling that is proposed to be developed through the teaching approaches. Mathematical modelling competencies are focused on because of the emergent emphasis being placed on mathematical modelling in school curricula across the world.

Brief comment on systematic reviews

Systematic reviews are approached from two traditions. The first is conventionalism which can be either a traditional realist approach or an interpretivist approach. The former deals primarily with evidence obtained via quantitative experimental methods and falls under the generic name of meta-analysis. Normally a generic inverse variance
method is used to combine study results to assess what works. The interpretivist approach deals with qualitative studies and the generic term associated with systematic reviews of qualitative studies is meta-synthesis. The most-used method for meta-synthesis is template analysis where a priori themes are identified. The method centres on identification of compliance or not of the findings to the themes.

The second tradition is the robust realist approach. This approach is based on critical realism or more appropriately critical naturalism (Bhaskar, 1978). This approach accepts that an object or phenomenon exists independent of our knowledge of it and a decision to study a phenomenon presupposes its existence. Systematic reviews in this tradition do not make the sharp distinction between qualitative and quantitative methods. It considers research findings forthcoming from both these research designs. The procedures use is firstly an examination of the empirical evidence in the literature “to map out in broad terms the conceptual and theoretical territory” (Pawson, Greenhalgh, Harvey & Walshe, 2004: v). This followed by a search for and appraisal of evidence to fill the constructed theoretical framework—the conceptual and theoretical territory. Finally the theoretical framework is used “for locating, integrating, comparing and contrasting empirical evidence” (Pawson, et al, 2004: v) to reach conclusions about what is working for whom under which conditions. (Pawson, et al., 2004: 25).

Accompanying these considerations is some appraisal of selected studies for both approaches along the lines of the methods employed, the appropriateness of the research design and analysis pertaining to the review question and the relevance of the study topic focus to the review question.

The review reported here leans towards the conventionalist approach and includes both qualitative and quantitative studies. It emulates a systematic review related to enhancing learners’ motivational effort for mathematics. (Kyriacou & Goulding, 2006). The overall endeavour of the project is the investigation of research evidence for successful practice in teaching with regard to mathematical modelling competency development for learners in grades 8 to 11. From this articulation the systematic review question being pursued is “What teaching approaches related to mathematical modelling enhances the development of mathematical modelling competencies of secondary school pupils?”

The next two sections describe some of the crucial constructs contained in the review question.
Mathematical modelling and associated competencies

Mathematical modelling competencies are linked to the ideal-typical cycle of mathematical modelling. These ideal-typical descriptions derive from observational and reflective reports on how mathematical modellers operate in practice. This practice entails “a mathematical modelling way of working”. From the ideal-typical mathematical modelling as, for example, articulated by Stillman (1998), Blomhøj & Jensen (2003), Niss (2010) the major characteristics of this way of working are (a) the actual problem is vaguely formulated although the ultimate outcome—an artefact to realise a particular objective as specified by a client—is known to both the model-developer and requester, (b) the modellers refine the problem into something consisting of a series of questions which are kept as near as possible to the context of the requester’s demands. There is a series of meetings and discussions of a qualitative nature between the two parties on these questions in order to reach eventual agreement on what the requester finally wants to receive from the model-developer, (c) the model-developer constructs the artefact making use of his/her experience, mathematical knowledge and insights and inputs from others in the modelling operation. If a group of model-developers work on the project, different partners might work on different aspects of the project with the division of labour distributed according to the strengths and weaknesses of the collaborators in the model-building process, (d) the delivery of the final artefact is in a format understandable to the requester. This final artefact might be a culmination of in-progress discussions on goodness-of-fit between the model-developer and requester on interim produced results and (e) the final artefact is stored by the model-builder for adapted use for near-analogous future requests or for refinement and offering of a finer-tuned artefact to the original requester.

From this description of the “mathematical modelling way of working” various diagrammatic representations evolved. Stillman (1998), Blomhøj & Jensen (2003), Niss (2010), for example, provide such diagrammatic depictions. The processes in the cycle of mathematical modelling forthcoming from these portrayals are: (1) the understanding and unravelling the complex extra-mathematical situation; (2) simplification of the extra-mathematical and generation of assumptions; (3) the formulation the mathematical problem; (4) the construction of the mathematical model; (5) interpretation of the model in terms of the simplified extra-mathematical situation; (6) goodness-of-fit checking to the complex extra-mathematical situation to ascertain fidelity and (7) the communication and demonstration of the model. These processes are not engaged with in a linear fashion, there are constant movements between and validations are done all along the model construction process.

A further aspect of modelling underpinning the systematic review is that of “modelling as content” described by Julie (2002) as
Mathematical modelling as content entails the construction of mathematical models for natural and social phenomena without the prescription that certain mathematical concepts or procedures should be the outcome of the model-building process. It also entails the scrutiny, dissection, critique, extension and adaptation of existing models with the view to come to grips with the underlying mechanisms of mathematical model construction.

This notion is to be distinguished from “modelling as a vehicle” where contexts are used in a variety of ways to introduce, develop understanding, etc. of mathematical ideas.

The competencies involved in the model-construction process are essentially derived from the idealised version of the modelling process. For this research competence is viewed as:

the ability to meet individual or social demands successfully, or to carry out an activity or task. This external, demand-oriented, or functional approach has the advantage of placing at the forefront the personal and social demands facing individuals. This demand-oriented definition needs to be complemented by a conceptualization of competencies as internal mental structures — in the sense of abilities, capacities or dispositions embedded in the individual. Each competence is built on a combination of interrelated cognitive and practical skills, knowledge (including tacit knowledge), motivation, value orientation, attitudes, emotions, and other social and behavioural components that together can be mobilized for effective action…(OECD, 2002, 8)

Blomhøj & Jensen (2003) and Maaß (2006) adopt a definition similar to that stated in the first paragraph of the quotation above for mathematical modelling competences. Blomhøj & Jensen (2003) view working with processes as engagement for competence development. Maaß (2006, 116 - 117), using a scheme for modelling processes developed by Blum and Kaizer (1997), further divides the processes as competences into sub-competences as follows:

Competencies to understand the real problem and to set up a model based on reality (making assumptions for the problem and simplify the situation; recognizing quantities that influence the situation, naming them and identifying key variables; constructing relations between the variables; looking for available information and to differentiate between relevant and irrelevant information).

Competencies to set up a mathematical model from the real model (mathematizing relevant quantities and their relations; simplifying relevant quantities and their relations if necessary and to reduce their number and complexity; choosing appropriate mathematical notations and to represent situations graphically).

Competencies to solve mathematical questions within this mathematical model (using heuristic strategies such as division of the problem into part problems, establishing relations to similar or analog problems, rephrasing the problem,
viewing the problem in a different form, varying the quantities or the available data etc.; using mathematical knowledge to solve the problem).

Competencies to interpret mathematical results in a real situation (interpreting mathematical results in extra-mathematical contexts; generalizing solutions that were developed for a special situation; viewing solutions to a problem by using appropriate mathematical language and/or to communicate about the solutions).

Competencies to validate the solution (critically checking and reflecting on found solutions; reviewing some parts of the model or again go through the modelling process if solutions do not fit the situation; reflecting on other ways of solving the problem or if solutions can be developed differently; generally questioning the model).

Mathematical modelling teaching approaches

In modelling teaching is generally identified by the problem types learners are exposed to. Blomhøj and Jensen (2003) give two problem types, and by implication teaching types. The one they term holistic—the problem is presented in a near similar way to how it is presented to adept modellers. An example is: What space is occupied by a tree? The second type is called atomistic—sub-problems, other support information and some guidance are provided to assist learners on their way to construct a model.

Blomhøj and Jensen (2003) see these two types as extremities of a holistic-atomistic continuum. From these considerations of problem types, the following teaching approaches for mathematical modelling can be formulated.

The immersion approach (Holistic)

This approach is just a renaming of the holistic approach and is characterised by the way students encounter the issue to be modelled. In this approach an environment is provided which near-simulates the environment of appliers and modellers of mathematics in industry, business, government, etc. The teacher plays the role of the requester of the service and provides, upon request, clarifications from the service-requester’s point of view. This type of teaching is normally preceded by a discussion on the applications of and modelling in mathematics where some version of the ideal-typical processes of modelling is presented to the students. An important aspect of this approach is that it is compulsory that students produce a final report in the same sense as is expected in practice. As an example one can think of the issue to be modelled given as: “How much water is used to produce a cup of tea?”

Holistic guided exploration and construction of models

Giordano and Weir’s (1985) textbook on modelling epitomises this approach. In this case the students are confronted with the problematic situation from the start and all the
components of the mathematical modelling process are integrated. The teacher in this case gives or recommends to the students the necessary resources (programs, books, and so on). The teaching is structured with due respect to the issue to be modelled and the processes involved in the model-construction process. Through explanatory discussions, questions and interrogative statements the students’ thoughts and actions are directed to attend to the issues of import and model-building processes.

**Mastery of stages (Atomistic)**

The emphasis in this approach is on the mastery of the different stages of the modelling process. The different stages of the modelling process are taught separately. So for example, Edwards and Hamson (1996: vii) assert that “Many of these [mathematical modelling] skills are gradually gained through practice and experience [and] students [must be directed] towards building up these skills in a systematic way, through carefully constructed exercises.” This approach privileges the mastery of different stages and once these different stages are mastered or students have worked through them then they should tackle problems through modelling holistically. Regarding the “water for a cup of tea” situation above, the problem may be started off as follows: “When a cup of tea is made, the water used is not only the amount of water used to fill the cup. One also has to consider if an electric kettle is used the minimum amount of water that must be used to cover the element of the kettle. Write down some other instances of water use to eventually make the cup of tea.” This problem is to sensitize and allow learners to generate possible issues to take into account when constructing and lead to the formulation of assumptions and declaration of variables and constants to be used in the process of constructing the model. This is followed by similar problems for other situations to consolidate the formulation of assumptions and declaration of variables and constants to be used—the initial competencies to be mastered to construct a mathematical model. The other competencies are dealt with in a similar manner.

**Mathematical topic sequenced approach**

This is a teaching approach where modelling is intimately linked to a mathematical topic. The modelling context is used to get students motivated, excited and interested in mathematics. Mastery of mathematics is the ultimate aim although it differs from the modelling as a vehicle view conception in that due respect is paid to elements of modelling process. The Sixth Form Mathematics Project adopted this approach and they view their approach as “to build up, and to consolidate, a thorough understanding of the exponential function. The content of the Units was determined with this end in mind: the material was not written as a free-wheeling exercise in modelling. In other
words the modelling presented in these books is highly selected modelling.” (Schools Council Sixth Form Mathematics Project, 1979: 3)

The above typology of different approaches for teaching the applications of and modelling in mathematics is a mere description of the major approaches that are followed in the teaching situation. They are, as alluded to, intrinsically linked to the formulation of problem types.

The systematic review

For systematic reviews data are mined from existing research studies. These studies are those that are relevant for the review. The sources searched include academic journals, conference proceedings, unpublished externally-examined publications such masters and doctoral theses and grey literature such as project reports. Much value is attached to research publications which have an external, independent blind review process for the acceptance for publication. This ostensibly enhances the quality of the research being reported. For the part of the project reported here the focus on the conference proceedings of the International Community on the Teaching of Mathematical Modelling and Applications (ICTMA). ICTMA is an affiliated study group of the International Commission on Mathematical Instruction (ICMI) which is a specialist group of the International Commission on Mathematical Instruction (ICMI). ICTMA is also the acronym for the biennial International Conference on the Teaching of Mathematical Modelling and Applications. The proceedings of ICTMA are published as a book by reputable publishing houses such as Horwood Publishers and Springer. To date 16 biennial conferences were held with 15 books of conference proceedings (ICTMA 1 to 15) published—the proceedings of the 16th conference is currently in the process of undergoing a review process for publication. The process followed by ICTMA for publication as a chapter in the published proceedings starts with a review of submitted abstracts for submissions to be presented at the conference. After the conference, presenters are requested to submit their reworked presentations as chapters for possible publication in the book of proceedings. These submissions are reviewed by two independent reviewers and based on these reviews the editors decide on which reviewed chapters will be included in the book. The accepted ones, some with suggested reworkings from the reviewers and editors, are then chapters published. This description indicates that the ICTMA process for publication is similar to those followed by peer-reviewed journals. This paper reports on the procedures followed and conclusions reached from one such book of proceedings—the 11th conference proceedings (Lamon, Parker & Houston, 2003). The larger study will cover proceedings from 1997 to 2014. The choice of start year is parochial in the sense that South Africa is now reasonably advanced in terms of her quest of a unitary and non-racial system of education. With 1997 proceeding as the choice to start the systematic review, the particular year 2003 was pragmatically selected for the article because it fell in the
middle of available proceedings and the 2009 proceedings focused primarily on mathematical modelling competences.

As for all systematic the inclusion and exclusion criteria were developed by the author and corroborated by three other mathematics educators, two from UWC and one from the University of Zululand (UNIZUL). These criteria are given in table 1.

<table>
<thead>
<tr>
<th>Inclusion criteria</th>
<th>Exclusion criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 It is a publication in the book publication of the ICTMA proceedings.</td>
<td>A chapter not presenting empirical data collected in a teaching setting.</td>
</tr>
<tr>
<td>2 The article reports a study presenting original data collected by the author(s).</td>
<td>A report of a research study not using original data collected by the author(s).</td>
</tr>
<tr>
<td>3 The article was published in the period January 1997 to March 2014.</td>
<td>Not about the teaching of mathematical modelling in grades 8 to 12.</td>
</tr>
<tr>
<td>4 The study deals with teaching ordinary high school pupils (grades 8 – 12).</td>
<td>Not relevant for the development of mathematical modelling competences.</td>
</tr>
<tr>
<td>5 Relevant for the development of mathematical modelling competencies.</td>
<td></td>
</tr>
<tr>
<td>6 The language of the published research studies is English</td>
<td></td>
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</tbody>
</table>

Table 1: Inclusion and exclusion criteria

A four phase process was followed to construct the systematic map, the articles (chapters) that will be subjected to an in-depth review. These were (1) the Table of Contents was scrutinised to identify from the titles possible chapters indicative of dealing with the review question, (2) the abstracts of the chapters identified in 1 were accessed and screened, (3) skim reading of excluded chapters for verification of exclusion and in-depth scrutiny of possible included chapters for relevance pertaining to the review question and (4) selection of chapters to constitute the systematic map. The outcome of this four phase procedure is given in Figure 1 below.

For all these phases quality assurance procedures were followed. For phase 1, the principal reviewer did the identification of the titles. The table of contents with the abstracts of the chapters was forwarded to two other members of an extended review group. The extended group was the principal reviewer, UWC collaborators, the UNIZUL collaborator, two mathematics educators from the Cape Peninsula University of Technology (CPUT) and one from the University of Cape Town (UCT). Each
member scrutinized the table of contents for further inclusions or exclusions of chapters other than those identified by the principal reviewer.

Figure 1: Phases to identify items for systematic map
No additions or exclusions were identified and phase 2 commenced. Three different abstracts were then independently screened by two different members. The primary reviewer screened the entire set of abstracts. This rendered that the abstracts of the 10 identified chapters were screened for inclusion by 3 independent reviewers. The reviewers had to give a judgement on the inclusion or exclusion of the chapters. A majority decision, 2 of the 3 reviewers agreeing on inclusion or exclusion, was applied. Four chapters were excluded based on exclusion criteria.

The principal reviewer skim-read the 4 excluded chapters for verification of exclusion and none was found to warrant re-insertion in the pool mentioned in phase 3. An in-depth scrutiny of the 6 possible included chapters was also done to ascertain its relevance pertaining to the review question. Two were found to be not relevant. The one dealt with teaching mathematical modelling to primary school learners. The other was more of an advocacy nature for the inclusion of mathematical modelling in school and empirical data are not really presented although there is reference to another study where findings emanated from empirical data. The four resulting chapters comprised the systematic map. These four chapters were by Yanagimoto, Carmona, Henn and Ikeda & Stephens.

**In-depth review of the four identified chapters**

The principal reviewer and one other member independently did the in-depth review of the full chapters. A template, consisting of three elements, exampled in table 2 below, was used to report the outcome for each of the chapters. Each component was assigned a value “low”, “medium” or “high” by the reviewer. The in-depth review rendered that the chapter by Henn did not satisfy the inclusion criterion of presenting empirical data on learners’ modelling competencies. It was removed from the systematic map of included studies.

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Ikeda, T &amp; Stephens</th>
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</thead>
<tbody>
<tr>
<td>Component</td>
<td>Low</td>
</tr>
<tr>
<td>A:Comprehensiveness of the description of the processes and methods used in study to arrive at its findings</td>
<td></td>
</tr>
</tbody>
</table>
A simple weight of evidence procedure was followed by converting the response categories of “low”, “medium” or “high” to the numerical values 1, 2 and 3 respectively. The weight of evidence value for each reviewer was calculated as the average value of the three elements assigned by the reviewer. The average of weights of evidence of the two reviewers was taken as the overall weight of evidence. This procedure is similar to the one followed by Kyriacou & Goulding (2006). The outcome of this procedure for the three studies is presented in table 3 with PR the principal reviewer and R1, R2, R3 the independent reviewers.

<table>
<thead>
<tr>
<th></th>
<th>Yanagimoto</th>
<th>Carmona</th>
<th>Ikeda</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PR</strong></td>
<td><strong>R1</strong></td>
<td><strong>R2</strong></td>
<td><strong>R3</strong></td>
</tr>
<tr>
<td>A: Comprehensiveness of the description of the processes and methods used in study to arrive at its findings</td>
<td>1 3</td>
<td>2 3</td>
<td>3 3</td>
</tr>
<tr>
<td>B: Appropriateness of the research design and analysis.</td>
<td>1 3</td>
<td>1 3</td>
<td>2 3</td>
</tr>
<tr>
<td>C: Relevance of the study topic focus to the review question</td>
<td>2 3</td>
<td>2 3</td>
<td>3 3</td>
</tr>
<tr>
<td>Weight of Evidence (WoE)</td>
<td>1.33</td>
<td>3</td>
<td>2.6</td>
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<td>-------------------------</td>
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</tr>
<tr>
<td>Average WoE</td>
<td>2.17 (M)</td>
<td>2.84 (H)</td>
<td>2.84 (H)</td>
</tr>
</tbody>
</table>

Table 3: Weight of evidence for the selected studies

The weight of evidence shows that the studies are in the medium to high band for drawing interim conclusions for the review question.

**DISCUSSION**

The review question is concerned about teaching approaches appropriate for the development of competencies related mathematical modelling. In order to ascertain such each study was summarised. As an example the summary for Yanagimoto’s study is given below.

Yanagimoto describes and explains the implementation of modelling activities to two different classes in different areas in Japan. In the one area a grade 8 class engaged in the modelling activity. In the other location grade 8 and 9 students were involved. The modelling was not done during a mathematics lesson but as part of a newly introduced interdisciplinary subject, *Sougou*, in the Japanese schooling system. The grade 8 class had to develop a model “to estimate the average global temperature in 100 years,…” (p 54). Using data on yearly global temperature averages since 1860, data points were graphed and the line of best fit determined. The average global temperatures for 2050 and 2100 were calculated for the obtained model. These temperatures were compared to the average temperature rise estimated by the Japanese panel for climate change. It is evident that a guided holistic approach was followed with the author referring to “We considered…” (p 54 and “The students graphed…” (p 54). Related to mathematical competences, engagement with the competence “Construction of the mathematical model” is evidenced “Global temperature average could be predicted by as simple as a linear function” (p 57).

The learners in the second school had to model the situation of halving in 10 years the number of bluegills in a lake in a district. Information of the number of bluegills in the lake and the plan for an annual catch of 3000 tons were provided. The assumptions were developed and a model was developed from these using a recurrence formula. Graphing calculators were used and for the assumption of a breeding rate of 15%, it was found that the “bluegills will increase slightly” (p 57). This resulted in the breeding rate being changed to 5%. It was now found that the “number of bluegills begins to decrease immediately and the fish would die off in
eight years” (p 57). Through further experimentation with different breeding rates, one was found for which the population would half in 10 years. As with the other class the teaching approach can be characterised as a guided holistic approach. For competence engagement is of the kind “Interpretation of the model in terms of the simplified extra-mathematical situation” evidenced by “Students tried various breeding rates” (pp 57 – 58).

Two claims related to the development of mathematical modelling competence are apparent. These are “Construction of the mathematical model (Global temperature average could be predicted by as simple as a linear function” (p 57) and “Goodness-of-fit checking (validation) of the model to the complex extra-mathematical situation to ascertain fidelity “They weren’t satisfied with the models and realized more complicated models were necessary” (p 59).

The above description renders that a guided holistic teaching moderately support the development of the competencies for the construction of the mathematical model and the goodness-of-fit checking (validation) of the model to the extra-mathematical situation.

For the Carmona study a guided holistic teaching approach was followed. The findings support the development of the competencies of understanding, unravelling understanding and unravelling the complex extra-mathematical situation. Ikeda and Stephens’s study compared two groups of grade Japanese learners with one group following an analytic and another group a constructive approach to mathematical modelling. From the descriptions it comes through that for the analytic group an atomistic teaching approach was used and for the constructive group a guided holistic one. The conclusion reached is that the guided holistic teaching approach epitomised by constructive problem formulation fosters engagement with consideration of the assumptions whilst the atomistic teaching approach the focus of the students were exclusively on the meaning of the supplied mathematical model.

**CONCLUSION**

This study, limited to systematically reviewing research studies appearing in one set of conference proceedings points in the direction that a guided holistic teaching approach contributes to the development of a limited number of competencies. Obviously this finding is tentative and the inclusion of the entire corpus of proceedings decided might change the picture.
Acknowledgement

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REFERENCES


EXTENDED ABSTRACT

Mathematics is a subject recognised as the mother of all learning. It is essential in almost every field of learning. Story problems (traditionally referred to as ‘word problems’) have been viewed by people in the field as a way to be used globally by curriculum policy designers to connect informal out-of-school mathematics knowledge with formal written primary and high school mathematics. However, such a call for bridging the gap between informal and formal knowledge of mathematics has an effect on teacher development across the South African primary schooling system. Recently the Department of Basic Education called on primary school teachers to put more emphasis on teaching learners how to solve and make sense of contextual story problems. In a report presented by the Department of Basic Education (2011b) on Grade 6 learners’ ability to solve story problems, it was reported that learners experienced solving story problems as the most difficult skill to master. According to this report, learners attained a 9% score nationally in solving word problems. We argue that a key skill that learners need in solving story problems is to identify the problem and find functions that link several entities. It is therefore against this background that the study reported in this article explored the factors that affect academic achievements in Grade 6 mathematics story problems within the context of a rural school.

Furthermore, the article discusses errors made by the learners when they solve word problems. The study used document analysis in a form of learners’ written work (or test) in order to collect quantitative data. The results obtained from a test consisting of six word problem tasks showed that learners struggled with realistic considerations of problem statement as well as with making meaning of situations embedded in the task. In brief, reading instructions aloud repeatedly and explaining key mathematical concepts have emerged as key strategies in understanding and solving word problems in mathematics.
This paper examines the solution strategies used by pre-service teachers\(^1\) when asked to solve a number problem as a means to engage them with the notion of structure sense. I report on how pre-service teachers’ flawed solution strategies provided opportunities for critical inquiry into the structure of the number context and engagement with the process of validating their solution strategies. I will show how the process of validating moves the pre-service teachers through higher levels of generalisation. In this paper I argue that the notion of developing structure sense within number contexts provides primary school pre-service teachers with a rich context in which to engage with structure sense and generalisation as interrelated mathematical processes.

BACKGROUND TO THE STUDY

This case study is an exploration of primary school pre-service teachers’ engagement with a number problem in which alternative solution strategies to the standard algorithm is used as an opportunity to develop the notion of structure sense. The purpose of this exploration is to explore how a number problem context develops structure sense when engaging students with errors in their solution strategies.

The Curriculum and Assessment Policy Statement (CAPS) defines number sense development as including: the meaning of different kinds of numbers; the relationship between different kinds of numbers; the relative size of numbers; the representation of numbers in various ways; the effect of operating with numbers and the ability to estimate and check solutions (National Curriculum Statement, Senior Phase Grades 7-9, p10). The definition of structure sense is not made explicit in the CAPS. The development of number sense within CAPS emphasises the representation of numbers in different ways which includes the recognition of equivalent forms, for example, 3 + 4 and 2 + 2 + 3, which can be considered a move to focusing on the structural elements of the expressions.

A review of the literature (Wagner & Kieran, 1989; Booth, 1989; Linchevski & Herscovics, 1996; Linchevski & Livneh, 1996) suggests that structure sense requires learners to recognise that two number expressions have the same value because of common structural elements. For example, while 3+4 and \(2+2+3\) have different

\(^1\) In this paper the pre-service teachers will be referred to as the students.
surface features, the structure of the number expressions has the form \( 3 + a \), and \( a + 3 \), which represents the same number because of the commutative property of addition. Structure sense will demand that the learners explain why the two numbers expressions have the same value without having to compute the value of the expressions but instead focusing on the properties of the number system. The latter aspect of structure sense is not made explicit: CAPS focusses on using number properties to compute rather than to compare the structure of equivalent number expressions.

In this study pre-service teachers were introduced to these differences between number and structure sense to engage them with ideas of how these notions are related and to develop structure sense. The literature review in the next session provides a brief overview of some key research studies into the notion of structure sense in school mathematics.

**LITERATURE REVIEW OF STRUCTURE SENSE**

The notion of structure sense as it applies to school mathematics developed from research that explored the connections between arithmetic and algebra (Lee & Wheeler, 1989). Lincheski and Herscovics’s (1996) research focused on learners’ solutions of equations and found that many of the errors in learners’ solutions were linked to incorrect manipulation of the numerical parts of the equation. For example, the equation \( 4 + n - 2 + 5 = 11 + 3 + 5 \) was solved incorrectly because learners grouped the 2 and 5 on the left to produce \( 4 + n + 7 = 18 \). This error was referred to as the 'detachment of a term from the indicated operation' by Linchevski and Herscovics (1996) and they argue that this error is associated with learners’ difficulty in uncovering the mathematical structure of the expression. Booth (1998) argues that learners’ difficulties in algebra are in part due to their lack of understanding of structural notions of arithmetic.

Kirshner (1989) argues that for some learners the surface features and visual cues of expressions dominate their manipulation strategy rather than applying the correct mathematical properties. The research of Liebenberg, Linchevski, Sasman and Olivier (1999) suggest similar findings when learners performed calculations in a numerical context. For example, learners know that within the numerical expressions \( 14 + 2 \times 6 \) and \( 14 \times 5 + 5 \) that multiplication is the first order of operation and may even know why it is important for there to be order of operation. Yet there is a greater likelihood that learners will comprise this mathematical property in the second expression, \( 14 \times 5 + 5 \), adding \( 5 + 5 = 10 \), and then multiplying.

Lüken’s (2012) research focussed on structure sense of very young learners in their first year of school. The learners in Lüken’s (2012) research were given geometrical patterns in which the pattern needed to be continued or to count the number of objects in a
geometrical structure. The key finding of Lüken’s (2012) research is that while all the learners could recognise the sub-structures in a pattern, the difficulty for some learners is recognising and establishing mutual connections and relations between sub-structures which support numerical procedures.

Kieran (1988) defined structural knowledge as the ability to identify equivalent forms of expressions. Linchevski and Vinner (1990) argued that this definition should be modified to include the ability to discriminate between the forms relevant to the task – generally one or two forms – and all the others. Hoch and Dreyfus (2004) define the notion of structure sense as it applies to high school algebra as a collection of the following abilities:

- To see an algebraic expression or sentence as an entity
- To recognise an algebraic expression or sentence as a previously met structure
- To divide an entity into sub structures
- To recognise mutual connections between structures
- To recognise which manipulations it is possible to perform
- To recognise which manipulations it is useful to perform

Hoch and Dreyfus (2004, p3.)

In this paper I will draw on the notions of structure sense as developed within the research described in this literature review to explore how structure sense can be developed in a numerical context to create opportunities for developing the thinking processes involved in generalisation.

**METHODOLOGY**

**Participants**

Prospective mathematics teachers at our institution² are streamed into the specific phases of the schooling system: foundation; intermediate – senior phase³ and the further education and training phase. The students involved in this case study are first year and

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² A university of technology
³ Intermediate-Senior Phase (ISP) includes grade 4 to 9. Grade 4 to 7 is part of the primary school. Most teachers in the ISP opt to teach in primary schools.
fourth year students who are in the intermediate-senior phase stream (ISP students).
The fourth year students are a group of 17 students who specialize in mathematics. The
first year students are students who did mathematics at school. Most of the students
are from the historically privileged group in South Africa and attended high schools
that are referred to as ex-Model C schools. The majority of the current prospective
teachers at our institution followed the outcomes-based curriculum. Critical inquiry was one of the core outcomes of the outcomes-based curriculum to develop critically-minded citizens for the new democracy.

Teaching Context

The mathematics education course for ISP students includes non-routine mathematics problems which students solve collaboratively and students are given the opportunity to present their solutions to the class. The non-routine problem that is the focus of this paper was given to the first and fourth year mathematics students in the first term of the academic year. The first year students had completed topics on number that included working with numbers in different bases; place value and number theory. The fourth year students completed topics in algebra that focussed on the solving of different kinds of equations. Some of the algebraic equations that were given to the fourth year students were taken from Hoch and Dreyfus’s (2004) research to see whether students’ solution strategies would reflect a structural approach or not. The fourth year students in this study produced the same lack of structural sense as the students in Dreyfus’s (2004) research.

The students were asked to complete the problem during the lecture period and to submit their solution to the problem at the end of the period. Written feedback was given to the students the following day and students were challenged to reflect on the feedback and to resubmit their work on the day the feedback was given. The purpose of the feedback was to engage the students in the importance of working with their productions as opportunities for critical inquiry into the mathematics embedded in their solution. This form of feedback draws on the research of Borasie (1994) in which

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4 In South African schools learners have a choice between mathematics and mathematics literacy. The learners make the choice between mathematics and mathematics literacy at the end of grade 9 and must complete either one of the two subjects to obtain the grade 12 matriculation certificate.

5 Ex-model C schools are schools in geographic areas designated for the white population during the apartheid era in South Africa.

6 Critical inquiry features explicitly in the post-apartheid school curricula.

7 The following algebraic problem from Dreyfus’s(2004) research was given to the students; solve for $n$ in $1 - \frac{1}{n+2} - \left(1 - \frac{1}{n+2}\right) = \frac{1}{132}$.
students’ mathematical errors are engaged with to extend the mathematical inquiry to present students with a view of knowledge as a dynamic process of inquiry.

**THE PROBLEM**

The problem was selected from a publication on mathematics competitions by the Association of Mathematics Education for South Africa (AMESA). The problem given below was adapted in the sense that students were asked to do the computation without using the standard algorithm but to apply their knowledge of place value and other properties of the real number system:

\[ 1997 \times 1995 - 1999 \times 1993 \]

The students were also asked not to use calculators. The problem was chosen to see whether students would recognise relationships between the numbers in the numerical structure of the problem and apply their knowledge of the properties of number to find the solution.

**RESULTS**

None of the first year students and fourth year students was able to produce a correct solution to the problem. Many of the fourth year students were resistant to solve the problem as they did not think that it was possible to solve the problem without following the standard algorithmic route.

The table below shows the solutions of students and the reasons that they gave to support their strategy:

<table>
<thead>
<tr>
<th>Student</th>
<th>Solution Strategy</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: 1st Year</td>
<td>Because ((1990 \times 1990) - (1990 \times 1990) = 0) therefore, (1997 \times 1995 - 1999 \times 1993) ((7\times5) \times (9\times3)) (=35 - 27) (=8)</td>
<td>Student A reasoned that since the numerical structures differed only in their unit digits, the calculation claim which was applied to the one structure must work for the other structure</td>
</tr>
</tbody>
</table>

The students’ solutions are representative of the groups in which they worked. Students who provided reasons for their strategies spoke on behalf of their group.
B: 1st Year

\[(2000 \times 2000) - (2000 \times 2000)\]
\[= (-3 \times 5) - (-1 \times 7)\]
\[= 8\]

Student B reasoned that a quick way to add is to round up but that one must take into consideration what was added

C: 1st Year

\[1997 - 1995 = 2\]
\[1999 - 1993 = 6\]
\[2 + 6 = 8\]

Student C reasoned that the quickest way to calculate is to find the difference and then to add

D: 1st Year

\[= (7 \times 5) \times (9 \times 3)\]

Students D and E reasoned that if one subtracts 1990 throughput, the numerical expression would not change its value and that \((7 \times 5) - (9 \times 3)\) would be an equivalent expression of \(1997 \times 1995 - 1999 \times 1993\)

E: 4th Year

\[= 35 - 27\]
\[= 8\]

Table 1: Students Solution Strategies and Reasons

After the students in the first year class presented their solution strategies to the class, a student remarked that since so many of the groups got the same answer through different strategies, their strategies must be correct. At this point students used a calculator to check the solution to the problem and this provided further conviction to the students that their solution strategies were correct.

Students were encouraged to reflect on additional ways of convincing others that their solution strategies were correct and to present their arguments in the following lesson. Only the fourth year students produced an additional argument for how they would convince others that their solution was correct:

‘We let 1990 = a, then 1997 \times 1995 - 1999 \times 1993 can be written as

\[(a + 7) (a + 5) - (a + 9) (a + 3)\]

Now we can just take out the a’s

\[\cancel{a} + 7) (\cancel{a} + 5) - (\cancel{a} + 9) (\cancel{a} + 3)\]

\[= (7 \times 5) - (9 \times 3)\]

\[= 8’\]
Analysis and discussion of students’ solutions

The analysis is structured into two sections. The first section is an analysis of the students’ solution strategies. The second section analyses the opportunities for generalisation and justification that emerged and present the moves to different levels of generalisation.

Solution Strategies of Students

The solution strategies of the students suggest that students gave up the knowledge that they have of number properties and tend to focus on the surface features of numerical expression and struggle to recognise the deeper structural features of the numerical expressions. Student A’s strategy shows that it is based on surface features, namely the units digits of the expressions. Student A’s explanation that the solution to \((1990 \times 1990) - (1990 \times 1990) = 0\) because \((0 \times 0) - (0 \times 0)\) is a means to find a justification for the student’s strategy. The student may know that \((1990 \times 1990) - (1990 \times 1990) = 0\) because the expression can be generalised to \((x - x)\).

Student B’s strategy and reason suggests that the student is aware of strategies such as rounding up and applying the principle of conservation but does not represent the strategy as \((2000 - 3) \times (2000 - 5) - (2000 - 1) \times (2000 - 7)\) that prompts exploration of the deeper structure of the expression. Student C’s strategy does not reflect application of a previously learnt strategy and that the student was simply focussed on generating an answer through operating on the numbers in a way that adhered to the instruction of not applying the standard algorithm.

Students D and E’s strategy reflects the application of applying the principle of generating equivalent equations to expressions. The students know that if \(x - 1990 = y - 1990\), then \(x = y\) and uses this information as the basis on which to argue that if 1990 is subtracted from each of the numbers in the given problem \((1997 - 1990) \times (1995 - 1990) - (1999 - 1990) \times (1993 - 1990)\), then the problem can be reduced to \((7 \times 5) - (9 \times 3)\). The fourth year students persisted in applying this incorrect strategy even when the students introduced a letter to present the problem in a more generalised form as \((a + 7) (a + 5) - (a + 9) (a + 3)\).

Opportunities for engaging students in generalisation and justification

A common element that emerged in engaging with the students’ reasoning about their strategies is that they did not see the need to check their strategies with problem types that had the same structure. Student A’s strategy will be used to illustrate the way in which the students were asked to explore, in a systematic way, whether their strategies could be applied to problem structures with different unit digits. For example, in the
case of student A the student was first challenged to see if the strategy worked if and only if one of the last digits was changed to a zero before considering other cases in which the last digit is changed. The table below shows the different problems used to check student A’s strategy:

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</tr>
</thead>
<tbody>
<tr>
<td>(0 × 4) - (5 × 0)</td>
<td>-1990</td>
<td>(7 × 1) - (1 × 5)</td>
<td>3982</td>
<td>(7 × 5) - (9 × 3)</td>
<td>8</td>
<td>(7 × 5) - (9 × 3)</td>
<td>2007</td>
</tr>
</tbody>
</table>

The systematic process of checking whether the strategy could be applied to other cases did convince the students that for their strategy to be correct it had to work for all cases. But this did not automatically prompt the students to consider what it was about the structure of the problem that allowed their specific strategy to work. Students needed to be asked to focus explicitly on the relationship between the numbers in the specific structure that allowed their strategy to work for the specific problem. Student A first generated the same structure as the fourth year student and then applied the distributive property of numbers to provide a valid argument for the strategy as follows:

‘I let 1990 = a

Then (a + 7) (a + 5) - (a + 9) (a + 3) = a² + 12a + 36 - a² -12a -27 = 8’

Student A proceeded to show that provided a number has a place value structure in which only the digit value differs, and the sum of the unit digits are equal to twelve in the numbers in each term, the calculation could be reduced to the subtraction of the product of the unit digits. The student illustrated with a specific example as follows:

‘I let 34590 = a in the expression 34597 × 34595 – 34594 × 34598

And the expression then becomes (a + 7) (a + 5) - (a +4) (a + 8)’

The student points out that 7 + 5 =12 and 4 + 8 = 12.

Student A has a partial generalisation schema in being only able to see that the unit digits must be equal to twelve. Further generalisation would require the student to see (a + 7) (a+ 5) – (a+9) (a+3) and (a + 7) (a + 5) – (a +4) (a + 8) as special cases of the general structure (a+d) (a+e) – (a+f) (a+g).

The special case (a + 7) (a+ 5) – (a+9) (a+3) when operated on produces the equivalent structure, a² + 12a + 35 - a² - 12a -27, which does reveal the generality of the structure since the “a” term will always have equal coefficients if d + e = f + g in the general
structure \((a+d)(a+e)-(a+f)(a+g)\). This is an instance of generalisation in which multiple cases are not required as in inductive reasoning which student A displayed (i.e. through trying examples) in an attempt to prove the validity of the calculation strategy.

Student A did not realise that the equivalent structure that was produced, \(a^2 + 12a + 35 - a^2 - 12a - 27\), reveals that the strategy will only work if and only if digit units differ in the place value structure of the number and that there is a special relationship between the digit units.

The solution to \(1997 \times 1995 - 1999 \times 1993\) within the AMESA competition publication provided a different solution strategy which none of the students in the first and fourth year produced:

Let \(t = 1997\), then \(1997 \times 1995 - 1999 \times 1993\)

\[
= t(t - 2) - (t + 2)(t - 4)
= t^2 - 2t - t^2 - 2t + 4t + 8
= 8
\]

The problem posed to the students as part of establishing a classroom inquiry culture was whether they could generate questions that could lead to further exploration of AMESA solution strategy. Most students acknowledged that the school classroom culture is generally one in which they are always given questions for which they have to find solutions but are not expected to generate questions to explore mathematical features of the problem. Students struggled to come up with questions for further exploration. The question, what kind of expressions with the same structure would always generate a constant when using the AMESA strategy, \(t(t + a) - (t + b)(t + c)\), was presented to the students. This led to the testing of the AMESA strategy with a specific case in which one of the unit digits was changed (the unit digit of the last number was changed from 3 to 4):

Let \(t = 1997\), then \(1997 \times 1995 - 1999 \times 1994\)

\[
= t(t - 2) - (t + 2)(t - 3)
= t^2 - 2t - t^2 - 2t + 3t + 6
= -t + 6
\]

The testing with a single case did not reveal the relationship between \(a\), \(b\) and \(c\) in the structure \(t(t + a) - (t + b)(t + c)\) that would generate a constant solution for the students as they focussed on the final equivalent expression \(-t + 6\). The relationship between \(a\), \(b\) and \(c\) that will produce a constant can be deduced from the specific case if the students
focussed on the following equivalent form \(-2t - 2t + 3t + 6\) and generated the following equivalent form
\[ t (-2 - 2 + 3) + 6 \]
from which they could have argued that \(-2 - 2 + 3\) is not equal to zero and to produce a constant the sum of the three numbers must be zero. This once again illustrates that the generalisation of the strategy for producing the constant did not necessarily require a logically deductive route from the general structure:

\[
\begin{align*}
& t (t + a) - (t + b) (t + c) \\
= & t^2 + at - t^2 - bt - ct - bc \\
= & at - bt - ct - bc \\
= & t (a - b - c) - bc \\
\end{align*}
\]

Therefore \((a - b - c) = 0\)
\[
\to \quad a = b + c
\]

On establishing the relationship between \(a, b\) and \(c\) in the structure, \(t (t + a) - (t + b) (t + c)\), the students could revisit the special case in which only the last digit of the original problem was changed from 3 to 4 (1997 \(\times\) 1995 \(-\) 1999 \(\times\) 1994) and generate different examples changing the last digit as well as other digits that would generate a constant, for example:

Let \(t = 1997\), then \(1997 \times 1996 - 1999 \times 1994\)
\[
\begin{align*}
= & t (t -1) - (t + 2) (t - 3) \quad \ldots\ldots\ldots\ldots[ -1 = 2 + -3] \\
= & t^2 -t - t^2 - 2t + 3t + 6 \\
= & 6 \\
\end{align*}
\]

**CONCLUSION**

The incorrect solutions strategies of the students in this case study reflect findings about students’ difficulty in applying the properties of numbers to create equivalent forms of number expressions. The key difficulty is the recognition of the relationship between the numbers in the expressions and to use a variable to express the general structure of the number expression. In this case study when students were not able to identify the relationships between numbers in the expression they often apply the properties of number incorrectly simply to get an answer.

CAPS place a strong focus on recognising patterns in number sequences to introduce learners to early experiences in the process of generalisation. In this case study the focus was on engaging pre-service teachers in the process of generalisation in a different kind
of number context that involves a numerical expression and recognising its structure to support computation strategies. The challenge for pre-service teachers being prepared for primary school is to consider different kinds of number contexts which support learners in engaging with generalisation processes and to recognise that generalisation and recognising mathematical structure are interrelated.

The incorrect solution strategies of the pre-service teacher students presented a context for challenging them through a critical inquiry approach to reflect on their strategies. In a pre-service teacher training course students’ engagement with their mathematics errors is an important component of preparing students to work with learners’ errors. Engaging with students’ errors in a way that allowed students not only to correct their errors but to consider the role of errors in exploring further mathematics in the problem is important if we want them to extend their learners’ thinking through engagement with learners’ errors.

REFERENCES


A comparison of South African with Indian education developments over the past 30 years highlights some important similarities but also some stark differences. Both countries attained independence from colonial rule: India in 1948 and South Africa in 1994. The South African constitution enacted in 1994, states explicitly a commitment to democratic principles guiding education. India has in policy similar commitment in various policy documents. However judged from current education publications the will to focus on social transformation explicitly in the South African educational landscape has waned, while the Indian policy makers have since 2009 embarked on a concerted effort at social transformation. Social transformation begins with the empowerment of people, and in the view of Batra (2009) necessarily with the empowerment of teachers as teachers have a pivotal role in society. In pockets of educational reform from India to Scotland to Alberta, Canada, we note the consistent reference to the idea of teacher agency in debates about social transformation. Various definitions have been circulated, the most pertinent of which is the ecological view proposed by Biesta and Tedder (2006), and extended by Priestley and Biesta (2013). Essentially agency does not primarily reside in the teacher but is an outcome of the teacher acting meaningfully within the educational and social milieu. A natural consequence of this outcome is that policy decisions with regard to the day to day role of teachers can either support or suppress agency. Likewise the professional development of pre-service teachers is potentially supportive or destructive with regard to agency. Our proposition here is that it is only the agentic teacher who can inspire cohorts of youth at school (and not just the elite few) to aspire to the high goal of an engaged citizenry leading to social transformation. A non-agentic teacher, without vision for a transformed society, without the full range of learner interest at heart has little or no potential to transform her engagement with her subject or her learners to effect meaningful and sustained social transformation. A clear and full understanding to the teacher as professional has to become the starting point for any teacher education programme. The teacher envisaged by Biesta, Priestley, Batra and within the South African context cannot function within the strings of curricula statements, especially not in the enabling of independent and autonomous thinkers, the purpose of education advanced by Biesta (2009) and others. We take the characteristics supporting the professional and agentic teacher, as identified by Batra, and reflect the current mathematics course for final year Foundation Phase teachers against these characteristics. A second reflection maps the characteristics supporting the professional teacher onto the “Minimal Requirements for Teacher Education Qualifications” (MRTEQ). From a third reflection where we map the elements of the mathematics course against MRTEQ, we propose that there are essential and
mathematics course against MRTEQ, we propose that there are essential and indispensable elements for teacher education that should be considered for inclusion into South African teacher education policy. We argue that without these elements teacher education development does not meet the democratic imperative of enabling participation in societal functions and social transformation. A full description and evaluation of the mathematics course is in process. Further research in the form of a survey of mathematics teacher educators is in process to gauge views on social transformation, professional development and teacher agency.
INTRODUCTION

A comparison of South African with Indian education developments over the past 30 years highlights some important similarities but also some stark differences.


Both countries have had ambitious educational goals aligned with a democratic imperative, and both countries have soon resorted to short term solutions vitiating their goals as a result of public concerns about implementation, and research focussed on short term goals such as immediate pass rates. In South Africa both the Review of Curriculum 2005 (Chisholm et al., 2000) and the Review of the National Curriculum Statement (Dada et al., 2009) resulted in conservative policy decisions, that saw a return to narrow, content based curricula within the first ten years of an endeavour towards meaningful change. While judging from current education publications the will to focus explicitly on social transformation in the South African educational landscape has waned, the Indian policy makers have since 2009 embarked on a concerted effort at social transformation.

The most striking similarity between South Africa and India has been the lack of attention to professional development of teachers that could sustain curriculum and social transformation. In South Africa a noticeable trend across policy making, educational research, and public media has been the blanket depiction of teachers as incompetent and in need of constant supervision. Much of the “evidence” for teacher incompetence has been given as the results of international and systemic testing of learner performance. While the results are not interrogated here, and nor are the political and educational debates around international large scale testing, it is the

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9 The issue of “pass rates” has been debated in South African education circles. We note too that the inferences to be made from test results often miss the mark.

10 The Right of Children to Free and Compulsory Education Bill was passed on the 3rd August, 2009 (Batra, 2009). The National Curriculum Framework (NCF) 2005 is described as “learner centred and progressive” (Batra, 2009, p. 126).and though teachers were required to address social inequalities, there was still a sharp divide between the teacher and the curriculum, meaning that there was little engagement of teachers with the curriculum (p. 129).
inferences and actions emerging from the results of systemic testing that we question. We also note that while there is a great deal of attention to results there is little attention to social transformation and the education-society dialectic in much of the research, whereas in India we see a focus on critical societal factors impacting on education and so included in teacher education.

While educationists such as Batra\textsuperscript{11} (2009) and the authors of this paper do not deny that problems exist the question to be debated is in Batra’s case how to traverse the education entitlement- social transformation divide. To imagine social transformation begins with the empowerment of people, and in the view of Batra (2009) the empowerment of teachers. In our case we question whether current policy will address our problems. In the South African teacher education policy documents we note that there is no explicit reference to either social transformation or the enabling of teacher agency which would support such a goal.

In this paper we reflect the elements of a mathematics course for final year Foundation Phase students against the characteristics of the professional teacher education curriculum presented by Batra as having the potential to transform society.

\textbf{The role of the teacher and social reform}

A key distinction in envisaging the role of the teacher must be between the teacher as \textit{implementing agent} and the teacher as \textit{agent of change}. In the first enactment we have the teacher executing the demands of the authorities keeping daily records of topics taught (but not necessarily learned) much as the waitress would execute the demands of the manager in a busy restaurant, or the mechanic on a production line. In the second enactment, the teacher keeps in mind the goal of education as the development of autonomous and independent thinkers and here engages the learners’ creative resources

\textsuperscript{11} Poonam Batra is a leading player in the National Council for Teacher Education (NCTE), whose own professional teacher education course (Bachelor in Elementary Education – BelEd, Batra, 1995) is a beacon exhibiting a route to social transformation. She has also recently chaired the Poonam Batra report (Batra, 2014) responsible for implementing the recommendations of the Justice Verma commission on teacher education.
in the learning of mathematics and literacy. This distinction applies equally to teacher education.

In the professional development programme at University of Delhi, the importance of teacher empowerment is acknowledged as a critical factor enabling the teacher to play a role in social transformation. In this context the school curriculum is perceived as “the inclusive space that extends beyond the realm of textbooks into that of classroom process that enable the agency of the child and her educator” (Batra, 2009, p. 129).

The short term objective of the mathematics course\textsuperscript{12} referenced in this paper is to improve the mathematical competence of the students such that they are comfortable with the critical concepts in primary school mathematics. This objective seeks to ensure that these aspirant teachers of Grades 1, 2, and 3 learners have a vision of the mathematical path that their learners will follow over the five years after leaving the Foundation Phase. In the mathematics course we take the aspirant teacher from the first steps of engaging observation and reasoning skills in the investigation of number patterns and relationships, for example, the multiples of three, setting in process mathematical thinking skills that we hope will generalise to other mathematics topics, and take most of the students to understanding more advanced proofs, for example the proving of divisibility rules.

A longer term objective was to create the conditions for dynamic competence, or agency, in the mathematics classroom.

**The concept of Agency**

In pockets of educational reform from India (Batra, 2009) to Scotland (Priestley, Biesta & Robinson, 2013) to Alberta, Canada (Simmt, 2011), we note references to teacher agency. Various definitions have been circulated, the most interesting of which is the ecological view proposed by Biesta and Tedder (2006), and extended by Priestley, Biesta & Robinson, (2013). The concept of agency is here defined as the dynamic competence of human beings to act independently, and to make choices (Priestley et

\textsuperscript{12} The precursor to this course was developed by the mathematics education group at Wits in 2004/5, by the authors, Louise Sheinuk, Craig Pournara & Bhathi Parshotam.
al., 2013) in order to advance toward their goals. Two additional ideas are key to this concept: the first is that agency is not intrinsic to a person, but rather perceived as occurring interactively with the environment, and secondly that the environment in which individuals find themselves may enable or constrain agentive action (Biesta & Tedder, 2006).

Essentially agency does not reside entirely in the person or in this case the teacher, but is a product of the teacher engaging with the environment. A further point is that dynamic competence (agency) has historical features, in that the teacher may draw on previous experience, a future in that action may be taken towards a future goal, and the present where the teacher has to negotiate the educational and social milieu. This view of agency, as interactive with the environment, also informs the view proposed by Batra (2009). A natural extension of this definition is that the policy decisions with regard to teachers can either support or suppress agency.

Here we give support to the elements of the professional development model proposed by Batra, in particular developing the self through reflecting on own attitudes and beliefs, interrogating one’s own educational experience and then “probing the education-society dialectic”. In addition to a focus on the teacher herself, there is attention to epistemological matters that have a bearing on teaching and a critical approach to knowledge development and acquisition.

In the mathematics course the lecturers took the position of both teachers and lecturers as agents of change. The purpose was to engage the obvious or latent agency of the students in the mathematics classroom. From the outset the focus was on developing and honing the problem solving skills needed to engage with the algebraic reasoning required of the students to engage with further mathematics.

Our proposition here is that it is only the agentic teacher that can lead the school cohorts of India and South Africa (and not just the elite few) in the process towards an engaged citizenry leading to social transformation.
Professional Identity and Teacher Agency

The teacher as professional then (agentic, with vision, with the interests of the children at heart, determined to make a difference) has to become the starting point. No other starting point is possible as without this dynamic action no amount of knowledge bits will find their way into a managed classroom experience. The teacher envisaged by Batra, Priestley, Biesta & Simmt cannot function when attached to strings of curricular statements.

So then what is the teacher education programme that will support such agency in teachers? Here we have to assert that the teaching profession differs from that of medicine, and law and engineering, in that teaching is encountering daily human beings who we argue have a trajectory that is essential creative. Bio ethics philosophers are exploring the notion of client agency in the medical profession in relation to informed consent claiming that it is not so much the substance of the interaction between doctor and patient but the communication that is enabled (Manson & O’Neill, 2007). In the legal fraternity the profession mostly means “looking after the interests of the professionals”, as the cartoon depicting a farmer pulling at each end of a cow, while the lawyer is milking the cow, satirises! There is certainly an eye on bringing in the required quota of the company’s income! 13

For teachers and teaching the situation is markedly different from both medicine and law. While all three professions serve the public it is teaching which potentially impacts broadly on the lives of children, the community and the society. We concur with Higgins (2013) that “teaching forces us to confront the inevitable tensions arising when subjects meet and each retains his or her agency” (p.8). “Subjectivity emerges in the matrix of intersubjectivity, in relationships where we successfully maintain the complex tensions between self assertion and recognition, independence and

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13 Of course there are lawyers who make an unselfish contribution to society, and educational institutions that are narrowly focussed on the next international ranking, and therefore the number of articles written rather than the number of aspirant teachers equipped to make that difference in society proposed by Batra (2009).
dependence, separation and connection” (p.9) The challenge then becomes how to find the balance between existing for ourselves and existing for others.

A critical feature of the course was that the students’ voice was given expression through the teaching style which encourages exchanges between students as well as with the lecturer, but also through students sharing their prior experiences with learning mathematics their fears and their hopes for their own teaching through writing and submitting a reflective essay. This exchange was a critical moment in the course where the direction changed in some respects to address their expressed hopes. A change made possible through the dynamic interaction of teacher educators and aspirant teachers.

**Teacher education**

Education curricula generally are made up of excerpts of other disciplines for example psychology, sociology and philosophy. Because these disciplines are not the primary focus they are often presented as a “given body of knowledge” rather than a science with a history, a mode of interrogation and a body of contesting literature and arguments. In the teacher education curriculum proposed by Batra (2009) there is an integration of educational theory, pedagogical theory, social sciences and human development all of which are subjected to critique. She notes that there is a difference between engaging with theoretical concepts and frameworks as given conceptions to be applied in the classroom or to learners, and of using concepts and frameworks as theoretical tools for analytic purposes, which then enable teachers “to intervene in the experiential and social realities” in which they find themselves (ref).

Batra (2009) is explicit about the social transformation agenda in the teacher professional development programme at the University of Delhi. The type of professional development curriculum envisaged by Batra for empowering teachers, and for charting the traverse across the social divide (Batra, 2009, p. 136) consists of eight characteristics embedded across the four years of teacher education. The key features of what she describes as “a well tested curriculum framework for the professional development of teachers” and which could “enable the realisation of the full potential of ideas” expressed in the National Curriculum Framework 2005¹⁴ document, and also support the goal of social transformation (Batra, 2009).

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¹⁴ National Curriculum Framework (NCF) 2005 is
We have grouped these characteristics into two broad categories, the first of which is a focus on the teacher and her environment, and the second of which is focused on knowledge, the curriculum and related issues. As will be shown in the discussion below we do not regard these categories as distinct in the professional development of teachers but rather as useful for analytic purposes. Four of the characteristics focus on the teacher, the children, the school and society. The second cluster give attention to knowledge (two key points), and its relationship to teaching (PCK, two key points). For a full description of these characteristics see Batra (2009, pp. 136 to 146). In this paper we highlight aspects of these characteristics which we believe will strengthen professional teaching development programmes. Also, because our approaches to professional development include teacher agency and to some extent promote a social transformation agenda, we illustrate these characteristics with elements from our courses. Our challenge of course is to integrate these broad characteristics to the teaching and learning of mathematics, which from some perspectives may be perceived as separate from a social transformation agenda.

The teacher

The focus on the teacher herself in Batra’s work is entitled “developing the self”. In philosophical treatise on teachers we find reference to the importance of personal fulfilment on the part of the teacher (Higgins, 2013). While teaching is seen as a “helping profession”, it is rather the fulfilled teacher, who is able to share her interest, joie de vivre and vision with her students. The teaching profession is in trouble if we are expecting self-sacrifice without the balance of fulfilment. Batra (2009) notes some characteristics of exemplary teachers that are worth discussing here. Exemplary teachers believe in the potential of their students and rather than talk about the shortcomings of the learners, reflect on what they, the teacher, could do better. Here we note the saying “The better I know my subject, the cleverer the students become”.

It is not the fulfilled teacher who starts with the assumption that the students are limited in some way either cognitively or socially. It is not the fulfilled professor that asks of

15 Discussion with Maryna du Plooy. This saying became our motto, and was evident in the test outcomes (see Du Plooy & Long, in process).
students a difficult mathematics question, and then shames the student for not being able to answer (personal discussion with a professor of education).

What distinguishes the fulfilled teacher from the disillusioned teacher? We believe that in addition to a rigorous and demanding mathematics agenda, personal reflection on own school experiences with mathematics, current feelings of inadequacy and dreams and hopes for their future careers are important.\textsuperscript{16} The teacher-educator as role model upholds the profession and invites the students in as colleagues who together engage the challenges. The hierarchical relationship between teacher educator and aspirant teacher does not contribute to professional collegiality. Batra (2009) talks about the “democratisation of space”; this characteristic is supportive of agency in that the aspirant teacher’s voice is given expression.

In the student teacher reflections\textsuperscript{17} it emerged that many had unsatisfactory mathematics teaching that left them fearful of mathematics. Here some reflection on the types of schools, those highly resourced and those moderately resourced in terms of both teachers, materials and facilities demand interrogation that engages the social divide.

On the other hand despite negative experiences, the majority of the students hoped that they would be able to instil a love for mathematics in their children and were prepared to work very hard to achieve that goal, the evidence of which could be found in the portfolios compiled during the course.

**Children and the school**

The action of “placing children in context” is one of the characteristics critical in the Batra’s professional teacher education programme. Here she notes the importance of “engag(ing) directly with children and their questions rather than about them” in order to grasp the finer nuances of developmental differences related to the social and political milieu. The African and South African context differs from communities in

\textsuperscript{16} Attention across time, past, present and future aligns with the view of agency presented Priestley et al., (2013)

\textsuperscript{17} These reflections are in the process of being formally analysed and reported.
the United States, the United Kingdom and Europe, and the contextual differences are not all weighted in favour of the North.

Batra advocates a closer relationship with schools – “engaging with schools”. She believes that a loosening of these boundaries may serve both the interests of the institutions (and the aspirant teachers), and the schools themselves.

We note here that in many South African teacher education programmes the student involvement at schools is a key feature with great care taken to cultivate relationships with schools. There is also the danger however, with all the pressure to publish\textsuperscript{18}, that the core education staff “buy out” the time that they would normally spend at schools to spend time on research that does not necessarily impact on the development of teachers, though it might.

### Society

True to a “critical pedagogy” perspective, Batra focuses on “probing the education society dialectic”. It is inevitable that in societies such as India and South Africa, where the disparities between communities are beyond imagination, that these inequalities are going to find their ways into the teacher education classrooms. These cycles of inequality are perpetuated. In a typical lecture theatre in South Africa there will be students from private schools, good government schools, “township” and rural schools. These children have all worked hard to get into university and their perseverance has seen them through the programme to their final year, but the differences in output differ. What is to be done about this imbalance? Is it simply a case of “they are lazy, they don’t come to class and they don’t listen”, as I was told (by one of the high achievers in the class) when I reflected aloud about the group of 7 out of 240 who had not passed the test.

\textsuperscript{18} The requirement at University of Pretoria is for each staff member to publish 2.5 papers per year. Even working with a low estimate of 100 hours for a paper, this means that 250 hours (8 weeks) are taken out of the academic year. The practice of “buying in” people to do one’s marking so that lecturers can focus on research also seems to me short-sighted as it is in the engagement with student writing that a teacher gets to know her students.
Here we need to alert students to the danger of perpetuating cycles of inequality, when it is possible to break these cycles with conscious engagement with students who are in danger of “disappearing under the radar”. This “disappearance” of students from the teacher-educators eye may be due to shame or embarrassment at not understanding, but whatever the reason this group of students have to be integrated into the engaged classroom.

Knowledge

Under this category Batra has two characteristics “rethinking knowledge and learning” and “reconstructing disciplinary knowledge”.

In whatever form knowledge is presented, the students have to engage with that knowledge to make it their own. Here we find the important threshold in academia where students are caught between personal opinion and plagiarism. What for many students who have attended good schools find relatively easy, those new to academia find impossible, and that is how to understand texts, engage with these texts and then find their own voice with which to express the confluence of ideas.

The counterpart in the mathematics classroom is understanding the mathematics, engaging with the mathematics such that one makes it one’s own, and then applying this mathematical understanding in other contexts.

Here Batra (2009) alerts the teacher-educator to two points, the first is that teachers’ opinions should be respected, the “right to know” should be reinforced, and the second point is that nothing should be asserted on the “basis of authority” but rather through reasoning and debate.

One answer here is the reconstructing of disciplinary knowledge through carefully designed activities and processes. Here we note the importance of “identifying misconceptions against a framework of developing competency” and alerting aspirant teachers to such a mechanism.
Pedagogical content knowledge

Two additional characteristics identified by Batra (2009) are “interrogating theory and practice” and “forging links between pedagogy and the curriculum”. We note that these two topics are critically important but do not have the space to deal with them here. We will however, illustrate how in a final assessment in the course referred to throughout this paper, we integrated aspects of theory and practice and made links between pedagogy and the curriculum.

In the mathematics class taught by the first author it was starkly evident that these aspirant teachers lacked confidence when it comes to mathematics and to “being a teacher”. *What will I do if someone asks me a difficult question? What if my own fear of mathematics comes over to the children? We are only Foundation Phase teachers we do not need higher levels of mathematics. I failed mathematics at school. I panic when I see a calculation because I do not know my tables.*

Allowing the teacher voice, in this case through a reflective essay, brought a myriad of experiences “into the classroom” so to speak. The double reflection, now of the lecturer could through planned activities and through some personal engagement to deal with the fears.

The somewhat common practice of testing the teachers on content that they might have known at school, but may have forgotten through unfamiliarity or through not having understood the concept properly in the beginning, would have been counterproductive and served to further erode their confidence. We assert with Matters (2009) that assessment information plays a powerful role in educational debate in the 21st century. However, the importance of the assessment “can be justified only if two conditions (at least) are met”. The first condition is that the assessment product is of high enough quality and adheres to measurement properties such that the “users of the assessment data – the analysts, the teachers, the administrators, the policy makers” can interpret the outcomes of tests in a meaningful and productive way (Matters, 2009, p.222).

Given the above statement, it was important that there was coherence across the conceptualisation of the course, the planning and execution of the course, the design of
the materials and the assessment. A first document was shared with the students outlining what components of the course would be tested, in other words those aspects of the course we considered most important. The test was then designed by the coordinator of the course (the first author) and reviewed by the co-teacher on the course. The test was also reviewed by the mathematics lecturer for the Foundation Phase. The test was marked by the co-teacher who noted the areas where we the teacher educators had not fully succeeded in getting the concept across.

In order to check whether the test was sufficiently robust to warrant a valid assessment, we conducted a Rasch analysis on the test data, assigning a person factor “Afrikaans” for one group, and “English” for the second group. The fact that there were two language groups was not as important as the information that could be gleaned from the comparison with regard to teaching styles. Hidden in this information is that in the “English” group were a proportion of students for whom English was not their home language. For our purposes all factors, other than that the outcomes reported for the two teaching groups were ignored. Our prime purpose was to provide feedback on the course, the test design, and feedback on the success or otherwise of our teaching. The full analysis will be reported elsewhere (DuPlooy & Long, in process); here we discuss only one output from the analysis, the Person-Item Threshold Distribution, with two levels identified, the “English” group and the “Afrikaans” group.

From a cursory look at the graph you will notice that the item distribution spans from relatively very easy (-1.7 logits) to very difficult (3.5 logits). The item mean is set at zero by the model. The person distribution ranges from low proficiency (-0.5 logits) to high proficiency (3.5 logits), about 1 logit.
From a test design perspective, the targeting of the test looks inadequate – the test is too easy for the cohort as a whole and there are not enough difficult items to provide information on the students with greatest proficiency.

We argue however that the cut point for this test is 70%, about 1 logit.\(^{19}\) Above one logit (about 70%) we regard the student as competent and able to function independently with respect to the mathematics that they will encounter when teaching at the Foundation Phase. However, below 70% means that more work is required. Note also that the mean location of the “Afrikaans” group is higher than the “English” group, and that the range of the “English” group is wider, with both the highest and the lowest achievers on this test. From a teaching perspective, we are concerned that there is a cluster of learners below the item mean of zero (below 50%) and surmise that the teacher of the English group could adjust her teaching to ensure that this group are supported.

In summary, the components of the professional development programmes reported by Batra (2009) support teacher agency in some respects in that their voice is heard, the theory and practice are engaged with critically, and we trust that all information given to the students is subjected to critique. Elements of our current teaching, in particular, a course in mathematics for Foundation Phase teachers, have been shared to illustrate...

\(^{19}\) The Rasch model is explained elsewhere. See Dunne, Long, Craig & Venter (2012) for explanation of the ideas presented here.
how some of these components have been taken on board. We have extended Batra’s work through inserting reflection on the assessment process.

In the following section of the paper we compare the components of Batra’s professional development programme with the Minimum Requirements for Teacher Education Qualification (MRTEQ) document.

**South African Minimum Requirements for Teacher Education Qualification**

In this section we present a comparison of the characteristics of the Professional Development Programme (BEIEd) (Batra, 2009) and the MRTEQ document. The comparison is somewhat artificial, as the documents serve different purposes, however the purpose is to investigate whether there are elements of the Professional Development Programme that should be given attention in policy documents.

Broadly speaking the categories into which we placed the characteristics, components of the professional development curriculum can be used to organise the MRTEQ elements. That is focusing on the teacher herself (Teacher, 4; 5; 10; 11), attention to the children (Learner/Children 3), focussing on the schools (School/management 8; 9) and society (Society/Diversity 7), attention to knowledge (Knowledge, 1; 6), and its relationship to teaching (Pedagogic Content Knowledge, 2).

**Table 1: The 11 basic competences reordered (MRTEQ, Appendix C)**

<table>
<thead>
<tr>
<th>Batra (2009)</th>
<th>MRTEQ</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Characteristics of a Professional Development Programme</strong></td>
<td><strong>Newly qualified teachers must have/be/know how to/understand/be able to ...</strong></td>
</tr>
<tr>
<td><strong>Teacher</strong></td>
<td><strong>Developing the self</strong></td>
</tr>
<tr>
<td>Sense of positive professional identity (in the face of negative media reports)</td>
<td><strong>Communicate effectively in general and in relation to their subjects, in order to mediate learning (4)</strong></td>
</tr>
<tr>
<td></td>
<td><strong>Highly developed numeracy, literacy and IT skills (5)</strong></td>
</tr>
</tbody>
</table>
| Knowledge | Reconstructing disciplinary knowledge  
Rethinking knowledge and learning | Sound subject knowledge (1), knowledgeable about the curriculum, plan and design learning programmes (6) |
| --- | --- | --- |
| **Children/learners** | *Placing the children in context*  
Engage directly with learners and their questions rather than about them | Know the learners and how they learn, understand their individual needs and tailor needs accordingly (3) |
| **Schools** | *Engaging with schools* | Manage classrooms effectively across diverse contexts, in order to ensure a conducive learning environment (8) |
| **Society** | *Probing the education-society dialectic*  
Understand the social construction of knowledge; How schools can perpetuate inequality; Interrogate own position in society | Understand diversity and include all learners, identify learning and social problems and work with professional service providers to address these (7). |
We ask the following questions:

1. In the MRTEQ document is the aspirant teacher positioned as a professional in the sense that the teacher has agency, the dynamic competence to act in her educational and social environment?

2. To what extent are social issues interrogated?

3. Is there evidence of a social reform agenda?

4. What components of the Batra Professional Development Programme (BEIEd) could be incorporated into a policy document such as MRTEQ?

In answer to question 1 we note that the teacher is position more as the implementer of the curriculum than an agent of change.

To question 2 we note that with the exception of number 7, where the focus is on “diversity” and “identifying learning and social problems” no attention is paid to social issues. Even here we note that it is intimated that the problem lies with the child rather than the society. It is interesting her to note that the Indian National Curriculum Framework was critiqued on the same grounds (Batra 2005).

To question 3 our answer is “No”.

In answer to question 4 we feel that the course as a whole, with its eight characteristics, could be adapted to the South African context, in particular the attention to “developing the self” as a primary goal, and also in relation to society, but also a critical engagement with the term knowledge.
Concluding remarks: professional identity and teacher agency

The paper begins with outlining the essential role of the teacher in society and particularly her role in the transformation of society. We assert that the teacher’s role in society is pivotal, that the teacher be seen as an agent of change rather than the implementing agent of decisions made far from her current context, and that this role be acknowledged. The professional status of teaching must be reasserted. Here we note that this professional teacher role is not possible without the agency required to observe her current environment, engage with the children and their questions, and make decisions concerning the best possible plan of action.

We reflect on the type of professional development required to support this teacher. Here we draw on the experience of Batra (2009) who describes the characteristics of a teacher education programme whose focus is the enabling of teachers who can act within society to make a difference.

We have adapted these features of the BEIEd programme to conform to our current views of what constitute the essential components in our context of a professional development programme that supports teacher agency and therefore positions prospective teachers to make a difference in the environments in which they find themselves in the next five years. We have also slanted the general professional development to a particular focus on the professional development of mathematics teachers as this is the focus of the authors.

A comparison with the South African MRTEQ document shows that the essential element of interrogation of the society in which we live is missing. In fact the critical engagement with own personal experience, critique of knowledge as currently presented is not made explicit.

Further research is envisaged and planned along two trajectories. The first is to conduct a survey based on the characteristics incorporated in the professional development programme (Batra) in order to gauge the perspective of teacher educators in three institutions.
A formal analysis of the Foundation Phase mathematics course will be conducted. Tracking students into their teaching practice will enable us to gauge what aspects of the course carry over into the classroom of Foundation Phase final year student.

REFERENCES


Appendix A: Excerpts from student reflections

Student A

“At the age of 3 I was well ahead of others my age with mathematics and problem solving”

“One of my biggest weaknesses in relation to problem solving is my attitude towards mathematics. I associate maths with feelings of negativity”.

“I am worried that I will slip back into having a negative view of maths. I also fear that my own views of mathematics will affect my learners negatively and cause them not to like maths. I fear that I will not be “good enough” to teach maths to young children effectively and that I will struggle to communicate that maths is a positive thing because I have struggled with maths myself”.

Student B

“Having started this module, I was initially not interested as I have never, at any point in my years of schooling, enjoyed mathematics. The first lesson seemed to remind me of my primary and high school days and how much I actually hated being in a...
mathematics classroom, but after the third lesson, things seemed to shift in a more positive direction. The lecturer engages us, allows us to ask questions, gives us enough time to work on tasks and is understanding and accommodating. I have not only started to enjoy the activities but I look forward to being a teacher that implements some of these activities in my own classroom someday. This module is really important for the foundation phase teacher because we can take our ideas into the classroom.”

Appendix B: MRTEQ document

|   | A sound subject knowledge
|---|--------------------------|
| 2 | Know how to teach, sequence and pace content in accordance with subject knowledge and learner needs
| 3 | Know their learners and how they learn; understand their individual needs and tailor their learning accordingly
| 4 | Communicate effectively in general and in relation to their subjects, in order to mediate learning
| 5 | Highly developed literacy, numeracy and IT skills
| 6 | Knowledgeable about the school curriculum and be able to unpack its specialised content; use available resources appropriately; plan and design suitable learning programmes
| 7 | Understand diversity and include all learners; identify learning or social problems and work in partnership with professional service providers to address these.
| 8 | Manage classrooms effectively across diverse contexts in order to ensure a conducive learning environment.
| 9 | Assess learners in reliable and varied ways; use the results of assessment to improve teaching and learning
| 10 | A positive work ethic, display appropriate values and conduct themselves in a manner that befits, enhances and develops the teaching profession
|   | Reflect critically, in theoretically informed ways and in conjunction with their professional community of colleagues on their own practice in order to constantly improve it and adapt their practice to evolving circumstances. |
INTRODUCTION

In South Africa, debates in Mathematics specifically and in Mathematical Literacy generally have centred on achievement issues rather than issues linked to examining the nature of disciplines and discipline specific orientations. We pose the question: To what extent would mathematics teachers readily view tasks that have a mathematical orientation as mathematics tasks? And, to what extent would mathematical literacy (ML) teachers more readily view tasks that have a ML orientation as ML tasks? How would mathematics (M) teachers view tasks that have Mathematical Literacy as their dominant orientation and vice versa? These are key questions the broader study explored linked to obtaining teachers’ perspectives on the subjects they are teaching. In order to respond to these questions, mathematics and mathematical literacy teachers were provided with a combination of M and ML tasks to see how they would interpret them. This article reports on an analysis of teachers’ views on mathematics (M) and mathematical literacy (ML) tasks, in some of the selected schools in Gauteng, South Africa. Bernstein’s (1996) constructs (classification and framing) and Graven and Venkat’s (2007) conceptual framework of a spectrum of pedagogic agendas were used to analyse the National Statements for Mathematics and Mathematical Literacy curricula and teachers’ views on Mathematics and Mathematical Literacy.

Study design and tasks

We administered an activity consisting of four tasks. All four tasks had aspects of Mathematics and Mathematical Literacy embedded in them. The tasks were selected on the basis that the learners of both Mathematics and Mathematical Literacy would be able to solve the problems. The four tasks below were taken from previous national question papers of both learning areas. The first three tasks were from Grade 12 previous years mathematics national question papers, while the fourth task was from Grade 12 previous years mathematical literacy national question paper. We decided to have one task in Mathematical Literacy (Task 4) because it had two sub-questions.

The four tasks presented to learners were as follows:
1. Solve for \( x \)

\[
3x + \frac{1}{x} = 4
\]

2. Calculate the value of \( 1234567893 \times 1234567894 - 1234567895 \times 1234567892 \)

3. A sequence of squares, each having side 1, is drawn as shown below. The first square is shaded, and the length of the side of each shaded square is half the length of the side of the shaded square in the previous diagram.

3.1 Determine the area of the unshaded region in diagram 3

3.2 What is the sum of the areas of the unshaded regions on the first seven squares?

4. Thandi washes her dishes by hand three times daily in two identical basins. She uses one basin for washing the dishes and the other for rinsing the dishes. Each basin has a radius of 30 cm and a depth of 40 cm, as shown in the diagram below.

Thandi is considering buying a dishwasher that she will use to wash the dishes daily.

4.1 Calculate the volume of one cylindrical basin in \( cm^3 \)

4.2 Thandi fills each basin to half its capacity whenever she washes or rinses the dishes. Calculate how much water (in litres) she will use daily to wash and rinse the dishes by hand (250 \( cm^3 = 0.025 \ell \)).

4.3 A manufacturer of a dishwasher claims that their dishwasher uses \textbf{nine times} less water in comparison to washing the same number of dishes by hand. How much water would this dishwasher use to wash Thandi’s dishes daily? Is the claim of the manufacturer realistic? Justify your answer by giving a reason(s).
Four mathematics teachers from four schools (MTS1, MTS2, MTS3 and MTS4) and 4 mathematical literacy teachers (MLTS1, MLTS2, MLTS3 and MLTS4) were interviewed in order to obtain their views on the four tasks.

Findings

Table 1 below shows the Grand-totals of both M and ML teachers who labelled a task as an only M or ML or both M and ML.

**Table 1: Summary of M and ML teachers’ comments on Task 1 to 4**

<table>
<thead>
<tr>
<th></th>
<th>M only</th>
<th>ML only</th>
<th>Both M &amp; ML</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task 1</td>
<td>7 (3+4)</td>
<td>0</td>
<td>1 (1+0)</td>
</tr>
<tr>
<td>Task 2</td>
<td>0</td>
<td>2 (1+1)</td>
<td>5 (2+3)</td>
</tr>
<tr>
<td>Task 3</td>
<td>0</td>
<td>1 (0+1)</td>
<td>7 (4+3)</td>
</tr>
<tr>
<td>Task 4</td>
<td>0</td>
<td>4 (2+2)</td>
<td>4 (2+2)</td>
</tr>
</tbody>
</table>

Looking at Table 1 above, Task 3 appears to have attracted multiple interpretations from the most number of teachers across the Mathematics and Mathematical Literacy groups. Four (4) Mathematics teachers and three (3) Mathematical Literacy teachers viewed task 3 as both a Mathematics and a Mathematical Literacy task. However, this task was selected from a previous National Mathematics question paper. Similarly, Task 2 was selected from a previous National Mathematics question paper. However, 5 of the 8 teachers viewed the task as both a Mathematics and Mathematical Literacy task.

The findings revealed that there were no clear distinctions between M and ML tasks as seen by teachers who participated in the study. Tasks that were part of Mathematics were deemed to emanate from ML papers. Similarly, tasks that were from ML papers were viewed as M tasks. Therefore, we recommend that in order to minimise confusion there should be adequate training for ML teachers to establish what Mathematical Literacy is and how it differs from Mathematics.

**CONCLUSION**

This article focused on a selected group of teachers’ views and experiences of Mathematics and Mathematical Literacy and tasks linked to these learning areas in the South African curriculum. The work of Bernstein (1982, 1996, 2000) and Graven and Venkat’s (2007) conceptual framework of a spectrum of pedagogical agendas were used to analyse the NCS for Mathematics and Mathematical Literacy and discuss teachers’ views on tasks linked to Mathematics and Mathematical Literacy.

Teachers appeared to lack a deep knowledge and understanding of relationships between content and everyday context in the two subjects. Such a limited understanding
of relationships appears to have shaped the ways in which teachers viewed tasks presented to them.

Issues of balance and orientation are critical to the ways in which teachers view Mathematics and Mathematical Literacy. Mathematics and Mathematical Literacy teachers responded similarly in spite of their supposed dominant orientations, and that this occurred because all the teachers’ orientations had mathematics as their foundational frame, and that Mathematics and Mathematical Literacy are inseparable because both have mathematics as their originating frame.

REFERENCES


LEARNER ERRORS AND MISCONCEPTIONS ON RATIO AND PROPORTION

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Ratio and proportion are central to proportional reasoning. In turn, proportional reasoning is the cornerstone of mathematics learning. The paper, through a case study, identifies errors and misconceptions executed by a sample of 30 learners from a selected school in KwaZulu-Natal when solving problems on ratio and proportion. The paper also looks at strategies that learners used to solve the problems, to which identified errors and misconceptions could be attributed. It then suggests teaching approaches that teachers could use to teach ratio and proportion so that identified errors and misconceptions are averted.

INTRODUCTION

Hart (1988, p.198) points out that ratio and proportion have always been important topics in the school mathematics curriculum. Science, Geography and Art need knowledge and understanding of the concept of proportion. Chick & Harris (2007), Van de Walle (2007, p. 353) support this view when they also state that “ratio and proportion is central to the development of proportional reasoning and proportional reasoning is the cornerstone of algebra and a wide variety of topics in mathematics”.

Many researchers agree that the study of proportional reasoning is one of the most important perform areas of mathematics for everyday, workplace and scientific tasks. In fact, according to the Australian Department of Education and Early Childhood Development, (2009, p. 1), proportional reasoning “underlies much of the mathematics curriculum, including work on percentages, ratio, reading and making scales, reduction and enlargement, similar triangles, construction of pie charts, linear functions, trigonometry, etc”.

Performance of South African learners in mathematics is poor, and has been so for a long time. This is evident from our National Senior Certificate results and learner performance in studies such as Trends in International Mathematics and Science Studies (TIMSS) (Department of Basic Education, 2011). Learner errors and misconceptions on ratio, rate and proportion are likely to be among central causes of this performance. These are the arguments, among others, that inspired the study to explore selected Grade 9 learners’ errors and misconceptions on ratio, rate and proportion.
The questions that the research study sought to determine answers to are: How do learners in the selected school perform in assessment items based on ratio and proportion? What errors and misconceptions do these learners commit when they solve problems on ratio and proportion? Why do these learners commit the identified errors and misconceptions?

The theoretical framework of the study

This research drew on a constructivist perspective, within which learners are viewed as actively constructing their own mathematical understanding as they participate in practices and whilst interacting with others (Cobb, Jaworski, & Presmeg, 1996). The theory of constructivism posits that students learn by actively constructing their own knowledge; knowledge is created not passively received; and views learning as a social process (Clements & Battista, 1990; Jaworski, 1994). Furthermore all knowledge is seen to be constructed by individuals rather than transferred directly by an expert, such as a teacher, parent or book, to the learner.

This means that the learner plays the primary role in organising input from outside into meaningful knowledge (Clements & Battista, 1990). The role of prior knowledge is crucial in constructivism, and learners’ interpretation of tasks and instructional activities involving new concepts is filtered in terms of their prior knowledge (Smith, di Sessa & Roschelle, 1993).

It is a generally accepted adage that we learn through mistakes. This means that, if learners are made aware of the mistakes that they made, they are unlikely to repeat them. Lannin, Barker and Townsend (2007) describe errors from a quotation by Salvador Dali “as opportunities for deepening understanding”. Lannin et al.,( 2007) also describe errors “as important components of learning process”. They then argue that “the view of errors as a vehicle for learning, rather than an activity to eradicate, continues to gain momentum in mathematics education” (p. 44).

How do constructivist’s theorists view misconceptions? How do they define a misconception? Brodie and Berger (2010) cite the work of Confrey (1990), Nesher (1987), Smith et al. (1993) to explain that within a constructivist perspective, “a misconception is a conceptual structure, constructed by the learner, which makes sense in relation to her/his current knowledge, but which is not aligned with conventional mathematical knowledge” (p. 169).

Smith et al. (1993) have identified a field that they call “misconceptions research”, which has been largely concerned with identifying misconceptions in many science and mathematics domains. They claim that many assertions emanating from the field are
inconsistent with constructivism. For example, misconceptions research has seen the results of learners’ learning as flawed when learners’ exhibit a particular misconception. Smith et al., point out that constructivism “characterizes the process of learning as the gradual re-crafting of existing knowledge that, despite many intermediate difficulties is eventually successful” (p. 17). Thus a perception that views misconceptions as interfering with learning, needing to be confronted and replaced and resisting instruction (emanating from the misconceptions research field) is inconsistent with the constructivist view that learners build more advanced knowledge from prior learning. They argue that a constructivist theory of learning must interpret learners’ prior conceptions as resources for cognitive growth. Therefore, while this study draws from constructivism, the study also takes cognisance of misconceptions research.

LITERATURE REVIEW

Introduction

Long (2007) found that South African Grade 8 learners do not have the operational understanding of TIMMS items which fell into the domain ratio, proportion and percentage. Learners, thus, found it difficult to solve problems on ratio, rate and proportion. Hart (1988) attributes errors observed in learners when solving problems on ratio and rate to the incorrect strategies or incorrect use of correct strategies. The paragraphs that follow outline strategies observed in learners as they solve problems on ratio, rate and proportion and to establish some relationship between learner strategies and observed errors and misconceptions.

Learner Strategies

Misailidou and Williams (2002; 2003) conducted a study on strategies used to solve ratio problems among 232 learners from four schools in the North West of England, and nine trainee teachers of mathematics. The main purpose of the study was to contribute to teachers’ awareness of their pupils’ strategies and misconceptions in the field of ratio: a topic that is difficult to teach and learn in middle school years. Three correct strategies observed among learners and teachers were for every strategy or unit value method; the multiplicative strategy (within measure space approach); and the cross multiplication method. The incorrect strategies observed were constant sum strategy; the constant difference or additive strategy; the incomplete strategy; the incomplete application of build up method; and the magical doubling. The strategies are briefly explained in the paragraphs that follow.

The for every strategy

The for every strategy or unit value method entails finding the simplest ratio first, and then multiplying by a factor that yields the required result (Misailidou and Williams,
Simplest ratio in this case refers to comparing unit value of one quantity to the other quantity. Hart (1988) observed use of the strategy among English learners during the Concepts in Secondary Mathematics and Science (CSMS) and Strategies and Errors in Secondary Mathematics (SEMS) projects. She however, points out that the prevalence of the strategy could be attributed to the fact that most British schools teach the unitary method.

Md-Nor (1997) also identified the use of this strategy when she investigated learning and teaching of ratio and proportion in a group of 160 students and 5 teachers from 2 randomly selected Malaysian schools. The strategy was used by about 24% of the sampled learners. Chunlian (2008), also observed the use of the for every strategy in a study that he conducted among 1002 students from Singapore and 1070 students from China. He refers to this strategy as a unitary method.

The multiplicative strategy

The multiplicative strategy (within measure space approach) entails determining a ratio of measures from the same space and using it as a factor (Misailidou and Williams, 2002, 2003). For example, if second quantity is double the first one, then all quantities from the same measure space as the second quantity are obtained through doubling quantities from the measure space of the first quantity. Hart (1988) also identified a strategy similar to this one in her projects which she refers to as the within or between comparison strategy. She explains between comparison strategy as comparing measurements of the same unit to come up with a ratio. That ratio will then be used as a factor to multiply by. The within comparison strategy is comparison of quantities from different measure spaces. Md-Nor (1997) refers to this strategy as the scalar strategy because a multiplicative scalar relationship between the two quantities in the same unit is established.

Cross multiplication

The cross multiplication method is based on setting up a proportion. Hart (1984) and Olivier (1992a) refer to this strategy as the use of the formula, \( \frac{x}{a} = \frac{y}{b} \). Md-Nor (1997) identified the strategy among Malaysian children. She refers to it as the rule of 3 since it involves use of three known values to find the fourth value (the unknown). In her study, use of the strategy was observed among seven percent of the participants. Chunlian (2008) refers to this strategy as algebraic method. He says one or more unknowns are chosen as variables and the equation(s) is (are) set up.

The constant sum strategy

The constant sum strategy assumes that the total number of compared items should be the same. For example, if 3 cans : 6 cans = x cans : 7 cans, then 3 + 6 = x + 7. Thus 3 :
A strategy that is almost similar to this one is the constant difference or additive strategy. In this strategy, the relationship within the ratios is computed by subtracting one term from another, and then the difference is applied to the second ratio. The strategy is often just referred to as addition strategy.

This misconception is linked to” incorrect reasoning that results from viewing an enlargement as requiring an operation of addition and not multiplication” (Hart, 1984, p. 8). One important observation from the CSMS study was that the adders were very often able to correctly solve the easier items using this incorrect strategy. Hart (1988) refers to the strategy as the incorrect addition strategy and describes it as the addition of a fixed amount. In the CSMS study, the incorrect addition strategy was the most common misconception identified. Md-Nor (1997) identified this error among 20% of her participants.

The same strategy was used by 34.5% of the learners in one question in Misailidou and Williams (2002) study. Misailidou and Williams (2003) also identified the additive strategy to be the dominant erroneous strategy used by their sample of participants. The researchers also point out that the “additive strategy is the most commonly reported erroneous strategy in the research literature related to ratio and proportion” (Misailidou & Williams, 2003, p. 346). In their study, 53% of the participants used this strategy to solve the problem on Mr Short and Mr Tall, also used in this study.

Long (2007) also found that in the TIMSS, the significant error made by South African learners was the “use of additive reasoning where multiplicative reasoning was required”, and she attributes this to a developmental issue because “additive reasoning is the precursor of multiplicative reasoning” (p. 16).

The incomplete strategy
The incomplete strategy constitutes using the same number given for the measure space. For example, if a learner is given $2 : 3 = x : 6$, the learner thus concludes that $x = 2$. This strategy links to what Hart (1988) terms “naïve” strategies, which incorporates “intuition, guessing or using the data in an illogical way” (p. 201).

Long (2007) found that in the TIMSS some responses were incorrect because of “incomplete reasoning”. She says that learner “reasoning took them part way towards the answer” (p. 16). Brodie and Berger (2010) categorise this error as an “error of routine”, and they call it a “halting signal”.

The building-up strategy
Hart (1988) refers to the strategy as the “addition and scaling” strategy, because it involves a multiplicative strategy combined with an additive one. Incomplete
application of build up method refers to an incorrect application of the build up method. Md-Nor (1997, p. 34) describes the building-up strategy as establishing a relationship within a ratio and then extending it to the second ratio by addition.

**Magical doubling**

The last incorrect strategy observed was the magical doubling method. The method refers to doubling to obtain an answer when doubling is inappropriate since there are problems on ratio and proportion where doubling is a correct strategy. Hart (1988) incorporates this strategy under the category of using multiplication but not by the correct factor, which the study will refer to as the incorrect use of the multiplicative method.

**Other strategies**

Some studies attribute learner misconceptions of ratio and proportion to the teaching of this topic. Md-Nor (1997) investigated the relationship between teachers’ pedagogical content knowledge (PCK), instructional classroom practice and students’ learning; with a particular focus on the teaching and learning of ratio and proportion. Chick and Harris (2007) also undertook a case study on the role of PCK in the teaching of ratio. The focus of their study was on the actions of the teacher in the classroom and the examples the teacher used to illustrate ideas on ratio. The conclusion from both studies is that PCK is critical for the successful teaching and learning of ratio.

A study undertaken by The Gatsby Charitable Foundation on ratio and proportion identified learners fail to realise that 2 : 8 is the same ratio as 1 : 4. The study also observed that learners believe that increasing a map scale increases map distance. Lastly; the study found that learners generally use direct proportion instead of proportional division. The last misconception refers to situations where, if learners are told that 4 people do a job in two hours, they then conclude that 2 people will do the same job in one hour (Graham, 2003).

Chunlian (2008), also mentions arithmetic methods; model drawing methods; guess and check methods; looking for a pattern; and logical reasoning as other strategies he observed in his study. These strategies partly overlap with some of the strategies discussed above, and the researcher attributes the strategies to the teaching of various heuristics in Singapore.

**CONCLUSION**

Nesher (1987) also argues that “the notion of misconception denotes a line of thinking that causes a series of errors all resulting from an incorrect underlying premise, rather
than sporadic and non-systematic errors” (p. 35). The literature reviewed thus clearly illustrates that learner errors and misconceptions result from both learning and teaching. Spooner (2002) maintains that while an error may result from a misconception or other factors such as carelessness, misconceptions are a product of a lack of understanding. The above literature is highly relevant for the study because some questions used in the study were taken and adapted from the instruments used in these studies. The findings of the study may be similar to those of the studies discussed above.

**RESEARCH METHODOLOGY**

The case study is the methodology used for this research. Some of the purposes of a case study are “to portray, analyse and interpret the uniqueness of individuals and situations through accessible accounts, and to catch the complexity and situatedness of behaviour” (Cohen, Manion and Morrison, 2007, p. 85). Creswell (1998) gives the purpose of a case study as the “exploration of a bounded system, or case (or multiple cases), over time through detailed, in-depth data collection involving multiple sources of information rich of context” (p. 61). The research study also explored and then portrays the situation in the selected school. It identified the nature of evident errors, and underlying misconceptions so as to understand what misconceptions can be attributed to the observed errors.

**DATA COLLECTION AND ANALYSIS**

30 Grade 9 learners from a particular school were sampled for the study. The learners wrote a versioned Concepts in Secondary Mathematics and Science (CSMS) study test items (see Annexure). Five learners were thereafter interviewed.

For data analysis, items in the assessment tool were categorized into four cognitive levels, borrowed from Hart’s (1981) study, as shown in the table below.

<table>
<thead>
<tr>
<th>Level</th>
<th>Description</th>
<th>Test items</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>No rate needed or rate given. Multiplication by 2, 3 or halving</td>
<td>1(a), 1(b), 1(c), 1(d), 2(a)(i) and 2(a)(ii)</td>
</tr>
<tr>
<td>2</td>
<td>Rate easy to find or answer can be found by taking half an amount, then half as much again</td>
<td>2(b), 2(c), 2(d), 3 and 8</td>
</tr>
<tr>
<td>3</td>
<td>Rate must be found and is harder to find than above. Fraction operation also in this group</td>
<td>1(e), 2(e), 2(f)(i), 2(f)(ii) and 5</td>
</tr>
</tbody>
</table>
The framework used in the CSMS study (Hart, 1981) was also adopted and adapted for the analysis of learner performance in the test.

**FINDINGS OF THE STUDY**

Learner performance indicates that they struggled to solve problems on ratio and proportion. 2 participants performed at level 0. This means that there were learners that could not solve elementary problems, in which rate was given. The learners struggled with problems that required multiplication by 2, 3 or halving. 13 learners performed at level 3 or above. This also means that less than half of the participants could correctly solve problems in which rate was not given and the rate was not easy to find. Only 3 learners could solve level 4 problems, that is, problems that required them to recognize that a ratio was needed to get a solution.

Learners apparently had a deficient ratio sense. For example, they were given a recipe for making soup for 8 people. In the recipe, 2 pints of water are needed to make the soup for 8 people. They then had to determine the amount of water needed to make soup for 4 people. 3 leaners could not recognize that to make soup for fewer people one needs less water. Here is an example of a learner response to Question 1(a).

Example 1: Learner response

What is obvious is that this learner did not ask himself or herself the crucial question: Is more or less water needed to make soup for four people? Also evident from the above response is incorrect conceptualizations of cross multiplication. For this learner, cross multiplication is perceived as $\text{numerator} \times \text{numerator} = \text{denominator} \times \text{denominator}$. The error was observed across research tool items.

What was also often apparent was the learners’ perception of a ratio as a fraction. Of course a ratio and a fraction are related, but subtly distinct from one another (Long, 2009). Incorrect conceptualization of ratios was also observed in the incorrect answers to Question 3. Solving problems similar to this one requires learners to know

| 4 | Must recognize that ratio is needed; the questions are complex in either numbers needed or setting | 4, 6(a), 6(b), 7(a) and 7(b) |

Table 1: Level descriptors
partitioning. That is, unlike a fraction, a ratio may be comparison of part to part. Hereunder is a response from one of the participants.

Example 2: Learner response

Evident from the calculation is that the learners worked from an incorrect premise, that one of the people was paid R350. For example, in the calculation next to the name Nkosi, the first part indicates that Nkosi got R350 for working for 12 hours \((12 \rightarrow R350)\). The learner then worked out what amount Rajen should get, and arrived at R233.33. In the latter part, Rajen was now paid R350 for working for 8 hours, hence Nkosi would get R525. Learners seemed to be trying to create a situation in which they could use the formula \(x/a = y/b\), although this was an inappropriate strategy.

The study also found that some learner responses were incorrect because of “incomplete reasoning”. Learner “reasoning took them part way towards the answer” ((Long, 2007, p. 16). Brodie and Berger (2010) categorise this error as an “error of routine”, and they call it a “halting signal”. For some learners, Questions 1(d) to 1(e) required them to halve (to obtain ingredients for four people); halve again (obtain ingredients for two people) and then add the result to first halving to the result obtained after halving for the second time to get the ingredients required for 6 people. This strategy is known as the build-up strategy. For example, to determine the amount of water needed to make soup for six people, learners had to halve 2 pints of water (water needed for to make soup for eight people) to get 1 pint (amount of water needed to make soup for four people). They then had to halve 1 pint of water again to get 1/2 pint (amount of water needed to make soup for two people). The amount of water needed to make soup for six people is then 1 pint of water + 1/2 pint of water = 1½ pints of water. In Question 1(c), for example, 2 learners either halved once to get 1, or halved twice to get 1/2, Learners that halved twice did not add the first answer to the second answer. The error resulted from an incomplete build-up strategy.

The study also found that quite an extensive proportion of learners also applied a strategy Misailidu and William (2002, 2003) refer to as an incomplete strategy (using the same number given for the measure space). For example, in Question 1(e) it is stated that 1/2 pint of cream to make soup for eight people. 4 learners said that 1/2 pint of cream was needed to make soup for six people too. In Question 6(a) learners were given ratios mercury : copper = 1 : 5 and tin : copper = 3 : 10 and requested to determine
ratio mercury : tin. The most popular response to this question was 1 : 3. This solution constituted a third of the responses to the question.

Use of *additive strategy* was observed in quite a number of questions. This is an indication that learners do not perceive a ratio as a multiplicative relationship. In Question 4, 7cm was outstandingly the most frequently obtained incorrect result. This incorrect response mainly reflects that learners did not see enlargement of 6cm to 9cm as multiplication by 1½. They see the enlargement as addition of 3cm to 6cm, hence, they have increased the length of 4cm to 7cm by 3cm. The *additive strategy* was used by 11 learners to arrive at an incorrect answer of 7cm. The strategy is sometimes referred to as the *constant difference strategy*. This error was identified in 14 learner responses to Question 5 (famous Mr Short and Mr Tall question) and more than 15 learners in Question 7(a).

What the findings mean for mathematics teaching and learning

The findings of the study regarding strategies used by learners do not differ from those of the other studies discussed in the literature review. Learner performance was also poor as was the case with the TIMSS and SACMEQ. Although the study does not claim that the findings are applicable to all Grade 9 learners in South Africa, there will be commonalities between this group and others from similar backgrounds. Therefore, the recommendations made below could be relevant for mathematics teaching and learning.

Everyday contexts that require intuitive applications of ratio and proportion should be used to teach ratio, rate and proportion. The recipe for preparing soup is a good example of such contexts. In the use of such contexts, learners should be encouraged to think critically through situations by asking themselves probing questions. For example, am I making soup for more people? Do I need more or less of an ingredient? Learners should be able to assess the reasonableness of their answers. A strong use of problem solving in the teaching of ratio, rate and proportion is thus proposed. The South African schools’ curriculum strongly prescribes problem solving in the teaching of mathematics.

Participants in this study liked cross multiplication. Some used it correctly and others did not. It is likely that it was the only strategy that they were exposed to. There is nothing wrong with cross multiplication. The study recommends that cross multiplication be not just taught as a meaningless rule that learners should use. Learners should know what a proportion is. They should know the Law of proportions and how it relates to cross multiplication. The study also recommends that learners be not restricted to a single strategy. They need to be exposed to a variety of strategies that they could use to solve problems on ratio and proportion.
Some participants demonstrated a weak understanding of the concept ratio. They seem to regard a ratio as a fraction. Long (2009) cautions us about the subtle difference between the two concepts. We therefore need to establish a clear relationship between a fraction and a ratio. We should make learners aware that while a fraction is limited to comparison of part(s) to the whole, a ratio does compare parts to parts. Beyond just comparison of parts, we need to emphasise the significance of the multiplicative relationship in a ratio. Teaching of ratio and proportion should ensure that these important facts are correctly conceptualised by learners. Enlargements provide good contexts for establishing this relationship.

CONCLUSION

Teachers should note that ratio and proportion are not the only cornerstones for mathematics learning and teaching. Errors and misconceptions should be identified in all key topics. The identified errors and misconceptions should be used as pillars for intervention. They should be scrutinised and ameliorated as early as possible to ensure that meaningful learning takes place. In fact all teaching should ensure that pre identified misconceptions are totally quarantined.

REFERENCES


Department of Education and Early Childhood Development. (2009). About Proportional Reasoning [Electronic Version], from


**Annexure: Research questions**

1. Onion soup recipe for 8 people
   
   8 onions 2 pints of water 4 cubes of chicken soup
   2 spoons of butter ½ pint of cream
I am cooking onion soup for 4 people.
(a) How much water do I need?
(b) How many cubes of chicken soup do I need?

I am cooking onion soup for 6 people.
(c) How much water do I need?
(d) How many cubes of chicken soup do I need?
(e) How much cream do I need?

2. Eels A, B and C are fed sprats according to their length.

A  5 cm long
B  10 cm
C  15 cm

(a) If eel A is fed 2 sprats,
   (i) How many sprats should eel B be fed?
   ________________________________________________________________

(ii) How many sprats should eel C be fed?
   ________________________________________________________________

(b) If eel B gets 12 sprats, how many sprats should eel C be fed?
   ________________________________________________________________

(c) If eel C gets 9 sprats, how many sprats should eel B get?
   ________________________________________________________________

Three other eels X, Y and Z are fed fishfingers. The mass of the fishfinger depends on the length of the eel.

X  10 cm
Y  15 cm
Z  25 cm
(d) If eel X gets 2 grams of fishfingers, how much fishfingers should be given to eel Z?

______________________________________________________________

(e) If eel Y gets 9 grams of fishfingers, how much fishfingers should be given to eel Z?

______________________________________________________________

(f) If eel Z gets 10g of fishfingers,

(i) how much should eel X get?

_________________________________________________________

(ii) how much should eel Y get?

_________________________________________________________

3. Nkosi and Rajen work on a job together.
Nkosi works for 12 hours.
Rajen for 8 hours.
How should they share the payment of R350 for the job?
Nkosi:_________________________________________________
Rajen:_________________________________________________

4. The rectangle below has a 6 cm base.

The rectangle is enlarged so that it keeps the shape (see diagram below).

The base of the enlarged rectangle is 9 cm.
What is the length of the other side?

5. Below, we are shown the height of Mr Short measured in paper-clips. Mr Short has a friend called Mr Tall.

Mr Short’s height is 4 matchsticks.
Mr Tall’s height is 6 matchsticks.
What is Mr Tall’s height in paper-clips?

6. In a particular metal alloy (mixture of metals):
   mercury : copper = 1 : 5,
   tin : copper = 3 : 10, and
   zinc : copper = 8 : 15.
Complete by filling in the missing numbers.
   (a) mercury : tin = _______________.
   (b) zinc : tin = _______________

7. The letters below are the same shape.
   One is larger than the other.
   The curve AC is 8 units.
   The curve RT is 12 units.
   The curve AB is 9 units. How long is the curve RS?
The curve UV is 18 units. How long is the curve DE?

8. On the 24 May 2010, $1 (1 US Dollar) could be exchanged for approximately R8. If I have R3 200, how much money do I have in US Dollars?
The exploratory study using descriptive method established educators and learners views about environmental variables and factors that contribute to poor performance in mathematics and science in rural Secondary Schools. Participants were purposefully selected from five secondary schools in the Waterberg school district, Lephalale in Limpopo with poor results. Data collection involved focus group interviews with four Grade 8 learners, three educators, and two parents from each school. In addition, a questionnaire was used to collect data from Grade 8 classes from participating schools. The study revealed that learning environments, teachers’ method of teaching and parental socio-economic status mediated on learners’ academic performance. Results identified two factors. The first identified to have a direct influence related to content knowledge, motivation, teaching strategies, laboratory use, and pile of administrative task which results to non-completion of the syllabus in a year. The second factor, was attributed to the parental culture, parents socio economic status, trend of youth culture, in general.

INTRODUCTION

Mathematics and science is not a compulsory school subject for all learners in South African secondary schools except learners enrolled for the two subjects. A category, conceptual understanding of mathematics and science is a key component and basic for modern scientific development and technology (De la Fuente, 2002; Department of Basic Education, DBE, 2010; Smith, 2011). Education stakeholders in South Africa, continue to invest heavily in the education of mathematics and science with the hope that the input will be equivalent to the output if not better. However, the performance continues to be poor in general. Of great concern is the learners’ performance in mathematics and science, given that these subjects are key to the attainment of the national goal of industrialization by the year 2030. This poor performance in mathematics and science was also established in rural schools of two circuits in Lephalale district.

This study aims examined the influence of three environmental variables namely, learning environment, educators’ instructional method and family condition of learners on academic performance in rural secondary schools.
LITERATURE REVIEW

Recent studies reported that poor academic performance in mathematics and science is of the result of outdated teaching practices, under qualified educators and lack of basic content knowledge have resulted into poor class instructional methods (e.g. Makgato & Mji, 2006). Tachie and Chireshe’s (2013) study revealed that learner poor performance is mainly affected by external factors such as lack of human and material resources, poor teachers, poor teaching methods and bad teacher behaviour. Some of the learners attributed to their failure in mathematics examinations to internal factors like laziness, lack of interest and absenteeism. Mathematics is important in daily lives since it deals with real life situations in daily activities (Ojose, 2011). Despite the fact that mathematics is important in daily lives, learners continue to fail the subject (Feza-Piyose, 2012). South African learners are generally not performing well in mathematics, the situation is even worse among blacks. The five schools in this study perform badly in mathematics and science, the low performance in mathematics and science prompted my curiosity to focus on the environmental variables indicated earlier if truly play a role in learner academic achievement.

For the country’s level of development and the proportion of the budget that is spent on education, South Africa performs substantially below what is expected of it in terms of education performance (National Education Evaluation and Development Unit, NEEDU, 2012, p. 13). Learner performance in this country is markedly worse as compared to other countries which are poorer than South Africa (Armstrong, 2014). The third round of international tests conducted by Southern Africa Consortium for the Measurement of Educational Quality (SACMEQ) in 2007, South Africa performed worse than most countries in both mathematics and language (Spaull, 2011) and took position 10 out of 15 participated countries.

Teacher quality and low levels of teacher efforts are cited as major drivers of South Africa’s education crisis (e.g. NEEDU, 2012). NEEDU further argued that teachers are unable to ensure high quality for learners. Armstrong (2014) concurs that poor performance is the results of lack of discipline amongst staff members and any remedial support to change the behaviour. NEEDU (2014) suggested that intervention has to be employed to improve and enhance the knowledge base of teachers to equip them with skills necessary to ensure the quality teaching and learning in the classroom.

Opportunities to learn vary greatly in South Africa (Branson & Zuze, 2014). They said that only few learners from under privileged families get a better education to enter the top income jobs. Learners from rich families have a greater range of schools to choose when compared to learners of the poor. Schools from the rich families have a low learner-teacher ratio and are more likely to be higher quintile schools (Branson, Lam &
Those learners also have an opportunity to attend schools that are further from their homes (Branson, Lam & Nel, 2014). This implies that these learners from rich families have a greater range of schools to choose from and have opportunities for educational support as compared to the learners of the poor.

Learners who live in poor rural areas have a limited educational support outside of school (Consortium for Research on Educational Acess, Transitions and Equity, 2011). They don’t have enough access to reading materials in their homes and often live in communities without public library facilities (Branson & Zuze, 2014). The parents are less likely to provide assistance with homework and their living conditions make studying difficult. Furthermore, some learners are expected to assist with domestic chores before and after school. They might even have responsibilities of taking care of sick sibling and elders. These additional responsibilities reduce the time that they can dedicate to their studies (Branson & Zuze, 2014). In spite of all these setbacks, they must remain in school and perform well enough at competitive school-leaving examinations to earn the right passage to the better way of life.

The organisational and organisational conditions in rich and poor schools vary considerably (Branson & Zuze, 2014). Schools in the top quintiles have additional funds to employ more or better trained staff members because of the additional revenue they raise through school fees. However, increasing public funding to poor schools does not guarantee that available resources will be managed effectively. Organisational characteristics such as curriculum planning, regular learner assessment and high teacher attendance have been linked to better academic results. Many of these indicators of efficient management are lacking in poor schools (Taylor, van der Berg, Reddy & Janse Van Renburg, 2011).

RESEARCH METHODOLOGY

This is the first phase of the bigger project that would be conducted in Lephalale district. The project intends to use compujector as an intervention strategy to improve the learner performance in mathematics and science. This study followed exploratory research design in which data collected was descriptive in nature and not discussed statistically (Rule & John, 2011). The study was qualitative in nature in which the environmental factors and factors affecting the poor performance of learners in mathematics and science were explored.

Sampling strategy

Data were collected from 5 rural secondary schools in the two circuits of Lephalale district in Limpopo Province. The sample was purposive (Creswell, 2013; Johnson & Christensen, 2013) in that the goal was to explore the factors that affect poor learner
performance and environmental factors in the 5 secondary schools. The sample included 20 Grade 8 learners in which 4 focus groups of learners were sampled from each school, 15 grade 8 teachers in which 3 were sampled from each school and 10 parents in which 2 were sampled from each school to respond to the following questions: How do secondary school learners perceive their learning environment as an influence on their academic performance? How do secondary school learners perceive teachers’ teaching methods as influencing learners’ academic performance? This study also used questionnaires to get data that can supplement the focus groups data in order to triangulate the two sets of data during analysis.

**Ethical issues**

The permission letter to conduct this project was submitted to Department of basic Education, the 2 circuit managers, 15 teachers and 20 learners. The letter entailed the application for grade 8 learners to participate in the study. Learners were informed about the study, reasons and purpose of the research also discussed with them. The letter also discussed and defined informed consent, the right to withdraw without being prejudice and confidentiality as advised by (Bokdan & Biklen, 2003). The schools, teachers, parents and learners’ names were used as pseudonyms to protect their identity (Johnson & Christensen, 2013).

**FINDINGS**

**Learners’ behaviour in studying mathematics**

From the findings of this study, several learners observed that distraction from peers is common a potent weapon that hamper many learners’ academic performance. They say some learners don’t respect teaching lessons and in some cases disrupt the lessons; they make noise and some back-chat. This can result in negative modelling of peers who are not attentive and do not focus in learning activities in the class. Some of the learners leave classroom lessons when teaching and learning without permission to do so.

**Learning environment**

A teacher revealed that the chalkboard is old and it is difficult to give notes, some of the learners also supported that the chalkboard is invisible as is dilapidated. Other learners mentioned that the roof of their classroom is open with some windows broken resulting into high absenteeism during rainy and cold days. In this case both rain and cold weather will affect their comfort in the classroom and equally their academic performance will be affected. A student from school A observed that their school is located close to the road and the bus stop. Learners indicated that noise form passing motorists affect their learning as is distractive to them. The volume of traffic that passes
through the roads may be responsible for the noise. Ideally, schools are best located in serene environment away from the busy road in order to have calm and peaceful learning classroom. These five significant variables observed in learning environment, in this study can inhibit learners’ academic performance. This finding is in agreement with Isangedighi (1998), Danesty (2004), that learning environment significantly affects learners’ poor academic performance. It is possible that desk, un repaired windows, poor chalkboard and bad roof of the classroom were not considered as important variables by the respondents that could affect their learning.

**Family conditions of learners**

From the finding of this study, out of the ten variables examined under family conditions, only five (5) were perceived by the respondents to have effect on their academic performance. On the lack of textbooks, 34% of the respondents perceive it as militating factors for their learning. Also, 46% of the respondents perceived non – availability of a laboratory to perform science and biology experiments as a factor that could affect academic performance. Migrating of mathematics and science teachers to the cities was perceived by 45% as a factor that can affect the academic performance. Trekking to school was perceived by 41% of the respondents as a factor that can affect their academic performance. Truly, if the distance to a learner’s school is far, the child ought to use any means of transport that will help to prevent the tear and wear of the body that will be caused by trekking. To achieve this purpose, learners must have the financial support from their parents. The problem of food was perceived by 33% of the respondents as a factor that could hinder academic performance. Taking a lunch box to school is very important. Glucose derived from food is an essential ingredient that any child’s brain needs to function at optimal level (Bee 2000). It serves as primary fuel for the brain. Lastly, 47% of the respondents perceived lack of parental education as anathema to their performance in school. Several studies show that children from poor families or families where the parents are relatively uneducated and unemployed have lower academic performance than those from middle – class families (e.g. Bee 2000). Poverty has a significant effect on children’s academic performance over and above what the parents own genes may have contributed (Bee 2000). This finding is expected because learners who come from a high socio-economic background will perform better academically compared to their counterparts from low socio – economic performance (Bransen & Zuze, 2014). This is because an affluent home is a better environment for better academic performance. Parents with high socio-economic status are in a position to give more parental help and motivation to their children than those of low-economic status.

Children from high socio-economic background have greater access to learning resources at home such as extra lesson, computers, toys and textbook (Taylor et al.,
In addition, they provide good shelter, food and healthy environment. Such positive conditions may affect their physical and mental development.

This result is in consonance with previous findings of Morokinyo (2003) and Rohana et al (2009). It is pertinent to support this finding in that we have cogent reasons to show that material life of the middle and upper class homes are secure and rich. These parents have appropriate knowledge having gone through schools themselves know the kind of stimulating experiences to provide for their children. Children coming from better home will have advantage in learning as a result of opportunities provided for them (Branson, Lam & Nel, 2014). These parents see education in terms of self-improvement, character building, cognitive development and total personality development. There is no denying the fact that a learner is helped by the available resources for bringing desirable changes in his /her behaviour. How effectively such changes will take place in his/her behaviour depend on the quality and management of these resources. Therefore, availability of appropriate learning materials and facilities like reaching-learning aids, textbooks, library, calm and peaceful environment are indispensable elements that will enhance learners academic performance. From the lenses of educational sociologists, learning depends upon primary socialization homes which provide conducive learning environment and socio-emotional climate available in the school system. These are the variables that constitutes quality environment for learning.

CONCLUSIVE REMARKS

This is a continuing community based practice study which took place in two circuits of Lephalale district in Limpopo Province. The study involved five secondary schools and a number of factors contributing towards poor academic achievement of learners were revealed in that research process. Drawn from the data collected from the five schools used focused groups and questionnaires, the compujector was recommended to be used as an intervention in order to improve the learners’ academic achievement in mathematics and science. The compujector was purchases by the project facilitators and housed in circuit offices. Schools collect them to use them as a resource for teaching mathematics and science. A compujector is an ICT tool designed to be used in the classroom for teaching and learning. It consists of learning soft wares that could be used across the grades. The use of compujector still has to be observed if there will be benefits out of its use and how often and effectively teachers use it to teach learners.

REFERENCES


Many reports have been written on the gap that exists between education policy intent and its implementation particularly with respect to mathematics. The perennial message is that despite the good mathematics curriculum, many students fail mathematics because of problems in teaching the curriculum. This paper critiques the South African Teacher Education policy and advocates that the quality mediation of the mathematics curriculum begins with good mathematics lesson preparation during the teaching practicum of student teachers during Initial Teacher Education. In particular this paper suggests that quality mathematics lesson preparation through Tyler’s objective curriculum model goes a long way in enhancing not only mathematics teaching practicum but quality mathematics teaching in the country at large.

INTRODUCTION

This paper discusses teacher education policy in South Africa (see The National Framework for Teacher Education and Development in South Africa (NFTEDSA), 2006). In relation to this policy, the paper makes suggestions for enhancing quality Initial Teacher Education (ITE) through pre-service students’ teaching practicum. The Ministerial Committee on Teacher Education (MCTE) report (Department of Education, 2004) states “there is, in general, a lack of tradition of effective supervision and mentoring of novice teachers” (p. 12). In my opinion, this is too important an area of teacher formation that must be addressed.

I advocate the crucial role of pre-service students’ teaching practicum in ensuring mathematics teacher quality in South Africa. In doing so, I make some comparisons with initial teacher education teaching practicum in Zimbabwe from which South Africa obtains some of its mathematics (and science) teachers.

Teacher Education Policy in South Africa

In general, South Africa’s teacher education policy aims to counter the effects of the divisive past. The policy aims to enhance social equality and social equity via education and mathematics education in particular. The policy is drawn with the belief that:

‘Education … beyond all other devices of human origin, is the great equalizer of the conditions of men, the balance-wheel of the social
machinery… it gives each man the independence and … it prevents a person being poor’. (Mann, 1848)

Also as Mandela is often quoted as saying; "Education is the most powerful weapon which you can use to change the world." The South Africa Report of the Ministerial Committee on Teacher Education (MCTE) (2006) states that one of its overarching aims is the provision of competent teachers for all learners in the country.

The great importance teaching practice has on social transformation through education was emphasised. NFTEDSA reported that;

“… the practice of teaching is the centre of gravity of the whole education system … any other parts of the system can be justified only to the extent that they enable teaching to flourish… recognising that policy alone will not realise our transformation goals, and an acceptance that 'deep change' of teachers' practices (rather than superficial compliance, mimicry, or merely rhetorical acceptance) is a long-term enterprise” (p.4)

Participants in NFTEDSA (2006) decried that Higher Education Institutions (HEI) seemed not adequately prepare teachers for the tasks of school teaching.

The policy recognised the need to guide and equip teachers to do be able to implement their duties effectively. According to NFTEDSA (2006):

“The overriding aim of the policy is to properly equip teachers to undertake their essential and demanding tasks, to enable them to continually enhance their professional competence and performance, and to raise the esteem in which they are held by the people of South Africa.” (p.4)

I argue that one way to equip teachers is to ensure that they are properly trained during the Initial Professional Teacher education stage particularly on issue of teaching practicum.

Among the norms and standards of the policy Department of Education (DoE) (2000), it is argued that during their training, pre-service student teachers must develop into specialists, ‘in a particular subject or phase and … in teaching and learning’ (p.5). The policy is drawn on the belief that teachers being the curriculum implementers play a central role in the education system. It is recognised that teachers’ roles are of strategic importance not only for the young people’s holistic development (DBE, 2005) but for the nation as well.
NFTEDSA (2006) reported that ITE has two facets, formal education at Higher Education Institutions (HEI) as well as on-site training in the form of supervised teaching practice. Formal education during ITE includes studies on theory of education, curriculum, pedagogy, assessment, specialist teaching subject matter knowledge among other areas. All these are made to bear in the classroom to support the teacher to teach effectively and efficiently.

**CONCEPTUAL FRAMEWORK**

In designing the curriculum, education theorists, try to answer the questions, what, why and how? In doing so, theorists have proposed various curriculum models (see Tyler, 1949; Taba, 1962; Wheeler, 1967; Skilbeck, 1984; Morrison; Nolet & McLaughlin, 2000). In his seminal work, Tyler produced the Objective Model which will be the main lens in this paper. In Zimbabwe, the Tyler model is used as a framework for lesson preparation by both pre-service and in-service teachers in their day to day lessons throughout the country.

Taba proposed the “interactive model” of the curriculum. In particular Taba emphasized the importance of selecting appropriate learning experiences for learners. But what is curriculum? According to Tanner and Tanner (1980), curriculum is “the planned and guided experiences and intended learning outcomes, formulated through the systematic reconstruction of knowledge and experiences, under the auspices of the school, for the learners’ continuous and willful growth in personal social competence” (p.23).

In Zimbabwe more than any other curriculum model, Tyler’s objective model of the curriculum underlies curriculum processes including in initial teacher education. Tyler (1949) argues that it is crucial to first of all define appropriate learning objectives, which are specific, time oriented, achievable, observable and measurable. Then it was crucial to select subject matter content that enables achievement of those learning objectives. Then the pedagogy (Shulman, 1986, 2004) to teach that content was to be clarified. Lastly was the assessment, formative and summative to monitor and measure how far and how much the objectives are being achieved or have been achieved. He argued that evaluating the curriculum helps to revise those aspects of the curriculum that are not effective.

**The pre-service teachers’ mathematics curriculum: Subject matter knowledge**

A major precondition to ensure quality teaching of a subject such as mathematics is the subject matter knowledge that the teacher possesses (Shulman, 1996; Ball, Thames & Phelps, 2008). The National Mathematics Advisory Panel (NMAP) (2008); argues,
research on the relationship between teachers’ mathematical knowledge and students’ achievement confirms the importance of teachers’ content knowledge. It is self-evident that teachers cannot teach what they do not know. Teaching well requires substantial knowledge and skill’ (p.21).

Notwithstanding the importance of teacher mathematics subject matter knowledge singled out above, I argue that learning how to mediate that knowledge is critical for student teachers at the ITE phase. The best place for students to learn to teach mathematics is during their supervised teaching practicum. It is during teaching practicum that student teachers develop a repertoire of pedagogical content knowledge (PCK) to teach mathematics effectively.

Teaching, Teaching practicum and Pedagogical Content knowledge

Teaching practicum is at the centre of the knowledge mediation matrix. This teaching practicum is learnt during teaching practice. What is teacher education? In general, teacher education concerns equipping prospective teachers with curricula knowledge as well as how to effectively teach that curricula to learners (Ashby, Hobson, Tracey, Malderez, Tomlinson, Roper, Chambers, & Healy, 2008). As DBE (2005) documented, teacher education is a ‘form of professional education that has as its defining purpose to improve the professional practice of teachers’ (p.7).

Teaching is inseparable to learning (DBE, 2005). Consider John Dewey (1933)’s comparison between a merchant and a teacher:

Teaching may be compared to selling commodities. No one can sell unless someone buys. We should ridicule a merchant who said that he had sold a great many goods although no one had bought any. But perhaps there are teachers who think that they have done a good day’s teaching irrespective of what pupils have learned. There is the same exact equation between teaching and learning that there is between selling and buying (Dewey, 1933, pp. 35–36).

Despite the wise words of Dewey, (DBE, 2005, p.12) reported that ‘there is, in general, a lack of a tradition of effective … mentoring of novice teachers’. Also recommended was extensive teaching practice in which students would spend the whole year of formal induction during which the novice teacher would be employed and paid a salary. This was seen as a way of killing two birds with one stone; further on the job training for teachers as well simultaneous reducing the problem of teacher shortages in the country.
Thus it is universally agreed that teaching practice is a key component of an ITE course (DBE, 2005; Ashby, Hobson, Tracey, Malderez, Tomlinson, Roper, Chambers, & Healy, 2008; Smith, 1999, 2011). Teaching practice concerns a period whereby a student teacher under training is temporarily teaching at a school under the supervision of experienced practitioners, such as a college tutor or teacher. During teaching practice, student teachers learn firsthand how to teach. At this stage they apply what they have learnt in a real classroom situation. Teaching practice is enhanced by PCK (Shulman, 1996) and mathematical pedagogical content knowledge (Ball, Hill & Bass 2005; Ball, Thames & Phelps, 2008). According to Shulman (1986), PCK involves “the ways of representing and formulating the subject that make it comprehensible to others” (p. 9).

Wits Education Policy (WEP) (2008) have commented that while good teaching practice exists, this often occurs by chance when student teachers find themselves in favourable circumstances. It is reported that in general, links with teaching practice schools in general are not well structured. HEI do not have resources to reward the schools that mentor their students. Faculty visits students two or three times to give professional support but this is seen as episodic and may prove insufficient. Thus the student’s growth to a great extend depends on the conditions at the school they practice. WEP reports on some negative experiences of students that hardly provide them with the models needed for their professional growth.

I argue that student teachers and practicing teachers prepare for every lesson that they teach using the Tyler model. In the Zimbabwe case of Initial Teacher Education, one of the important aspects that college lecturers look for in every lesson is the lesson objective. The lesson objective must be time specific, achievable, measurable and must be stated using action verbs like construct, draw, summarise, compare, contrast, work out, list et cetera.

**Lesson Preparation and Pedagogical Content knowledge**

In many education systems, for example in Zimbabwe, a lesson plan is a mandatory for every teacher to implement the curriculum at classroom level. Good lesson preparation is viewed as evidence for quality teaching. O’Bannon (2008) defines a lesson plan as a document in which a teacher describes in detail the stages of delivering a lesson scheduled on a school time-table. The date the lesson will be taught, the time it will be taught and the class that will be taught must be clearly shown.

In a lesson guided by the Tyler model the rallying point of the lesson plan is the lesson objective/s. The teacher must clearly specify the behavioral objective/s of the lesson. It is considered key for the teacher to be able to assess whether a learner has achieved the
lesson objective/s or at the end of the lesson. In most cases the objectives are prefaced by; by the end of the lesson learners must be able to… For example; by the end of the lesson learners must be able to solve simple linear equations like $3x + 4 = 20$ or; by the end of the lesson, using a pencil, ruler and compass only, learners must be able to bisect a given angle. (Notice that the verbs used in stating objectives are action verbs. Thus it is considered improper to write the objective in this way; learners must know how to solve a simple linear equation because knowing is not specific, and also the objective is not time bound)

Then the Source of Matter (SOM) stipulating where the teaching and learning material is obtained must be stated clearly, usually these are mostly textbooks. The Source of Matter must show the name of the source/s, chapter/s and page numbers. The teaching approaching to be used in the pursuit of objectives are also suggested as well as learning activities. Lastly there is a section for lesson evaluation in which the teacher looks back on the lesson to reflect on the lesson’s strengths and weaknesses. The teacher records what went well in the lesson and what did not go so well. The idea is that the teacher can work on areas of improvement so that they can improve their teaching next time. In particular, the teacher must build on their demonstrated strengths in the future.

While some teachers believe that they know what they want to teach and see lesson planning as time wasting, I strongly disagree with that position. In lesson planning the teachers pre-lives the lesson. That way the teacher anticipates the difficulties learners may have in learning particular learning material. So the teacher is fore-armed and can devise strategies of addressing learner difficulties. Also, the teacher has time for reflecting how they may deliver particular subject matter so that it is understandable to learners, giving them space to reflect on pedagogical content knowledge.

In the Zimbabwean educational scene, it is regarded as a negligence of duty to teach any lesson without a lesson plan. Of course if a student teacher teaches without a lesson plan then he/she knows that he/she has failed. It is argued that if a lesson plan is well written any other conversant teacher can take that lesson and teach it without many problems should the teacher who prepared it fails to deliver that lesson for any reason. Indeed it is argued that should a supervisor visit a student teacher and for some reason the student teacher cannot teach that lesson then the student can be assessed on the basis of their documents alone including lesson plans.

Despite the advantages of lesson planning I have outlined above, many practising teachers and student teachers detest it. Yet a lesson plan guides management of the teaching and learning environment.
CONCLUSIONS

It seems self-evident that one of the most effective ways for successful teaching practicum and teaching mathematics is founded on good lesson preparation. Notwithstanding the format, all mathematics teachers need to think deeply about the lessons they teach. They need to make informed decisions about how they will present the learners the material that they want them to learn. They need to map-out well thought out approaches they will use to help students master subject material. Preparing mathematics lessons in detailed is good evidence of teacher organisation which can only be rewarded by students’ success in learning mathematics. The practice of conscientious and thorough mathematics lesson planning need to be cultivated during student teachers’ ITE years. It must become a life-long habit to all mathematics teachers to effect good mathematics teaching and learning. That way equity in mathematics learning and achievement could be achieved as the South African government desires.

REFERENCES


Mann, H. (1848). 12th Annual Report to the Massachusetts State Board of Education.


In this paper, the experiences of secondary school B.Ed. (in-service) mathematics teachers were explored. The intention was to find out in particular what opportunities and challenges secondary mathematics teachers are faced with regarding facilitating mathematics using a problem solving approach while participating in a hybrid distance learning model. The focus of the study was on four teachers who are in their third and final year of the B.Ed. (in-service) programme. The study made use of a questionnaire, semi-structured interviews and classroom lesson observations. Findings suggest that all four teachers have the knowledge of the theoretical aspects of a problem solving approach but the implementation aspects are still problematic, teachers continue to teach in predominantly teacher-centred ways.

INTRODUCTION

Much of secondary school mathematics focuses on developing learners’ understanding of, and skill in using, symbolic notations to reason and describe about equations, inequalities, functions, expressions and variables (Marcus & Fey, 2006). Despite hours of instruction and practice, learners often fail to master basic school algorithms or to apply them correctly in mathematical situations. For many learners, the experience with algebra is a meaningless and disconnected process consisting of rules for operations with symbols that do not represent anything real or useful (Kline, 1978). According to Koehler and Grouws (1992) these meaningless experiences might be as a result of the teaching they experience in learning mathematics at the secondary schooling level. Hence, Marcus and Fey (2006) claim that a problem solving approach to mathematics teaching and learning offers an alternative to such experiences.

The purpose of encouraging learners to solve problems is that they will acquire and be able to use the process of mathematical thinking, so that they will put these processes to work whenever they are needed. Schroeder and Lester (1989) describe three important areas of problem solving, namely, teaching for problem solving, teaching about problem solving and teaching via or through problem solving. Teaching for problem solving implies cases whereby teachers teach procedures first and then problems related to the taught concepts are solved. In teaching about problem solving learners are taught about various techniques as options when faced with a problem, these techniques can include, drawing of a table or graph. Teaching through problem solving implies that problems are used to teach important mathematical concepts.
Teaching through problem solving is more than just posing the correct type of problems and then allowing learners to solve them.

**RESEARCH QUESTIONS**

In order to realize the aim of the study the following research questions informed and guided the research:

**Main research question**

How do secondary school mathematics teachers in the John Taolo Gaetsewe district of the Kuruman area experience the facilitation of mathematics through a problem solving approach while participating in a B.Ed. (in-service) programme?

**Sub-questions**

- What are the views of in-service B. Ed mathematics teachers about a problem solving approach?
- How do in-service B. Ed mathematics teachers apply a problem solving approach in their own classrooms?

**Significance of the research**

This study was motivated by various factors, mainly, my personal experience as lecturer at RUMEP. It seemed as if RUMEP B.Ed. (in-service) teachers experience obstacles incorporating a problem solving approach in facilitating mathematics lessons despite being encouraged and taught through such strategies. Teachers still use drill and practice methods in order to teach mathematics. I, therefore, wanted to explore opportunities and challenges facing these teachers regarding the use of problem solving while studying at Rhodes University (RU). The focus was on the RUMEP teachers in the John Taolo Gaetsewe district of the Kuruman area in the Northern Cape Province where I am currently staying and working. It will be convenient for me to meet the teachers since they are also residing and working in the area.

It is foreseen that the findings that emerge from this study may make some contribution towards the improvement of teachers’ mathematics teaching skills. The study appealed as significant to me as a mathematics lecturer and may potentially provide information to other lecturers at RU who participate in mathematics teacher education. In addition, the study may be helpful to individual policy makers - such as the Department of Basic Education (DBE) and the Department of Higher Education and Training (DHET) - especially in terms of the South African DHET 2011 *Minimum Requirements for Teacher Education Qualifications*, Government Gazette 34467.
Distance education and RUMEP model

Distance education emerged in response to the need of providing access to individuals who would otherwise be unable to participate in face-to-face tuition (Beldarrain, 2006). Distance education practices around the world use a wide range of online and audio-visual technologies to overcome the lack of direct contact between lecturers and students (Baggaley, 2008). However, these practices are not universally adopted by all distance education lecturers nor even encouraged by their institutions. This is the case with the RUMEP model where the use of online and audio-visual technologies is replaced by contact sessions in Grahamstown (ten-day contact session each year), alternate Monday afternoon workshops and frequent interaction with the local facilitator (the researcher). The RUMEP model can be considered as a hybrid distance learning model.

Mathematical problem solving

Problem solving has different meanings to different people and it is difficult to reach a common meaning to problem solving. Broadly, to solve a problem means finding a way where no way is known, finding a way out of difficulty, finding a way around an obstacle, attaining a desired end that is not immediately attainable by appropriate means (Hatfield, Edwards, Bitter & Morrow, 2000). Others state that problem solving means ‘finding an appropriate response to a situation which is unique and novel to the problem solver’ (Johnson & Rising, 1967). According to Silver (1987) problem solving means the application of one’s knowledge to tasks that may be well structured or poorly structured, familiar or unfamiliar, simple or complex. Problem solving can also be perceived as the process of getting from givens to goals when the path is not obvious (Lesh & Doerr, 2003).

Not all learners in a class may view what is being taught as a problem. Orton and Frobisher (2000) put forward that what is a problem to one learner may be an exercise for another learner. For instance, those who have little understanding of a situation may view any mathematical idea arising from an activity associated with the situation as a problem. This means that learners who have already met a situation before and become reasonably familiar with the different aspects of the mathematics of the activity will view their work as a repetitive exercise.

For example, in an activity that requires learners to solve for \( x \) in \( 2^x = 3 \) using a calculator and leaving the answer in decimal form, a learner used to solve exponential
equations using logarithms is likely to obtain the answer as 1,584962501. Such a learner does not see exponential equations having prime bases as a problem. However, another learner who is at early stages of solving exponential equations can regard the equation $2^x = 3$ as a problem. Such a learner will employ many strategies, methods and processes to arrive at a solution. One learner may use trial and error methods substituting $x$ by 0.5; 1; 1.5 and 2 using a calculator and estimate the answer as lying between 1 and 2 instead of arriving at exact solution 1,584962501. But a less advanced learner may say we cannot solve the equation since the bases are not the same. Stacy (2005) states that successful mathematical problem solving depends upon deep mathematical knowledge, reasoning abilities, communication skills, abilities to work with others and personal attributes such as persistence, organisation and confidence.

**Problem solving frameworks**

In this section I describe two problem solving frameworks, namely, Polya’s and Schoenfeld’s problem solving frameworks.

**Polya’s framework of problem solving**

According to Polya (1988) there exist four stages of problem solving. He outlines the four stages as understanding the problem, devising a plan, carrying out the plan and looking back. In the stage of understanding the problem learners able to repeat the statement using their own words. Modes of representation can also be used to enable the teacher make the problem understandable and interesting to learners. In the stage of devising a plan, learners ought to find strategies of solving the problem with little interference from the teacher. For carrying out the plan, the learner should be convinced about the correctness of each step. In the last stage, looking back, learners re-examine and reconsider the path that led to the required solution. In essence, proponents of Polya’s framework follow a step-by-step instruction in teaching.

**Schoenfeld’s problem solving framework**

Schoenfeld’s framework is based on Gestaltism learning theory developed from a cognitive science perspective. Schoenfeld (1985) outlines categories such as heuristics, control and belief systems that are required when one is working on problems with mathematical content. According to Schoenfeld (1985) resources are tools, procedures, facts and skills potentially accessible to the problem solver. Heuristics are strategies, techniques or rules of thumb for successful problem solving, suggestions that help an individual to understand a problem better and make progress towards its solution. Control implies how the individuals use the information available at their disposal to
solve problems. The category of belief systems means one’s mathematical world view, the perspective with which one approaches mathematics and mathematical tasks.

**METHODOLOGY**

Since my aim was to determine the experiences of mathematics teachers in a limited number of school contexts within a limited time frame, I found the case study as a design type or genre appropriate for this study (Cohen, Manion, & Morrison, 2007). According to Yin (1993) case study research can be based on single- or multiple-case studies. A single-case study focuses on a single case only, while a multiple-case study include more than one cases (Yin, 1993). Thus, a multiple-case study consisting of four case studies (one case for each teacher) was preferred for this study. Hitchcock and Hughes (1995) who consider a case study as a suitable way to investigate practices in an everyday environment. Also, Yin (2009) regards case study research as an empirical inquiry that investigates a contemporary phenomenon within its real-life context. This case study thus investigated and reported the dynamics of problem solving in real-life secondary school and university tutorial situations.

An interpretive paradigm underpins this research in that it mainly centres on the significance of participants’ views and what meaning can be made from their views (Ary, Jacobs, Razavieh & Sorensen, 2006). In this case, I found this paradigm suitable for my research as I intended to explore the experiences of teachers regarding their facilitation of mathematics through problem solving. The main purpose of studies using an interpretive lens is to understand the experiences or the world of others. Within this paradigm, a case study design afforded me, as a researcher, the opportunity to interpret and comprehend the experiences within set contexts (Cohen et al., 2007). To retain the integrity of the phenomenon being investigated, efforts were made to get ‘inside’ the mathematical practices of participants and to understand such practices in real classroom situations (Cohen et al., 2007).

**Participants**

I purposefully and conveniently selected four teachers from a group of 12 B.Ed. (in-service) teachers in John Taolo Gaetsewe district teaching in four different schools. Currently there are 12 teachers participating in the B.Ed. (in-service) mathematics education programme in the John Taolo Gaetsewe district. The four teachers were likely to be informative about problem solving, because the Monday afternoon workshops, the Grahamstown contact sessions and the classroom observations already exposed teachers to a problem solving approach. Therefore, the four teachers have been chosen according to their interest and orientation towards problem solving. Convenience played a role in the selection because these four teachers were chosen due
to their closeness of where I, as researcher, was staying and working. This enabled me
chance to go back to the research participants to seek clarification if such need might
arise.

INSTRUMENTS

In this study, I used multiple data generation procedures. This gave me an opportunity
as the researcher to examine cases from several points of view (triangulation) because
multiple data sources provided information in the context, thereby providing rich data
for analysis. According to Yin (1993) the important aspect of case study data generation
is the use of multiple sources (interviews, observations, questionnaires) of evidence that
converge on the same set of issues. Thus, I video-recorded classroom lessons of the
four selected teachers, conducted face-to-face interviews with the four teachers and
allowed all 12 teachers, including the four cases, complete a questionnaire based on
their problem solving experiences. A video recorder was only used as a back-up later
if I want to look back of what happened in the classroom. The aim was not to analyse
the video captions in itself.

ETHICAL CONSIDERATIONS

Permission was obtained from the Northern Cape Education Department, school
principals of the schools participating in the study and Rhodes University’s Faculty of
Education. In addition, all ethical issues have been clarified and approved by the
Research Ethics Committee (Human and Social Sciences) of Stellenbosch University
and adhered to by the researcher. Alphabetical letters as pseudonyms have been used
to protect the identity of candidates.

FINDINGS AND DISCUSSION

Data was obtained from the questionnaire, the interview and the classroom
observations. Firstly, I analyse responses from the questionnaire completed by the 12
teachers registered in the RUMEPP programme including the four cases. Later, I will
analyse data collected from the four selected cases.

Summarized data from the questionnaire

The questionnaire constituted only a limited survey rather than a true complete
questionnaire survey. A questionnaire was, therefore, used to generate secondary data
for this study. The secondary data were generated in order to complement data from the
four cases. This allowed the researcher to analyse data from several points of view.
The closed-ended questions

37 closed-ended questions and two open-ended questions were asked. Data generated from the closed-ended questions were divided into eight sections, namely, Teaching method, Planning and preparation of lessons, Learner communication, Teacher questioning, Tasks and activities, Classroom discourse, Monday afternoon contact sessions and Study materials. The eight sections listed above were identified in the literature review as items that may contribute to better mathematics teaching with a focus towards problem solving for the B.Ed. (in-service) mathematics teachers studying via a hybrid distance learning model. All the 12 teachers in the RUMEP programme, including the four cases, completed the questionnaire. The four cases were not separated from the other teachers during the completion of the questionnaire.

Data generated from the closed-ended questionnaire indicated that all RUMEP teachers consider problem solving as one of the appropriate methods that can be used to facilitate mathematics lessons in order to learn new mathematical skills. This is positive and praise-worthy since it shows sensitivity for the potential of problem solving. In essence, active learning and teaching is promoted by effectively engaging teachers and learners in work on problem solving (Marcus & Fey, 2006).

The two open-ended questions:

(a) Comment on how problem solving has changed your view to mathematics teaching.

It seems as if RUMEP teachers’ views have shifted in favour of a problem solving approach. In general, their views appeared to be consistent with the opinions of the four selected cases as explored during the face-to-face interviews. For example, one teacher noted: “I used to work alone. Now they (learners) do a lot more than me. I should not talk too much in class and do everything”. Another teacher wrote: “I must allow learners to do tasks themselves, and my role is to guide them whenever they do not understand”.

These teachers appeared to belief that their role is not to transmit clear information, demonstrate procedures for solving problems and explain the process of solving sample problems.

(b) Comment on how you found problem solving useful in your own mathematics classroom.
From the teachers’ responses, it seems that problem solving is useful in their classrooms. One teacher noted:

“problem solving in my classroom has stimulated interaction in diverse ideas and strategies amongst my learners which has resulted in a wider range of solutions”.

This study indicated that views held by the 12 RUMEP teachers, including the four cases, appeared to be consistent with the data generated from the four cases during the face-to-face interviews. However, the teachers’ views are the opposite of what happens in their actual classrooms. Their classroom teaching methods is still heavily teacher-centred. The teacher still regards himself or herself as the main transmitter of information while learners are expected to act as listeners.

Summarized data from interviews and lesson observations

This section explored the four selected teachers’ views regarding problem solving in an attempt to answer the research question: What are the views of B.Ed. (in-service) mathematics teachers about facilitating mathematics using problem solving? Five codes were identified in this study through the theoretical framework explored earlier. The codes were categorized into those that depicted teacher responses during the interview and teacher actions depicted during the facilitation of lessons. This resulted in the five items, expressed in the form of action statements, namely, probing for understanding, sense-making, drive learning, exploratory discussions and learner-interaction.

One of the questions asked in the category, probing for understanding, was:

What do you think about the claim that “learners should always be encouraged to justify their thinking?” What follows are some extracts of the four cases’ responses.

Teacher A said:

“I am in full support of that claim for the reason that learners are always constructing their own meaning of the concepts”.

Even though Teacher A gives the impression that learners should be given opportunities to construct their own meaning, I observed that he seems not to adhere to his convictions during lesson facilitations. He will sometimes stop learners and interfere when they are working on a particular problem. He will say “listen, listen, before we start with anything . . . when you are looking ...” By continuously using terms such as ‘Listen’, ‘No’, ‘Before we start’ and ‘Look’ he seems to deny learners the opportunities to construct their own meaning.
**Teacher B** noted:

“If learners just say something out of the moon, you don’t know if it’s guessing or whatever, but if a learner understood something he will be able to justify it, so it’s a way of showing understanding”.

The verbal responses of Teacher B appears to be in line with Schroeder and Lester (1989) that facilitation of mathematics using problem solving is more than just posing the correct type of problems and then allowing learners to solve them without understanding. However, I observed that Teacher B is dominating the lesson and learners are contributing less in the conversation.

Table 1 represents one of the discussions that took place between Teacher B and the Grade 10 learners in an attempt to probe for understanding. The topic was sketching the linear graph.

<table>
<thead>
<tr>
<th><strong>Teacher B</strong></th>
<th><strong>Learners</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Now when you look at the three graphs ( y = 3x; \ y = 3x + 1 \text{ and } \ y = 3x + 2 )</td>
<td>They are the same</td>
</tr>
<tr>
<td>what can you say about their slopes, are they the same or are they not the same?</td>
<td></td>
</tr>
<tr>
<td>What cause them to be the same …? What is it that makes them to have the same slopes …?</td>
<td>The signs, they have positive signs</td>
</tr>
<tr>
<td>So with the positive signs they all slope in the same direction?</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Table 1: Sample conversation between Teacher B and Grade 10 learners

From Table 1, Teacher B ended the conversation by reminding learners that in \( y = mx + c \), the variable \( m \) represents the slope of the function. Even though the answers of the learners were not in detail and satisfactory, according to Teacher B he was following-up on learners’ answers to establish understanding of the concept of the slope. However, he is actually not establishing understanding. Learners are only saying ‘the signs, they have positive signs’ but he does not allow them to explore how the signs relate to the gradient or the slope. Learners give ‘yes’ answer without
motivating or justifying their responses instead he can allow them to expand on their responses without him saying a lot on their behalf.

**Teacher C** said:

“... *that is very good because learners answer or they see things differently, so I think they must be given a chance to justify their thinking...*”

It looks as if Teacher C wants learners to think about things in a new way, such as seeing new relationships between mathematical ideas. However, Teacher D does not seem to give learners a “chance” to justify their answers as she claims.

Table 2 depicts some of the discussions that took place between Teacher C and the grade 10 learners in exploring the value of the gradient in the equation: \( y = 2 \).

<table>
<thead>
<tr>
<th>Teacher C</th>
<th>Learners</th>
</tr>
</thead>
<tbody>
<tr>
<td>May, what is the gradient</td>
<td>Undefined</td>
</tr>
<tr>
<td>[teacher laughs] Learner A, what is the gradient?</td>
<td>Gradient is ( y )</td>
</tr>
<tr>
<td>Learner B, what is the gradient?</td>
<td>1</td>
</tr>
<tr>
<td>Who can help them? [teacher asking the whole class] Do we have the gradient in ( y = 2 )?</td>
<td>No no mam</td>
</tr>
<tr>
<td>We only have the the …</td>
<td>The ( y ) – intercept</td>
</tr>
<tr>
<td>That is where we are going to plot the graph at ( y = 2 )</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Sample conversation between Teacher C and Grade 10 learners

From Table 2, Teacher C end the conversation without finding out why learners are saying the gradient is ‘undefined’, ‘the gradient is 1’ or ‘the gradient is \( y \)’. She ends the discussion by introducing the concept of ‘\( y \)-intercept instead of finding out why at least three learners give the wrong answer. She denies learners an opportunity to take responsibility for their own learning as suggested by Malan (2000). Instead she laughs and does not find out why this learners say the gradient is undefined, 1 and \( y \). She then asks the whole class who response by saying there is no gradient. Actually in this case the gradient is 0. It is not mathematically convincing to say there is no gradient. Also at the end of this conversation, Teacher C does not provide learners with the correct answer and establish why they are giving such responses. Her teaching does not probe for understanding because she does not allow learners to provide reasons on the
responses. I do not think learners managed to grasp the concept of the gradient in this lesson.

**Teacher D** echoed:

> “if maybe you encourage learners to justify their thinking, then learners will have deeper understanding of mathematical concepts ...”

Teacher D seems to support Driver and Oldham (1986) that teachers’ key role should be to lead learners in their own discovery and understanding of mathematical concepts. However, from the lesson observation data, there is a mismatch between what Teacher D said and what happened in his classroom. In his lesson learners were given a four-sided figure and had to prove in one of the questions that $\text{PT} \perp \text{SR}$ using analytical methods. He then concluded the conversation by stating that “the two lines, $\text{PT}$ and $\text{SR}$, are perpendicular because the product of their gradients equals $-1$.” However, Manouchehri (2007) states that, ending learners’ discussions by giving them the correct formulas or answers, would likely close the door not only on their mathematical investigations but also on the formation of a learning community in which members willingly explore mathematics and engage in collaborative construction of knowledge.

From these conversations and observations it appears that the four cases seem to be positive about using a problem solving approach in their own classrooms. However, they are still experiencing difficulties in implementing the approach. I noticed that the teaching orientation of the four selected teachers was still dominated by too much talking and demonstrations from them while the learners were attentively listening. No efforts were made by the four cases to probe for understanding.

**CONCLUSIONS**

Based on the findings from the consulted literature and the empirical findings reported in this study, the following conclusions can be drawn regarding teacher experiences about problem solving.

Firstly, while the four teachers who participated in the study seemed enthusiastic about learner-centred practices and intended to implement such practices in their classrooms, they continued to teach in predominantly teacher-centred ways. The findings suggest that these teachers still facilitate mathematics lessons using a ‘traditional’ approach, namely, ‘telling and showing’, despite doing a problem solving oriented course in mathematics education for their further studies.

Secondly, the findings showed that there seems to be a mismatch between what the four selected teachers advocated and what actually happened in their classrooms. Interview findings thus did not correspond well with the observational. The latter findings illustrated that the teachers regularly intervened to show learners how to solve
mathematical problems and that little was done to promote problem solving techniques, such as enhancing deep understanding.

Thirdly, the findings indicated clearly that the participating teachers are willing to move towards a problem-oriented teaching. However, the four teachers still experience challenges to implement a full problem solving approach.

Lastly, the findings also showed the availability of time as a main obstacle to the implementation of a problem solving approach. However, teachers seem to have missed the point that the knowledge gained and the procedures used in solving a specific problem may be applied to other problems and thereby could address the time factor. This includes the view that no problem is completely solved and there may always be an improved understanding of a solution (Buschman, 2004).

The above conclusions seem to suggest a number of implications not only for teaching mathematics using problem solving but also for future research.

**IMPLICATIONS**

For mathematics education the findings of this study imply that the teaching of mathematics has to include more opportunities for promoting a problem solving approach. Mathematics education using problem solving could be carried out by teachers who were trained by university or other lecturers through such methods. Such training for mathematics education could also happen more regularly through teacher in-service programmes.

Classroom practices based on learner-centred approaches as opposed to teacher-centred approaches seem to be superior in terms of learning gain. This practice may allow learners to justify their answers while also motivated to listen to each other’s comments. In particular, the value of discussions among learners focusing on mathematical problems needs to be emphasized and thus learners’ answers, rather than teachers’ superior knowledge should be guiding instruction.

In-service programmes for teachers could create whereby positive attitudes and competence towards a problem solving approach in mathematics teaching are developed. Forms of in-service training for teachers may include locally-based afternoon workshops, contact teaching blocks and classroom support visits. It may also be important to point out that teachers find traditional ‘one-shot’ workshops as of little value (also see Buschman 2004) and such interventions may not result in the desired changes.
This study has raised a number of issues that are critical towards the facilitation of mathematics using a problem solving approach. Therefore, the study could potentially pave the way for further studies regarding the challenges facing mathematics teachers who attempt to incorporate problem solving methods in their teaching. An implication for further study could be an in-depth study to establish which factors currently prevent mathematics teachers to implement the problem solving approaches in the classrooms. Such a study would present a clearer picture of the misconceptions regarding the use of problem solving as well as why teachers are still reluctant to incorporate problem solving approaches in their teaching.

CONCLUSION

This study has shown how in-service secondary mathematics teachers can experience the benefits and challenges of using a problem solving approach to their teaching. In spite of the limitations of this case study, the researcher was able to note some small changes in the attitude of teachers towards a problem solving approach.

However, it must be noted that the use of a problem solving approach by teachers is a long term investment and cannot be achieved overnight. It may take a gradual approach to convince teachers that their present, traditional methods are less relevant and ineffective in relation to the needs of modern societies. To convince the majority of teachers of such a view, opportunities may be created where they successfully experience the actual methods used to enhance problem solving. Teachers also need to increasingly challenge and critically reflect on their own teaching methods more frequently.

I finally contend that series of regular workshops, contact teaching blocks and classroom support visits for B.Ed. in-service teachers, based on constructivist learning models and a problem solving approach in mathematics teaching may be a way to improve the quality and results of school mathematics in future. This may only materialize if universities, teachers and school authorities can synergistically muster their efforts and resources to achieve such an outcome.

REFERENCES


The importance of proof to the learning of mathematics is discussed. A perspective on proving in the South African context is provided. An example is utilized to explicate knowledge and reasoning requirements when constructing proof of a mathematical statement. Cognitive load theory is utilized to argue that learners are not provided with adequate reasoning experiences and knowledge in terms of proof at the high school level. Subsequently suggestions are made on possible ways to enhance the learning experience in terms of proof at the high school level. Possible gaps in the curriculum are discussed.

INTRODUCTION

The mathematics curriculum in South Africa has undergone a number of changes in the last few years. One of the major changes in the previous curriculum has been the removal of Euclidean geometry from the main curriculum. In the latest curriculum (Curriculum and Assessment Policy Statement (CAPS)) it has been reintroduced. In previous curricula as well as the present one Euclidean geometry provided the bulk of the proof and proving experiences for learners in the Further Education and Training (FET) phase. Proof and proving however permeates all of mathematics and is not restricted to instances in Euclidean geometry. Previous and current grade 12 national results indicate that questions on Euclidean geometry where proof and proving is required is some of the questions where South African learners perform worst. The concern of this paper however is not about Euclidean geometry in particular, but about proof and proving at the high school level in general. In particular this paper is concerned about how the perceived adequacy of provided knowledge impacts learning and understanding in the domain of mathematical proof.

There seems to be general consensus in the mathematics community that proof and proving plays a central role in mathematics. Proof is essential in mathematics education since it is a necessary aspect of knowledge construction (Herbst, 2002a) and is perceived to be a vehicle for the teaching and learning of deductive thinking (De Villiers, 1990; Stylianides & Stylianides, 2008). Furthermore proof is considered to be essential for developing, establishing and communicating mathematical knowledge (Hanna, 2000; Stylianides et al, 2008). There is also a view that learners’ proficiency in proof can improve their mathematical proficiency generally and in particular their deductive reasoning since proof is required in many mathematical problem solving...
situations where conclusions are to be reached, connections between concepts are to be made and decisions are to be made (Stylianides, et al, 2008).

THEORETICAL FRAMEWORK

The learning framework for this paper is based on a cognitive load perspective. Cognitive load refers to the amount of mental effort being used in the working memory. Sweller (1988) contends that increased amounts of information in working memory increases cognitive load.

Sweller (1994) argues that schema acquisition and the automatization of learned procedures are two essential mechanisms in the learning process. Automatic processing refers to cognitive activities that occur without conscious control. Schemas are cognitive constructs that organizes elements of information according to the manner in which it will be utilized. Furthermore Sweller (1994) maintains that learning and problem solving is premised on selective attention and cognitive load. The knowledge levels of learners in a specified domain may also impact on cognitive load. Based on cognitive load theory it is argued that instructional tasks that require the processing of many new elements of information simultaneously will not be very effective and the ability to use the information in related tasks will be severely constrained. Instructional designers should therefore always bear in mind the limitations of our cognitive architecture when designing tasks for instruction.

McNally (1974) in his study on Piaget and education use the example of a small child that sees sheep for the first time. The child then refers to the sheep as dogs. The reason McNally (1974) advanced for this erroneous classification is that the child saw the sheep in terms of that part of his cognitive structure which seemed to apply. A possible inference one can make from this is that the child at this point in his/her life having encountered four-legged woolly animals called dogs applied this schema to the sheep, since that was the only schema available to the child. McNally (1974) therefore argues that environmental events such as the above are assimilated into the cognitive structures not as a mechanistic transaction, but that the cognitive structures imposes its own organization, meaning or interpretation on an external stimulus. This means that the child compared the sheep to what was available in his cognitive structures for small woolly four-legged animals and since only dog was available interpreted it as dog. So assimilation is the intellectual process whereby the individual deals with the environment in terms of his present cognitive structures. This state of affairs however only exists until such time that accommodation in the cognitive structures of the child occurs. Accommodation occurs when the cognitive structures are forced to modify by the demands of the environmental event. In other words once the child have a more complete schema of a dog the child is forced to modify his cognitive structure in relation
McNally (1974) therefore argues that an essential feature of schemas is that they change as a consequence of the interaction of maturation and experience. One way of describing how new knowledge is discovered in mathematics is by utilizing Buchberger’s (1989) creativity Spiral. This creativity spiral provides the different aspects that are involved in the discovery of mathematical knowledge. Kutzler (2000) maintains that in the spiral there are three phases namely experimentation, exactification and application. Kutzler (2000) explains that during the phase of experimentation one applies known algorithms to generate examples then obtains conjectures through observation. During the phase of exactification conjectures are turned into theorems through the method of proving, then algorithmically useful knowledge is implemented as algorithms. During the phase of application one applies algorithms to real or fictitious data. Using the above explanation Kutzler (2000) asserts that the Egyptians and other ancient civilizations applied only the phases of experimentation and application in their construction of mathematical knowledge. Consequently they only used inductive reasoning. He argues that in about 500 B.C. the Greeks took the Egyptian mathematics and applied to it the deductive method of reasoning i.e. they added the phase of exactification. He maintains that from then on mathematics comprised of all three phases and that mathematics was thus established as the deductive science of today.

However from about 1950 on the French mathematician Dieudonne and his colleagues (known as the Bourbaki group) developed the system of “definition-theorem-proof-corollary-…”(Kutzler, 2000). The Bourbaki system therefore did not include the phase of experimentation and consists only of the phases of exactification and application. This Bourbaki system has now become part of the modern process of teaching and learning. Thus it has become customary to teach mathematics by deductively presenting mathematical knowledge and then asking students to learn it and then use it to solve homework and examination problems. This is especially so in the case of proof and proving.

Proof in the South African context

Herbst (2002b) contends that high school geometry courses in the USA were composed of two different types of propositions namely fundamental propositions and exercises. The fundamental propositions are described as the minimum which all learners should know. The purpose of the fundamentals was to develop the topic and to exemplify what it meant to prove a proposition. On the other hand the fundamentals themselves would be the information that students would use as they did proof exercises. Herbst (2002b) maintains that the perceived purpose of these exercises was to stimulate student reasoning and to practice what had already been learned. South Africa has a similar
that each learner has to know in order to do proof exercises. It is expected of the teacher to present proofs of these theorems in the statement and reason format to learners. Some of the reasons advanced for this format, was so that students could understand the deductive reasoning involved and also to serve as an example of how proving should be done. De Villiers (1990) argues that another possible reason is to expose learners to a formal axiomatic system. It is therefore not surprising then that some teachers and learners at school are under the impression that this is the only kind of proof there is, and that all proofs should be done in this way as the two-column proof form the bulk of their proving experiences at school level.

In South Africa learners encounter formal proof for the first time in the FET phase. This abrupt introduction of learners to proof at the high school level has been identified as a possible reason for their struggles with proof and proving (Stylianides & Stylianides, 2008; De Villiers, 1986). It has been my experience that the normal teaching strategy with proof is to deductively present learners with the theorems and their proofs and then expect them to learn and to regurgitate the proofs in assessment activities. De Villiers (1986) argues that this instructional method can lead to rote learning and memorization of theorems, proofs and definitions with little or no understanding. Besides the reasons presented above there are many other reasons for learners’ difficulties with proof. I will utilize an example of proof by contradiction to elucidate some of the other contributing factors that in my opinion contribute to learners’ struggles with proof.

Prove the following statement by contradiction:

For all integers \(a, b\) and \(c\), if \(a \nmid bc\), then \(a \nmid b\).

The logical structure of proof by contradiction is as follows:

Step 1: Suppose the statement to be proved is false (i.e. use negation)

Step 2: Show that this supposition leads logically to a contradiction

Step 3: Conclude that the statement to be proved is true.

Solution

Step 1

Suppose \(\exists\) integers \(a, b\) and \(c\) such that \(a \nmid bc\) and \(a / b\)

Step 2

Since \(a / b\) there exists an integer \(k\) such that \(b = ak\) - by definition of divide
Then $bc = (ak) \cdot c$ – closure of integers under multiplication

$bc = a(kc)$ - by associative law for multiplication

But $kc$ is an integer since it is a product of integers and so $a / bc$ - by definition of divide

Thus $a \nmid bc$ and $a / bc$ which is a contradiction

**Step 3**

Hence the supposition is false and the statement is true.

**Discussion of the example**

Based on cognitive load theory it is suggested that for effective learning to take place the learner needs to make the logical structure of proof by contradiction (as set out above) part of his/her long term memory. In order to do this example the learner also needs to have internalized the definition of divide and the associative law. The working memory can then access this information from the long term memory to make the indicated connections. The following delineates how this happens in practice:

- First the learner needs to realize that the statement that must be proven is written in the if-then format and that it is a compound statement. The statement therefore consists of an antecedent and a consequent. The antecedent in this case is $if a \nmid bc$ and the consequent is $a \nmid b$. The learner also has to be cognisant of the fact that in order to prove this statement by contradiction one has to start with the negation of the consequent.

- Subsequently step one has to be utilized to write the negation. This implies that the student at an earlier stage has engaged with negations and now has a complete mental schema of negation. This means that the student knows that negation in this case means accepting that the antecedent is true i.e. $a \nmid bc$ and the consequent is false i.e. $a / b$

- Second the student should know from prior experience that he/she next has to start with $a / b$ since the definition of divides is known. This definition should then be applied.

Next the equation that is a result of the application of the definition should be connected to the antecedent i.e. by multiplying by $c$ on both sides of the equation giving $bc = (ak) \cdot c$. Again the student can only know this as a result of prior experience that is brought about by practice.
The student should then realize that the definition of divide can be applied again to the equation $bc = (ak) \cdot c$ but this time in the opposite direction. In other words if the associative law for multiplication of integers is applied to give $= a(k \cdot c)$, then from this equation one can deduce that $a / bc$. This deduction can only be done by comparing the equation to the definition of divide that is stored in the long term memory.

- The fact that $a \nmid bc$ and $a / bc$ brings about the contradiction that was required.

- Step 3 can now be applied to conclude that the supposition is false and hence the original statement is true.

It is clear from the above outline that each successive step in the reasoning process is dependent on the previous step. Now if the learner has not internalized the structure of the proof then he/she will have to keep in the working memory the required definitions and the structure of the proof while at the same time devising a strategy to induce a contradiction. This obviously limits working memory capacity (cognitive overload) and is a much more difficult exercise and we therefore suggest that the learner internalize the structure of the proof by means of practicing on examples that require few deductions. As the student becomes more confident in applying the structure of the proof progressively more difficult examples can be provided. This should be the case for the other types of proof too.

Similar arguments can be advanced for proofs that form part of the South African FET curriculum. It is suggested therefore that learners should practice the different proof methods to gain experience, to become confident and importantly so that the proof structure can become part of their long term memory. Their working memory will then be freed for other important functions of proving like looking for conceptual connections in the provided information.

**CONCLUDING REMARKS**

Time constraints, overloaded curriculums and public and political expectations mitigate against all the phases of knowledge discovery being used on a regular basis in the average South African mathematics classroom. It is therefore unrealistic in my opinion to expect that instructional designers should endeavour to design tasks in such a way that learners ‘discover’ knowledge in all learning situations. I do however believe that curriculum designers should strive to include the majority of relevant knowledge in a mathematical domain even if it means that the knowledge is presented in a less complex way. Learners are not provided with all relevant knowledge in terms of proof in my opinion.
In the majority of cases the theorems of the FET curriculum in South Africa are compound if-then statements. It has been my experience that this fact is not made known to learners since it is not specified by the curriculum. Furthermore in most cases the type of proof that is utilized to construct a proof for the prescribed theorems is direct proof. Learners are also not made aware of this fact. When utilizing direct proof the normal starting point is the antecedent part of the statement. One then uses previously established results, definitions, inference, etc. to show that the consequent is true. Since learners are unaware of this way of working with proof they tend to memorize the proof with, the attendant, lack of understanding of the underpinning concepts.

In the South African mathematics curriculum the term theorem is introduced via the theorem of Pythagoras in grade 8. It has been my experience that in teaching situations the term theorem is not defined and no explanation is provided as to the fact that this type of geometry belongs to a specific type of axiomatic system and that other geometries exist.

Important questions therefore is, are learners of Euclidean geometry and proof in general in South Africa calling a sheep a dog because their cognitive structures have not been developed to maturation because of lack of relevant knowledge, experience and practice?

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Reflective symmetry is one concept in mathematics that is heavily embedded in the informal or everyday world because it is an aspect of nature. The assumption is that students draw from these everyday experiences to make sense of the more formal and scientific form of symmetry. Research however shows that when reflective symmetry is taken from its everyday context into the more scientific/mathematical form, a large portion of high school learners continue to exhibit very limited understanding. This study aimed at understanding the extent to which Grade 11 learners borrowed from the everyday world of symmetry and how this prior knowledge supported or inhibited their understanding of the formal/mathematical concept of reflective symmetry. A sample of 235 Grade 11 learners was selected purposively from 13 High Schools in the Eastern Cape. A written test was used as a data collection instrument. Analysis of data focused on learner responses to one reflective symmetry task. Results show that the visual perception embedded in the everyday conceptualization of reflective symmetry eluded learners when working with the formal conceptualization. Learner errors suggested that prior knowledge is not always helpful for learning new content in mathematics. Such results point to the need to reinterpret basic assumptions in the constructivist discourse which emphasizes building on prior knowledge.

INTRODUCTION

Children experience symmetry from a very early age because it is an aspect of their bodies, of nature, and of many person-made constructions. So when students begin to study formal proof (of symmetry or any other concept) at school or university, Tall (2004) would argue that they already have a wealth of preceding experiences or ‘met-befores’ on which to build. With specific reference to reflective symmetry the assumption is that since it is heavily embedded in the everyday; studying it might help students to draw upon their prior everyday knowledge in order not only to make sense of the more refined scientific form of symmetry but also to connect or apply this more refined form elsewhere. Despite this assumed symbiotic relationship, research shows that when reflective symmetry is taken from its everyday context into its more scientific/mathematical form, a large portion of high school learners continue to exhibit very limited understanding (e.g. Xistouri, 2007). Given that reflective symmetry is heavily embedded in the everyday, yet there is well documented evidence of discontinuities when learners use such prior knowledge (Bachelard, 1938; Roschelle, 1995; Hoyles & Healy, 1997) I then mused: ‘Is there a potential gap between the informal/everyday and formal knowledge of reflective symmetry? If so, what is the
nature of the gap and in what ways is it likely to be simultaneously necessary and problematic for students?’ So the paper addresses the following research questions:

1. What is the nature of conceptual development of reflective symmetry?

2. In what ways is this nature of reflective symmetry supportive as well as problematic for learners’ conceptual understanding?

3. What evidence is there in learners’ responses to support/refute the argument that learners’ met-befores might have been supportive and/or problematic?

THEORETICAL FRAMEWORK

Reports on misconceptions arising from a gap between prior knowledge and new learning content are not unique to reflective symmetry but are ubiquitous both within and outside mathematics (Roschelle, 1995). This juxtaposition of prior knowledge being supportive on one hand and yet problematic on the other creates a paradox whose solution has remained elusive mainly because researchers stress differences rather than commonalities. According to Roschelle on one side of this divide, educators rally to the slogan of constructivism; a discourse which has provided the most influential theory of learning in education. On the other hand, research tends to characterize prior knowledge as an epistemological obstacle (Bachelard, 1938) as it is seen as conflicting with the learning processes. In the context of mathematics education, critics argue that research on epistemological obstacles focuses on impediments that occur as learning shifts to new contexts (McGowen & Tall, 2010). On the other hand constructivism appears to take for granted that new levels of mathematical thinking are necessarily built logically and consistently on previous experience, just as they can be by reference to context or action (Hoyles & Healy, 1997). Neither side on its own has resolved the problem of the learning paradox because prior knowledge appears to be simultaneously necessary and problematic. According to Roschelle (1995) we therefore need an interpretive theoretical framework that accepts the flawed character of some prior knowledge, but still gives it a positive role. The notion of met-befores is presented as an alternative to the ‘Great Divide’ (Roschelle, 1995) created by the central tension that dominates the debates about prior knowledge.

The term ‘met before’ was given a working definition as a mental structure that we have now as a result of experiences we have ‘met-before’. Its basic premise is that there is a need to consider the roles of met-befores in different situations where sometimes they are helpful and where sometimes they are not. In the context of reflective symmetry the notion of ‘met-befores’ therefore relates to the knowledge gap that occurs as the student shifts from an earlier and useful informal understanding to a formal but
problematic understanding of reflective symmetry. McGowen & Tall (2010) further argued that without addressing the problematic met-befores that remain under the surface, any chance of conceptual understanding is suppressed and this leads to rote-learning that may exacerbate the problem in the longer term. It is in this context that I was interested to analyse the nature of conceptual development of reflective symmetry then tease out evidence from learners’ responses of met-befores of reflective symmetry that might have been supportive and/or problematic.

DISCUSSION OF THE RELATED LITERATURE

There appears to be general consensus that our mathematical growth of shape develops through a life-time of experiences which we gain from both powerful experiences of everyday examples as well as through formal mathematics instruction. Many frameworks are available to present this long-term journey from physical perception and action, to more complex ideas. Tall (2008) bounced his ideas against previous frameworks such as Piaget’s (1973) sensori-motor → pre-operational → concrete operational → formal operational stages; Bruner’s, (1966) enactive → iconic → symbolic stages; Fischbein’s (1987) intuitive → algorithmic → formal; and van Hiele’s (1986) visualization → analysis → abstraction → deduction → rigor. Tall concluded that all these ideas coalesced into three distinct threads:

1. An object-based **embodied** world - relating to our sensory perceptions of and the physical actions on the real world.

2. An action-based **symbolic** world - relating to our use of symbolism in arithmetic, algebra and more general analytic forms that enable us to calculate and manipulate to get answers.

3. A property-based **formal** world - relating to the formal axiomatic world of mathematicians which is the final bastion of presentation of coherent theories and logical proof (2008 p. 7).

He captured this development from the embodied, through symbolic to the formal world succinctly in his model shown in figure 1.
In this sense, school mathematics builds from embodiment of physical conceptions and actions (conceptual embodied). Once these actions or operations on physical objects are practiced and become routine, they can be symbolized and used dually as operations or as mental entities on which the operations can be performed. As the focus shifts from the embodiment to manipulation of symbols, mathematical thinking also shifts from the embodied to the (proceptual) symbolic world. The later transition to the formal axiomatic world builds on these experiences of embodiment and symbolism to formulate formal definitions and to prove theorems using mathematical proof. As one might notice, this development of reflective symmetry from the embodied through symbolic to the formal worlds then leads to very different methods of making arguments or what Rodd (2000) described as ‘warrant for truth for that which secures knowledge’. In the embodied world, it occurs first through ‘seeing’ something is true – we can see it. In the symbolic or proceptual world, it arises through calculating a correct result, or using the generalized arithmetic of algebraic manipulation to verify the required symbolic statement – we can calculate it. In the formal world it arises through specifying axioms and definitions set-theoretically - it is an axiom.

A major gap has been attributed to the shifts in thinking and warrants of truth as the learner moves from the embodied through the symbolic to the formal world of reflective symmetry. In the embodied world of reflections, since the results of both seeing real world examples and folding are produced by a practicable and observable process, the learner is likely to be easily convinced about its validity visually. Yet on the other hand, the arithmetic and algebraic computations deriving from the symbolic and formal worlds do not reveal, in a manner as obvious as the diagrammatic/visual solution. The
symbolic or formal way of solving may introduce a faster and more precise method where the intuitive approach is insufficient because the warrant of truth is not visible. Tall (2006) captures this sudden change metaphorically as he says that the formal-axiomatic world expresses properties in general set-theoretic terms to ‘turn mathematics on its head’. I was therefore interested to see how learners manipulated the concept of reflective symmetry within its different representations.

**METHODOLOGY**

This study followed a qualitative approach. Burns and Grove (2003:19) describe a qualitative approach as “a systematic subjective approach used to describe life experiences and situations to give them meaning”. The data collection instrument that was used for this study was a written test. A total of 235 Grade 11 learners purposively sample from 13 Eastern Cape – Grahamstown High Schools responded to a task on reflective symmetry. The task was designed in such a way that it would enable the researcher to see whether or not learners were reasoning using their object-based embodied experiences, or symbolic experiences or through the formal world of reflective symmetry. Questions to the task were phrased as follows:

*The diagram below shows a Cartesian plane with points A(-3; 6), B(4;-2) and C(-5;1).*

*Draw on the grid below, Aꞌ, Bꞌ, and Cꞌ, clearly labelling the coordinates of each point if:*

(a) *Aꞌ is the image of A reflected in the y – axis.*

(b) *Bꞌ is the image of B reflected in the line y = 0.*

(c) *Cꞌ is the image of C reflected in the line y = x.*
Figure 2 Task on Reflective Symmetry

In this task I argue that a learner reasoning from the object-based embodiment would perceive shapes, which have to be flipped over mainly in the vertical and horizontal axes because those are common in nature. A learner reasoning from the symbolic world would use algebraic formula and matrices to determine the image positions. A learner reasoning from the formal world of reflective symmetry would use such theorems as distance formula, midpoint theorem, perpendicularity etc. A tool for analysis based on these three worlds of mathematics was designed as shown in table 1.
Table 1 Reflective symmetry as seen through Tall’s three worlds.

<table>
<thead>
<tr>
<th>Embodied world</th>
<th>Symbolic world</th>
<th>Formal world</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Algebraic representation</strong></td>
<td><strong>Matrix Representation</strong></td>
<td></td>
</tr>
<tr>
<td>The embodied world is where learners think about the things around them in the physical world. It includes perceptions of shapes, diagrams, figures, points that are physically manipulated and whose images are of the ‘same size’ or ‘congruent’ to their originals, images which are ‘reversed’ or ‘flipped over’ or ‘laterally inverted’ in and equidistant from the mirror line.</td>
<td>(x; y) becomes (x; -y)</td>
<td>In the formal world, concepts are introduced by definitions, and their relationships delineated by stating and proving theorems. For this task, finding gradient, distances between points, midpoints, equations, perpendicularity, through formal proofs and theorems e.g. use of the Pythagoras theorem in the distance formula, would be indicative of a learner working in this formal world.</td>
</tr>
<tr>
<td>(x; y) becomes (-x; y)</td>
<td>1 0</td>
<td></td>
</tr>
<tr>
<td>(x; y) becomes (y; x)</td>
<td>0 -1</td>
<td></td>
</tr>
<tr>
<td>The use of such algebraic notation for solving the task would be indicative of a learner working in the symbolic world</td>
<td>-1 0</td>
<td></td>
</tr>
<tr>
<td>0 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 1</td>
<td></td>
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<tr>
<td>1 0</td>
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</tbody>
</table>
My immediate observation was that the entry point for 199 (85%) of these 235 learners was through joining the three points A, B, and C to form a triangle yet the task required learners to reflect 3 independent points in 3 different mirror line. That such regularity was observed across 13 different high schools with 13 different teachers, with a consistency far beyond chance; echoes Berieter's, (1985:24) concern who asked the question:

*Out of a magnitude of correspondences that might be noticed between one event and another, how does it happen that ....different children, with consistency far beyond chance, tend to notice the same correspondences?*

![Figure 3 Reflection mainly in the vertical axis](image)

Most learners then went on to reflect their incorrect triangle mainly in the x and y axes.

**DISCUSSION OF FINDINGS**

Despite their failure to identify the object that was to be reflected, their strategies however follow consistently from a definition of reflection in the embodied world. According to Hoyles & Healy (1997), if we focus on the regularities amongst the responses rather than on the errors, it is clear that the learners exhibit well-reasoned constructions. We can see a common set of invariant properties: the reflected images are the same size and shape as the original triangle that they drew; they are the same
distance away from the mirror as the image, and they are ‘opposite’ or ‘reversed’. Roschelle (1995) posits that many “misconceptions” are correct elements of knowledge which has been over generalized and so by specifying a narrower range of situations, the concept becomes correct. In fact I argue that if this task had required learners to reflect a given triangle (narrowing the range of situations) instead of three independent points in three different mirror lines, the majority of learners would have got a correct solution. This also confirms McGowen & Tall’s (2010) argument that they had preconceptions and not misconceptions.

CONCLUDING REMARKS

This study aimed at understanding the nature of conceptual development of reflective symmetry and how that development impacts on the learners’ sense making. In this paper I showed how this concept is heavily embedded in the everyday (object-based embodiment) from which it develops into the symbolic embodiment when introduced in the school then to the formal embodiment where theorems and axioms dominate. In the earlier experiences learners make insightful use of their everyday experiences thereby making sense visually. However, as the development of the concept moves into the symbolic and formal worlds, a number of gaps are created which need to be objectified because the visual perception eludes learners when working with the symbolic and the formal yet constructivism suggests prior knowledge is important for learners to make sense of new content. My analyses therefore point to the need to reinterpret basic assumptions in the constructivist discourse. Empirical evidence abounds that shows that in the apparent logical structure of mathematics curriculum, the biological brain will bring previous experiences to interpret the situations that are presented which can lead to unforeseen difficulties that arise through apparently sensible approaches.

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Human Sciences Research Council
Pretoria, South Africa

The Curriculum and Assessment Policy Statement for mathematics for all the phases (Foundation, Intermediate, Senior and Further Education and Training) identifies a number of specific aims that mathematics educators have to keep in mind as they prepare lessons for the mathematics classroom. One of the specific aims at the FET Phase requires that the educators must integrate mathematical modelling into the curriculum, drawing upon real life and contextual problems. This paper elaborates on the contextual teaching and learning theory as central to any efforts on the integration of mathematical modelling. Mathematical modelling is elaborated upon and discussed, indicating a few real life and contextual problems that are used as an illustration on how this can be introduced in the mathematics classrooms.

INTRODUCTION

The Curriculum and Assessment Policy Statement (CAPS) for mathematics for the Further Education and Training (FET) Phase identifies eight specific aims that teachers must take into account as they work with the learners to advance and enhance mathematical understanding (Department of Basic Education, 2011). All these specific aims are important for the teaching and learning of the mathematical content, however, I regard aim on mathematical modelling as central for the teachers to continuously reflect upon and find ways to integrate in their work:

Mathematical modelling is an important focal point of the curriculum. Real life problems should be incorporated into all sections whenever appropriate. Examples used should be realistic and not contrived. Contextual problems should include issues relating to health, social, economic, cultural, scientific, political and environmental issues whenever possible

The mathematical modelling aim in particular draws attention to the fact that mathematics cannot be taught only for the purpose of understanding the rules and procedures and equations without ensuring that they can find relevance and application in real life outside the classroom. This is in line with Boaler’s argument (2001:126) that if students only ever reproduce standard methods that they have been shown, then most of them will only learn that particular practice of procedure repetition, which has limited use outside the mathematics classroom. The purpose of this paper is intended to reflect further on this specific aim to look at the requirements and implications
thereof for the teachers and learners and to make suggestions on how it can be interpreted and made more meaningful.

The specific aim of mathematical modelling identifies two major thrusts which must be taken into account: Firstly, it requires teachers to incorporate real life problems into all sections whenever appropriate. Secondly, it emphasizes that contextual problems should include relating to health, social, economic, cultural, scientific, political and environmental issues wherever possible. Although this looks easy at face value, it requires a lot more from the both teachers and learners for proper implementation.

**Contextual teaching and learning theory**

According to the contextual learning theory, learning occurs only when students process new information or knowledge in such a way that it makes sense to them in their own frames of reference (their own inner worlds of memory, experience, and response). The mind naturally seeks meaning in context by searching for relationships that make sense and appear useful (Centre for Occupational Research and Development, 2012). Thompson (2012) expands the contextual learning theory to include the social component. In this sense, learning does not occur solely within the learner, but in the group and community in which they work. Learning is a shared process which takes place through observing, working together and being part of a larger group, which includes colleagues of varying levels of experience, able to stimulate each other's development. The theory is also looked upon in slightly different ways depending on the emphasis. For instance, it may be viewed as a theory of teaching and learning that helps teachers relate subject matter content to real world situations (Berns and Erickson, 2001). Here the emphasis is on the role of the teachers opposed to the role of the student. On the other hand, it may also be viewed as a diverse family of instructional strategies designed to more seamlessly link the learning of foundational skills and academic or occupational content by focusing teaching and learning squarely on concrete applications in a specific context that is of interest to the student. One of the primary principles of the contextual teaching and learning theory is that knowledge becomes the students' own when it is learned within the framework of an authentic context.

**Mathematical modelling**

Why is mathematical modelling a very important specific aim? Niss (2012: 49) provides a very compelling argument. “Every time mathematics is used to deal with issues, problems, situations and contexts in domains outside of mathematics, mathematical models and modelling are necessarily involved, be it implicitly or explicitly”. Stilman (2012) indicates that the term mathematical modelling is usually interpreted in a number of ways when used in curricular discussion and implementations. One of the interpretations sees mathematical modelling as motivating, developing, and illustrating the relevance of particular mathematical
Another interpretation views the teaching of modelling as a goal for educational purposes not a means for achieving some other mathematical learning end. Stilman sees the two approaches as involving the essential components of formulating, mathematising, solving, interpreting and evaluating. In line with Stilman, I would argue that in the contextual examples that are identified, most of these elements should be possible to identify and explored as far as it is possible. This requires a continuous process of identifying contextual problems that create meaning and can be related, in many ways to real life.

**Contextual problems**

‘The plethora of data that confronts us on a daily basis requires that we know more than simply being able to calculate. It demands that we understand the context in which the mathematical ideas are embedded and what those ideas are telling us in relation to the context’ (Hurst, 2007:25). Contextual learning aims to provide students with the knowledge that can be flexibly applied (transferred) from one problem to another and from one context to another (Tambelu, 2013:27). Gravemeijer and Doorman (1999) argue that the role of context problems used to be limited to the applications that would be addressed at the end of a learning sequence – as a kind of add on. ‘Nowadays, context problems have a more central role. They are endorsed because of today’s emphasis on the usefulness of what is learned and because of their presumed motivational power’. CAPS clearly spells out the importance and centrality of contextual problems as a component of mathematical modelling. This means that as argued by Gravemeijer and Doorman, these problems cannot just be viewed as an add on when there is time available to introduce them. If they are to be meaningful, they need to be identified during the planning phase and integrated throughout the content presentation as an integral part.

Berns and Erickson (2001) define contextual teaching and learning as conception that helps teachers relate subject matter or content to real world situations; and motivates students to make connections between knowledge and its applications to their lives. It is a focus on the context of what we teach from the students’ point of view. Contextual teaching and learning also emphasizes learning by doing. The students will learn better if they also get involved in the class activities. In contextual learning theory, learning occurs only when students (learners) process new information or knowledge in such a way that it makes sense to them. It emphasizes the learning process through “constructing” not memorizing and teaching is not only a process of transferring knowledge to the students.

**Cultural villages as a source of contextual problems and experiences**

The pictures attached below are examples of some of the artifacts that are found at many Cultural Villages in South Africa. This means that they are within reach in most provinces in South Africa as almost all Provinces have an establishment of a cultural
village. Amongst the villages that have been constructed in South Africa are: Lesedi Cultural Village in Gauteng Province; Kaya Lendaba in the Eastern Cape Province; Isinamva Cultural Village in Mount Frere, Eastern Cape Province; Basotho Cultural Village in the Eastern part of the Free State Province; Shakaland in Kwazulu-Natal Province; etc. Actually it is safe to say that every Province has at least one cultural village. These villages employ a number of the elderly people (not only elderly) who are knowledgeable in the various cultures that are exhibited at such villages. Their knowledge goes far beyond the making of various cultural artifacts but goes into the history, the science, the mathematics, the environment in which they are constructed. Mosimege (2007) identified many other artifacts that make up wide collection at various cultural villages. These and other artifacts have been analysed to reveal a lot of embedded mathematical concepts. Such mathematical concepts and related processes may be integrated in various way into the mathematical content. This requires that teachers familiarize themselves with the work at such villages in order to identify mathematical opportunities that may be integrated into the mathematical content.

Photograph One: Beadwork at Lesedi Cultural Village, Gauteng Province

The role of teachers and students in contextual problems and situations for the purpose of the realisation of the mathematical modelling aim
The few contextual examples based on ethnomathematical research discussed and illustrated in this paper have got the following implications for the implementation of mathematical modelling in mathematics classrooms:

(i) Teachers should continuously identify contexts that may provide rich experiences for the learners; Such context should include, but not confined to cultural settings

(ii) In the identification of the context and real life situations, the teachers should engage continuously with the learners to draw upon their individual experiences and interactions; This is bound to provide relevant and not contrived experiences and contexts

(iii) Teachers, together with the students, should identify various contexts and where such require mathematical analysis, should continue to identify and work further on such opportunities; These can be built on mathematization processes that have been undertaken in various contexts.

(iv) Teachers should ensure that contextual learning is closely related to the mathematical content to create better meaning for the experience of students in mathematics classroom i.e. these contexts should be related as much as possible to mathematics classroom experiences.

REFERENCES


Presented at the Mathematics Education for the Future Project Conference.
North Carolina, USA, September 2007


This paper reports on Grade 10 learners’ strategies in pattern recognition and generalization. A test was administered to 50 randomly selected learners in a school in Sekhukhune district, Limpopo province of South Africa. The study also sought to understand the difficulties that emerge from learners’ strategies and the role of different pattern representations in influencing learner strategies. Content analysis was carried out to assess learners’ strategies and the difficulties from their written test responses. The analysis focussed on: type of generalization strategies used; difficulties that emerged from learners’ responses; and the role of different pattern representations such as visual, geometric, numeric and mixed. The findings show that learners prefer to use numeric instead of visual approaches. They also incorrectly use the differencing method when generalizing.

INTRODUCTION

Pattern exploration has been identified as a central construct of mathematical inquiry and as a fundamental element of learners’ mathematical growth (Burns, 2000). Looking for a pattern or regularity is one of the actions which are performed in mathematics on the whole. The definition of mathematics as a pattern and systematic science emphasises the importance of pattern in mathematics. They are essential in the development of mathematical knowledge, concepts and processes (Fox, 2005). Olkun and Toluk-Uçar (2006) viewed patterns as a system which consists of the objects or shapes that are recurring or ordered regularly. In this study a pattern is defined as the relation between two consecutive terms in a sequence of numbers, pictures or geometric shapes. Patterns are included in most world-wide curricula in order to help learners to develop the skills of calculating, looking for order and structure their thinking strategies. In addition, they are essential in developing the skills of reasoning, communication, association and problem solving (Tanisli & Özdas, 2009).

Generalising a pattern is an essential aim of mathematics instruction (NCTM, 2000). Beatty, R & Bruce (2012) defined generalization as establishing a rule that enables learners to predict proceeding terms in a sequence. This meaning includes extension (empirical or mathematical formula) of existing sequence. However, recognising a pattern is an essential step in the formation of generalization (Nunes, Bryant, Evans, Bell, Gardner, Gardner, & Carraher, 2007). Generalization of patterns improves learners’ algebraic thinking and helps them to construct the concept of a variable
Furthermore, generalization helps learners to understand symbolic representations and interrelations among previous knowledge of arithmetic (Lannin, 2005). Patterns tasks also enable learners to observe and verbalize individuals’ own generalizations and translate them symbolically (Vale & Cabrita, 2011). Due to the fact that searching for patterns is a fundamental step in order to make generalization it is seen as a way of approaching algebra (Johnston-Wilder & Mason, 2005). In short, patterns have an important role as a bridge between generalization and algebra. The patterns represented in different forms and especially expressed as symbolic will make a real contribution to the understanding of basic algebra concepts (Akkan, 2013). It is therefore, important to study the strategies that learners use to generalise patterns.

Learners' efficacy in pattern recognition and generalisation is one concept in the study of sequences and series that needs attention. Ma (2007) mentions the importance of developing abilities like searching and exploring numeric and geometric patterns, as well as solving problems, looking for regularities, conjecturing and generalizing. South Africa’s new Curriculum Assessment Policy Statements (CAPS) (DBE, 2011) also emphasises the importance of generalisation as is evident from the specific outcomes for Mathematics. One of the specific aims of the CAPS mathematics curriculum is to provide learners with the opportunity to develop the ability to be methodical, to generalize, and make conjectures and try to justify or prove them.

The study of pattern is integrated in all the strands of mathematics. Patterns exit in various forms and contexts such as numeric, geometric, concrete and figural. Patterns allow learners to integrate and extend skills and knowledge of number, measurement, geometry, data collection and statistics, probability and algebraic thinking. The study of patterns also helps to bring together mathematics with a variety of curricular areas such as music, visual art and craft, vocabulary building, creative writing and verbal communication, social studies, science and environmental studies, talent and technology (Akkan, 2013). A pattern can be defined as something that remains constant within a group of numbers, shapes or attributes of mathematical symbols or concepts (Samson & Schafer, 2007). The aim of mathematical thought is to discover what is general in what is particular (Güner, Ersoy, & Temiz, 2013). Thus, an important objective when teaching this topic is that learners should find such generalisations from the particular cases of the pattern that they are exposed to.

The focus of this study is to provide insights into the strategies used by learners to generalise patterns. Very little is known about what influences learners to use particular generalisation strategies. The tasks used in the study require learners to recognise and generalise numeric, geometric and mixed patterns. Learners at this level are in a transition phase from senior phase to FET band and have not yet acquired formal algebraic skills. Thus it is important to analyse the nature of their approaches.
Therefore, the study aims to investigate Grade 10 learners’ strategies in generalising patterns and gain some insight into their thinking processes as well as difficulties emerging from their work.

PROBLEM STATEMENT

Current literature reveals that learners use varied and inconsistent pattern generalization strategies (Barbosa, Palhares, & Vale, 2007). Furthermore, learners’ generalisation competence using analytical and visual methods often produce inconsistent performance on pattern generalisation (Barbosa et al., 2007). However, it is not clear from the literature the type of generalisation strategies that learners use to get $n^{th}$ terms of given sequences and why such inconsistencies in learners’ generalization strategies. Therefore, this study sought to explore: the type of generalisation strategies that learners use; the effect of different pattern representations on learners’ generalizations strategies; and why learners experience noted difficulties.

RESEARCH QUESTIONS

What are the generalization strategies used by learners to find the $n^{th}$ term of a sequence?

What are the difficulties experienced by learners on pattern recognition and generalisation and why?

What is the effect of different pattern representations on learners’ pattern recognition and generalization?

PURPOSE OF THE STUDY

The purpose of this study was to explore the strategies used by Grade 10 learners to derive rules for finding the $n^{th}$ term of a sequence and the effect of different pattern representations on learners’ generalizations strategies. The study also sought to gain insights into the difficulties encountered by learners (and why) when finding the general ($n^{th}$) term of a sequence.

THEORETICAL FRAMEWORK

This study is guided by García-Cruz and Martinón’s (1997) theory of abstracting number patterns and formulating general relationships between the variables in the given situation. This theory in turn was based on Stacey’s (1989) work. The theory was developed to analyse the processes of generalization developed by secondary
school learners. According to the theory, generalization strategies are classified according to the nature of the pattern: visual, geometric, numeric and mixed. If the drawing played an essential role in finding the pattern it is considered as a visual strategy, on the other hand, if the basis for finding the pattern was the numeric sequence then the strategy is considered numeric. Learners who used mixed strategies acted mainly on the numeric sequence and used the drawing as a means to verify the validity of the solution. Rivera (2013) explains generalization of patterns in algebra in terms of a combined abduction-induction process. Abduction plays a significant role in the logic of discovering and establishing a generalization of a pattern sequence. According to Rivera (2013), learners’ generalisation strategies are predominantly numeric. They identified three types of generalisation: numerical, figural and pragmatic. Learners using numerical generalization employed trial and error with little sense of what the coefficients in the linear pattern represented. Those who used figural generalization focused on relations between numbers in the sequence and are capable of seeing variables within the context of a functional relationship. Learners who used pragmatic generalization employed both numerical and figural strategies, seeing sequences of numbers as consisting of both properties and relationships. Amit and Neri (2008) identified several strategies to deal with generalization in problems related to patterns. These include recursive, common ratio, counting, additive, explicit or non-explicit, whole-subject, linear, guess-and check, trial-and-error and contextual strategies.

**Generalisation and conjecturing strategies**

Great numbers of different types of patterns are studied using different strategies in order to generalize those (Nunes et al., 2007). Lannin, Barker and Townsend (2006) proposed a number of strategies that learners use when generalising and conjecturing mathematical patterns. Strategies such as recursive (counting strategy), common ratio, counting, additive, explicit or non-explicit, whole-subject, linear, guess-and check, trial-and-error and contextual strategies have been used to generalise patterns (Amit & Neri, 2008). They claim that learners’ strategies often emerge through different types of reasoning. It is fundamental that teachers and learners must understand the potential and limitations of these approaches. Some of these strategies may lead learners to make incorrect deductions. Lannin (2005) identified two classes of generalisation strategies: explicit or non-explicit. Explicit strategies (whole-object, contextual, guess and check, rate-adjust) establish a direct relationship between the dependent variable and the independent variable. As for non-explicit strategies (counting, recursion) the calculation of a specific term requires the calculation of all preceding terms.
**Guess and check**

This is a well-known and commonly used strategy for generalising patterns. The strategy involves guessing a rule and tries multiple input values to check its validity. This strategy is considered as a good problem solving numerical tool. However it is unreliable and sometimes leads learners to incorrect conclusions even if all conditions are not considered (Rivera, 2013). It involves guessing the rule without paying attention to whether it works or not. A learner may introduce an algebraic relationship (rule) representing the problem situation and does not take note of validity of his rule during process. Making a guess can help learners to better understand conditions of the problem; it can be a way of to try something when a learner is stuck. Learners may make random guesses, but over time, they learn to make more informed guesses (Collins, 2012).

**Visualization strategy**

A strategy is considered visual if the image/drawing plays a central role in obtaining the pattern, either directly or as a starting point for finding the rule. According to Polya (2014), visual representations are often used as a strategy that allows powerful and creative solutions. Techniques such as counting, whole-object with visual adjustment, difference with rate-adjustment and explicit are sub-categories of this strategy. Counting is a successful strategy though it is limited to near generalization questions. Drawing is another sub-category of the visualization strategy. Sometimes learners use a drawing to extend pattern, however this strategy is more applicable in questions involving far generalization.

**Explicit**

The explicit method refers to a strategy where a general formula is first derived for the nth term and the desired term is then calculated directly from the general formula by using the independent variable, namely, the position of the term. Provided the general term has been correctly formulated, the explicit method will yield any number of algebraically equivalent expressions for the nth term. Since this strategy provides to determine the functions by using equations and formulas it can be used to find near and far terms. Therefore it enables to obtain nth term and write general rule. One benefit of using the explicit strategy is that it encourages learners to examine the power of explicit reasoning (Lannin, 2005). The explicit strategy leads to a high level of efficacy. Learners based their work on the structure of the sequence, making reference to the relation between the variables reported in the problem.
Recursive and Additive

In this strategy learners recognise a core in a repeating pattern and extend it for several terms. Learners use previous term(s) in the pattern to find next term or terms. The strategy involves finding the common difference between two successive terms and then adds the common difference on last term in order to find next term. Lastly, learners will also be able to create a symbolic version of a repeating pattern using letters.

METHODOLOGY

Design

The study sought to establish and describe learners’ strategies in figuring out and generalizing patterns, difficulties emanating from learners’ strategies and the influence of different pattern representations on learner strategies. This may, in turn, help explain the extent to which learners use recursive, visualisation, difference method, explicit, guess and check strategies to derive the general formulae (the $n^{th}$ term) of a given pattern. Thus, the mixed methods approach employing case study design was chosen to carry out the research.

Population and sample

The population for the study comprised of 105 Grade 10 learners drawn from a secondary school in Mwaritsi circuit in Sekhukhune District, South Africa. Using the Rao Soft sample size calculator, a minimum recommended sample size of 50 learners was obtained. A probability sampling procedure was used and a simple random sample consisting of 19 male and 31 female learners was drawn. Thirty-seven (74%) of the participants were in the 13-15 years age category. 92% of the participants spoke Sepedi while 8% were fluent in other languages.

DATA COLLECTION

The sample form part of the classes of Grade 10 Mathematics learners allocated to one of the authors (working at the school) for day to day mathematics teaching. Therefore, a test was administered to 105 Grade 10 learners at the end of the topic on patterns. The researcher marked the scripts for the 50 learners selected.

INSTRUMENT

The researcher in collaboration with the grade teacher compiled a test based on patterns which was used as an instrument for data collection in this study. The test consisted of 9 questions (3 structured visual and geometric, 1 mixed, 5 numeric patterns) which were
selected from Grade 10 textbooks in line with CAPS mathematics curriculum guidelines. The test was validated by a panel of teachers and researchers in mathematics education. It was pilot surveyed with a group of 20 Grade 10 learners who were excluded from the main study.

DATA ANALYSIS

The content analysis of learners’ responses to the test focused on their recognition and generalisation strategies, difficulties and the effect of various pattern representational formats on their solutions. Qualitative content analysis is "probably the most prevalent approach to the qualitative analysis of documents" and that it "comprises a searching-out of underlying themes in the materials being analysed" (Bryman, 2004: 392). The emphasis of this approach was on allowing categories of strategies, difficulties and learners’ thinking processes on different pattern representational formats to emerge out of data.

FINDINGS

Pattern Recognition and Generalisation

It can be recalled that the purpose of this study was to explore pattern generalisation strategies used by Grade 10 learners when solving problems involving patterns. The research also looked into the difficulties that emerge from learners’ thinking processes and the role played by different pattern representational formats in influencing learners’ strategies. Thus the analysis of learners’ work focused on the following aspects as informed by the research questions: generalization strategies, the difficulties experienced by learners in approaching particular questions and the effect of representational formats on learners’ performance. The summary of the findings is shown in table 1 below.

Table 1 shows that on average 67.6% of the learners were able to recognise patterns and extend them to the next two or more terms without using a formula. On average 32.4% of the participants experienced difficulties in recognising patterns and consequently failed to generalise the patterns. However, a completely different trend of results was observed on learners’ generalisation abilities. Results reveal that on average 43.3% of the participants were able to derive $n^{th}$ formulae and predict future values of the sequences. On average more than half of the participants (56.7%) struggled to derive $n^{th}$ formulae of both numeric and geometric sequences.
**Generalisation strategies**

Content analysis of learners’ written work indicated that the following strategies were used to derive general (nth term) formulae for the tasks: recursive, visualisation, difference method, explicit, guess and check and other. On average visualisation (35.9%) and differencing method (25.6%) were the most dominant strategies employed by the majority of learners. Explicit (4.8%) and guess and check (8.8%) were the least preferred strategies and the success rates of learners who utilised these strategies were relatively low compared to the other strategies. Apart from the strategies discussed in the classroom during the build-up to the assessment tasks learners devised their own strategies which were classified as “other” and the average success rate of these strategies was 12.4%. A further analysis of the “other” strategies shows that learners combined both analytic and geometric methods to generalise some sequences. A sample of learners’ responses to one of the test items is shown below:

Find the nth term of the sequence: 15; 19; 23; 27; ….

<table>
<thead>
<tr>
<th>Correct generalization strategy</th>
<th>Sample of learner’s response</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T_1 = 15 = 15 + 4(0))</td>
<td>(T_1 = 15)</td>
</tr>
<tr>
<td>(T_2 = 19 = 15 + 4(1))</td>
<td>(T_2 = 19 = 15 + 4)</td>
</tr>
<tr>
<td>(T_3 = 23 = 15 + 4(2))</td>
<td>(T_3 = 23 = 19 + 4)</td>
</tr>
<tr>
<td>(T_4 = 27 = 15 + 4(3))</td>
<td>(T_4 = 27 = 23 + 4)</td>
</tr>
<tr>
<td>..................................</td>
<td>................................</td>
</tr>
<tr>
<td>(T_n = 15 + 4(n - 1))</td>
<td>(T_n = (n - 1) + 4)</td>
</tr>
<tr>
<td>(= 4n + 11)</td>
<td></td>
</tr>
</tbody>
</table>

The learner identified that the terms of the sequence are generated by adding 4 to the previous term. Thus the pattern was correctly recognised. However, such observation could not help the learner to make a correct algebraic generalisation. The learner used \((n - 1)\) instead of \(T_{n-1}\) to denote a previous term.

**Analytic versus visual methods**

The strategies used by learners can be classified into two major categories: analytic and visual methods. Tasks 1-5 involved mainly numeric patterns, task 6 was a mixture of geometric and numeric patterns, and tasks 7-9 were mainly geometric. Results in table 1 indicate that an average success rate of 63.6% of the participants correctly recognised numeric patterns while 46% of the participants successfully generalised numeric patterns. A similar trend was observed for geometric patterns, however, a high average success rate (76%) and was observed for recognition while a low (21%) success rate was recorded for generalisation. Learners used both numeric and geometric approaches
to recognise and generalise task 6. The success rates were 62% and 54% for recognition and generalisation respectively. Thus the overall picture emerging from learners’ work shows that they can successfully recognise patterns but struggle to generalise them in an algebraic formula. Although results show a higher (63.6 % and 76 %) success rates in pattern recognition in both visual and numeric tasks, generalisations seem to be a challenge in both pattern representations. The success rates for pattern generalisation in the two representations seen to favour numeric representations. However such a claim can be ascertained by conducting a test. The results for the test are shown in table 2 below
<table>
<thead>
<tr>
<th>Activity number &amp; representation format</th>
<th>Recognition</th>
<th>Strategy</th>
<th>Generalisation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Correct</td>
<td>Incorrect</td>
<td>Recursive</td>
</tr>
<tr>
<td>1</td>
<td>Numeric</td>
<td>43</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>Numeric</td>
<td>34</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>Numeric</td>
<td>37</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>Numeric</td>
<td>19</td>
<td>31</td>
</tr>
<tr>
<td>5</td>
<td>Mixed</td>
<td>26</td>
<td>24</td>
</tr>
<tr>
<td>6</td>
<td>Geometric &amp; Visual</td>
<td>31</td>
<td>19</td>
</tr>
<tr>
<td>7</td>
<td>Geometric &amp; Visual</td>
<td>33</td>
<td>17</td>
</tr>
<tr>
<td>9</td>
<td>Geometric &amp; Visual</td>
<td>41</td>
<td>9</td>
</tr>
<tr>
<td>Overall</td>
<td>(67.6%)</td>
<td>(32.4%)</td>
<td>12.5%</td>
</tr>
</tbody>
</table>
Table 2: t-Test

<table>
<thead>
<tr>
<th>Process</th>
<th>Pattern</th>
<th>n</th>
<th>N</th>
<th>Mean</th>
<th>Success Rate (%)</th>
<th>Standard Deviation</th>
<th>Df</th>
<th>t-value</th>
<th>p-value</th>
<th>Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognition</td>
<td>Numeric</td>
<td>5</td>
<td>63.6</td>
<td>9.14</td>
<td>1.6738</td>
<td>0.132</td>
<td>0.05</td>
<td>Insignificant (p&gt;0.05)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Geometric</td>
<td>3</td>
<td>76</td>
<td>4.36</td>
<td>2</td>
<td></td>
<td></td>
<td>T-tests for the difference of means were conducted to test the effects of different pattern representations on learners’ recognition and generalizations. The results for the two tests are shown in Table 2 above (df = 6; t = 1.6738, p = 0.1327). For recognition the results tend to favour the null hypothesis and therefore suggest that there is no significant difference between learners’ recognition in numeric and geometric patterns. However, results (df = 6; t =3.0698; p=0.0372) indicate that there is a significant difference between learners’ generalisation strategies in two representation formats. We, therefore, conclude that different pattern representations have significant effects on learners’ thinking processes and generalization strategies.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalisation</td>
<td>Numeric</td>
<td>5</td>
<td>46</td>
<td>6.63</td>
<td>3.0698</td>
<td>0.037</td>
<td></td>
<td>Significant (p &lt; 0.05)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
|                  | Geometric | 3  | 21  | 4.39 | 2                |                     |      | DISCUSSION

The purpose of the study was to explore the strategies used by Grade 10 learners to derive rules for finding the nth term of a sequence and the effect of different pattern representations on learners’ generalizations strategies. The study also sought to identify the difficulties learners exhibit and why. Learners who participated in this study demonstrated little difficulties in recognising both numeric and geometric patterns. Hypothesis tests also confirmed different pattern representation formats have no effect on learners’ thinking and comprehension strategies. These findings are consistent with the findings of the study conducted by Rivera (2013) who observed that learners experience minute difficulties in recognising and extending number patterns. However, the same cannot be said about the generalisation strategies for the two formats. Learners indicate that numeric formats pose little generalisation difficulties compared to geometric ones. Learners prefer analytic approaches to mathematical activities, converting problems that have a visual nature into numbers, and work in that mode and interpret the answers in geometric context. These
observations are consistent with the findings of Samson and Schafer (2007) where learners found it easier to generalise numeric number patterns than geometric patterns. Some learners even used their own knowledge of geometric objects instead of confining themselves to the emerging pattern in question. Two competing generalisation strategies namely visualisation and differencing method tend to dominate the strategies list. Learners associated numeric patterns with the difference method even in situations where it was not applicable. A number of learners incorrectly used the difference method to generalise task 4. Learners also indicated a low competency in making interconnections between numeric and geometric in tackling pattern problems. This finding concurs with Mwakapenda (2008) who observed that learners lack skills to make important connections between the strands of mathematics. Another difficulty that emerged from learners’ work was the lack of experience in the use of algebraic symbolism to reason about and to express those pattern generalizations.

**CONCLUSION**

This paper has given an account of the strategies used by learners in pattern generalisation involving five numeric, one mixed and three geometric problems. The purpose of this study was to explore the strategies, emerging difficulties from learners’ thinking processes and effects of representational formats on learners’ generalizations. Content analysis on learners’ pattern generalisation processes suggests that it is not generalisation tasks that are difficult, but rather the format in which they are presented. The main drawback observed from learners’ generalisation strategies was over-dependency on one strategy (the difference method) and the lack of algebraic symbolism to formulate $n^{th}$ terms. The pedagogical implication for mathematics teachers is that the teaching approaches to pattern generalisation have a bearing on learners’ strategies. Most teachers and even textbooks dwell more on the difference method than any other method. In addition, learners should have adequate background knowledge of algebra and algebraic symbolisation in order to be competent in pattern generalisation.

**REFERENCES**


INTRODUCTION

There is no shortage of theories to explain the learning of mathematics with some being more valuable than others. Understanding the mental mechanisms which are associated with conceptual development in mathematics lies at the centre of the Action, Process, Object, Schema (APOS) model of learning. The theory was introduced by Ed Dubinsky in the early 1991’s (Dubinsky, 1991) in a bid to investigate the ways in which students develop understanding in advanced mathematical topics studied at university level. The theory has been applied widely to most topics in mathematics. Dubinsky and McDonald explains that “a theory of learning mathematics can help us understand the learning process by providing explanations of phenomena that we can observe in students who are trying to construct their understandings of mathematical concepts and by suggesting directions for pedagogy that can help in this learning process.” (Dubinsky & McDonald, 2001). In this paper I look at how APOS theory can help to understand barriers to learning and teaching.

I firstly focus on expanding the view of learning as changes in conceptions before presenting a more detailed explanation of APOS theory. This is then followed by a discussion on the fragility of the learning process as exemplified by one learners’ struggles with gaining fluency in multiplication. I then focus on some demands of teaching mathematics and finally reflect on how we could contribute to improving the mathematics knowledge of teachers.

Learning as changes in conception

De Lima and Tall (2008) explain that learning occurs by making new connections in the brain, which can happen when a series of actions is repeated until it becomes automatic. The authors argue that this relegation of the routine to the subconscious allows conscious thought to focus on important issues. The process of making links leads to a compression of knowledge from complicated phenomena to rich concepts with useable properties and coherent links to other ideas (de Lima & Tall, 2008, p.4). This compression of thought into “thinkable concepts” is an effort to reduce the load of working
in primary and secondary school grades classroom where students are learning mathematics by doing mathematics. Through the use of manipulatives, the child develops a methodical understanding of concepts. Lett (2007) viewed manipulatives as any hands-on object that the student can physically move in order to discover the solution to the problem. Goldin and Shteingold (2001) used smile faces (positives) and frown faces (negatives) to manipulate addition and subtraction of directed numbers.

Directed numbers is a topic that is high-risk for memorizing rules without understanding the reasoning behind the rules (Altiparmak and Özdoğan, 2010). Teaching and learning signed or directed numbers without concrete materials has always presented teachers and learners with difficulties. Seemingly there is an unchallenged assumption that the use of manipulatives in the teaching of mathematics is a key to learning about mathematics concepts, and this is supported by the mathematics manipulatives industry (Marshall and Swan, 2008). The purpose of teaching through a concrete-to-representational-to-abstract sequence of instruction is to ensure students truly have a thorough understanding of the math concepts/skills they are learning. When students who have math learning problems are allowed to first develop a concrete understanding of the math concept/skill, then they are much more likely to perform that math skill and truly understand math concepts at the abstract level (Altiparmak and Özdoğan, 2010).

A number of researchers defined manipulatives differently. McNeil and Jarvin (2007) defined manipulatives as objects used to help students understand abstract concepts in the domain of mathematics. Manipulatives can also be defined as physical objects that are used as teaching tools to engage students in the hands-on learning of mathematics (Toptaş, 2012). Lira and Ezeife (2008) also defined manipulatives as objects designed to represent explicitly and concretely mathematical ideas that are abstract. Akkan (2012) defined manipulatives as objects that appeal to several senses and that can be touched, moved about, rearranged, and otherwise handled by children. Stein and Bovalino (2001) regarded manipulatives as a way of making mathematics learning more meaningful to students. They are used to introduce, practice, or remediate a math concept and bridge the gap between informal and formal mathematics (Boggan and Harper, 2010). Students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions which can be transformed into formal concepts in later grades. Douglas and Clements (2009) acknowledged that students who use manipulatives in their mathematics classes usually outperform those who do not, although the benefits may be slight.

Literature also reveals that manipulatives improve children’s long-term and short-term retention of mathematical concepts (Boggan, Harper and Whimire, 2010). Attitudes towards mathematics are improved when students have instruction with manipulatives provided by teachers knowledgeable about their use. However, the use of manipulatives does not guarantee success. A study conducted by Moyer (2001) showed that classes using
traditional approaches outperformed classes using manipulatives on a test of transfer. At times students learn to use manipulatives only in a rote manner. They perform the correct steps, but have learned little more. Educational research indicated that the most valuable learning occurs when students actively construct their own mathematical understanding, which is often accomplished through the use of manipulatives (Seefeldt and Wasik, 2006). Kamina and Iyer (2009) summarized the benefits of concrete materials as follows: (1) they provide an additional resource in learning mathematics, (2) they help children connect with real-world knowledge, and (3) they help increase memory and understanding.

Other researchers hold the view that kinesthetic experiences can enhance perception and thinking, but argued that understanding does not travel through the finger tips and up the arm to the brains (Nathan and Koedinger, 2004). Haylock and Cockburn (2003) suggested the network of connections between concrete experiences, pictures; language and symbols could be significant to the understanding of a mathematical concept. Moyer (2001) supports the idea by stating that the active manipulation of materials allows learners to develop images that can be used in the mental manipulation of abstract concepts. Manipulatives can provide images that help pupils contextualise mathematical ideas. They also provide experiences from which pupils can abstract mathematics (National Council of Teachers of Mathematics, 2000). However Delaney (2001) warned that mental imagining of a given resource, and any action undertaken with the resource, need to be ‘internalised and used to process mathematics when the resource is not physically present’, otherwise the understanding of a mathematical concept cannot be guaranteed. In support of this Kaminski, Sloutsky and Heckler (2009) noted that students tend to focus on irrelevant aspects of the manipulatives and this pulls them away from the mathematical structure. In addition students do not see the same structure in concrete materials as experts. Thus, when learning new mathematical concepts, there is a risk that students will not see the intended structure in the manipulatives, and therefore, the materials will not effectively ground the symbolic representations.

Researchers seem to concur that effective mathematics instruction in the lower secondary school grades incorporates substantial use of manipulatives. A study conducted by Blair and Schwartz (2012) on the effectiveness of using manipulatives and produced mixed results while Bluhm (2013) revealed that concrete are effective especially with beginning learners while maintaining that older learners would not necessarily benefit from them. Evidently, just using manipulatives is not enough to guarantee success. To abstract mathematical ideas from manipulatives can be challenging. The material may be concrete, but the idea you intend that students see is not in the material. The teacher and students may not share the same idea as depicted by the manipulatives. Mathematics, like beauty, is in the eye of the beholder, and the eye sees what the mind conceives. Bergeson (2000) warned that the teacher needs to be aware of multiple interpretations of materials selected to illustrate concepts. Without this awareness one can presume that students see what we
intend they see, and communication between teacher and student can break down when students see something other than what is intended.

English and Sriraman (2010) emphasised that manipulatives must fit the cognitive level and the mathematical ability of the child otherwise they are rendered useless. He further stated that there are probably as many wrong ways to teach with manipulatives as there are to teach without them. Manipulatives, when used appropriately enable the teacher and students to have grounded conversations. Their use provides something “concrete” about which the teacher and learners can talk about. The nature of the talk should be how to think about the materials and on the meanings of various actions with them. Lyn (2012) call such conversations the connecting phase of mathematical learning. According to Brown (2009) the use of manipulatives is not a sure-fire strategy for helping children succeed in the classroom. Instead, manipulatives can help or hinder learning, depending on a number of different factors.

Durmuş and Karakirik (2006) suggested that manipulatives can be effectively used as an intermediary between the real world and the mathematical world. He contends that such use would tend to promote problem-solving ability by providing a vehicle through which children can model real-world situations. Manipulatives can also be viewed as isomorphic structures that represent the more abstract mathematical notions that children should learn (Siler & Willows, 2014). The materials should “foster children’s concepts of numbers and operations, patterns, geometry, measurement, data analysis, problem solving, reasoning, connections, and representations” (Seefeldt and Wasik, 2006, p.93). The effective use of manipulatives can help students connect ideas and integrate their knowledge so that they gain a deep understanding of mathematical concepts.

The aim of this study is to investigate the role of manipulatives in teaching directed numbers. Specifically the study seeks to shed light on how students make sense of negative and positive numbers with the aid of manipulatives. The purpose of this research is to examine the effectiveness of the use of manipulatives on students’ understanding of directed numbers. It is hoped that the ideas presented in this study can be a source of on-going attempts to recreate instructional activities that enhance the use of teaching of directed numbers using manipulatives.

**PROBLEM STATEMENT**

The learning and teaching of directed numbers has at all times been problematic and the most effective teaching method has not yet been found. Directed numbers are readily accepted by learners as a natural extension of the number line. Problems arise when pupils are to carry out the four basic operations with positive and negative integers. The teaching method used in many cases is abstract and relies on ‘rules’. Rules, when not well understood
have the tendency to be forgotten and mixed up with each other. An abstract method for teaching directed numbers is accessible to a minority of secondary school pupils. To work consistently within an abstract mathematical system of formal rules requires abstract operational thinking. However this level of thinking is attained by few pupils at lower secondary level. The majority requires concrete models to make sense of the operations with positive and negative integers. Therefore this study seeks to examine the effect of the manipulatives use on students’ understanding of directed numbers.

PURPOSE OF STUDY

The purpose of this research is to examine the effectiveness of using manipulatives on students’ understanding of directed numbers. The study seeks to determine whether the use of manipulatives in mathematics yields a statistically significant increase in student performance.

OBJECTIVES OF THE STUDY

The objectives of the study were:

i. To investigate whether the use of manipulatives have an effect on students’ understanding of directed numbers.

ii. To investigate if there are any differences in performance between students who use manipulatives and those who do not.

iii. To give recommendations to the educators and policy makers on the best practices that promotes students’ understanding through the use of manipulatives.

HYPOTHESES

H1: There is a relationship between the use of manipulatives and conceptual understanding and performance (understanding and problem-solving skills) in directed numbers.

H2: There is a significant difference in performance between use of manipulatives and non-manipulatives learning materials.

THEORETICAL FRAMEWORK

A theory that guides this research builds on the constructivist assumptions that mathematical knowledge, like all knowledge, is not directly absorbed by the learner from the teacher but is actively constructed by each individual learner. Learning theorists have suggested for some time that children's concepts evolve through direct interaction with the environment, and manipulatives provide a medium through which this can happen (Parsons,
2011). For learning to be effective, it has to be active, manifested itself in a veritable explosion of ‘activity methods’ and ‘learning by doing’ in the belief that unless children are physically acting on concrete materials they could not be learning (Effie, 1997). Piaget (1973), Bruner (1966), van Hieles (1958), and Dienes (1967) developed the strongest arguments in favour of concrete models. Piaget (1971) suggested that concepts are formed by children through a reconstruction of reality, not through an imitation of it. In Piaget's view, children actively construct their mathematical knowledge by interacting with their physical and social world. Dewey (1938) argued for the provision of first hand experiences in a child's learning; Bruner (1960) indicated that knowing is a process, not a product; while Dienes (1969) suggested that children need to build or construct their own concepts from within rather than having those concepts imposed upon them. On the basis of evidence advanced by these theorists, it sounds fair to suggest that the duty of the teacher is to introduce students to symbolic representations of concepts they already possess than to teach completely new ideas (Madden, 1999).

Resnick and Ford (1981)’s associationist theory of identical elements where simple concrete tasks assist in transfer of complex learning can also be used to explain students’ understanding of mathematical concepts via manipulatives. This theory suggests that teachers should engage students in the learning process by mediating between the concrete object and the characteristics of the problem situation (Lehtinen and Hannula, 2006). Wookfolk (2008) shared the same sentiments that unless prompted or guided, learners fail to apply the problem solving procedures and learning strategies that they have mastered.

MODELS FOR TEACHING DIRECTED NUMBERS

Number line model

Number lines are often used to teach directed numbers, and are frequently included in textbooks. They are sometimes drawn free from any context, but there are also several different ‘real-life’ situations which are modelled by a number line, and with which students may already be familiar. Common models are temperature, sea-level, and building floor levels. The bottom-to-top orientation matches with students’ perception of smaller numbers being lower than bigger numbers, and the left-to-right orientation matches students’ knowledge of a non-negative number line, with 0 being at the left or start when reading from left to right. Addition and subtraction can both be modelled as movements along the number line. Multiplication can be performed easily as repeated addition: starting at the origin zero, the negative number is added a number of times corresponding to the positive number. Multiplication of negative numbers by negative numbers needs to be thought of as repeated subtraction: the first negative number is subtracted a number of times corresponding to the magnitude of the second number. Dividing a negative number by a positive number is best thought of as partition or sharing division: breaking the negative
dividend into the number of groups represented by the positive divisor, whereas dividing a
negative number by a negative makes less sense in this context: students cannot think of a
negative number of groups. Kuchemann (1981) noted that the model is extremely straight
forward and effective for addition, however, for subtraction the model is far more difficult
to use, not only because the operation is not seen as a simple sequence but also because the
meanings given to the integers differ and are not consistent with the simple meaning used
for addition.

**Debts and Assets Model**

It is possible to model positive numbers as money and negative numbers as debts. Students
are generally familiar with the idea of owing somebody money, so they have some concrete
frame of reference for this model. It is also possible to provide physical manipulatives, in
the form of model currency and debt-notes (Janvier, 1985). However, they will also have
familiar ways of working with money, which will probably not utilise negative numbers,
and may interfere with the imposition of a negative number system onto a known framework
(Rousset, 2010). Gilderdale and Kiddle (2008) referred to this model as Strange Bank
Account that uses the context of depositing and withdrawing money; however it does not
have such a strong analogy to explain all four operations. However, it can be used to
introduce the concept of directed number, and then used in conjunction with another model.

**The chip model**

The concept of negativity and positivity can be difficult to teach and explain to learners.
This is partly because, from a child’s perspective, there are no obvious examples of negative
numbers naturally found in nature. Research has indicated that this counter / chip model is
the most appropriate one to be used as it can be used to model all the four basic operations
(Kesianye, Durwaarder and Sichinga, 2001). Integers are regarded as discrete entities or
objects, constructed in such a way that the positive integers cancel out the negative integers.
The clear advantage of such a model is that the same meaning can be used for the integers
both within and across the operations of addition and subtraction, and it seems likely that
this would enhance children's understanding of subtraction in particular. In this study the
researcher used the three models alternatively. For the purpose of this study, the above three
models were used interchangeably.

**METHODOLOGY**

**Approach**

This study is explorative and uses a quantitative design to explore students’ construction of
mathematical meaning through the use of manipulatives. A quantitative research is based
on the measurements and the analysis of causal relationships between variables (Bryman, 2012). An explorative design is one in which the major emphasis is on gaining ideas and insights (Cresswell, 2013). Exploratory research is conducted to provide a better understanding of a situation. It is not designed to come up with final answers or decisions but to produce hypotheses and explanations about what is going on in a situation. The overall goal of this study fits well with the general intention of the exploratory aspect as it sought to provide a basis for formulating more precise questions about whether the use of manipulatives have an effect on student achievement in mathematics.

Participants

The study consisted of 48 randomly selected grade 8 urban high schools students around Polokwane. The research was conducted in two phases. The researchers conducted a pre-test, followed by a three week instructional period using manipulatives and then conduct a post-test.

The research variables

To achieve the objectives of this study, two variables were used. The use of manipulatives to demonstrate operations involving directed numbers was used as the independent variable and the change in academic achievement as the dependent variable. The group received the instructions on how to add, subtract, multiply and divide directed numbers using manipulatives and also were given the opportunity to use them. The lessons were conducted by the researcher in to avoid external influence.

DATA COLLECTION METHOD

The data for this study was generated using a pre-test and post-test on adding subtracting, multiplying and dividing directed numbers. The pre-test and the post-test items were different but were based on a similar scope. As in most studies, the post-test will not be identical to the pre-test. This allows for identifying change caused by the treatment by comparing before and after results from a similar test (change in performance).

Reliability and Validity

Reliability refers to the degree of consistency of the data gathering instrument in measuring that which it is supposed to measure. This degree of consistency is measured using Cronbach’s alpha coefficient. It is a measure of internal consistency that shows the degree to which all the items in a test measure the same attribute (Masitsa, 2011). It is mandatory that assessors and researchers should estimate this quantity to add validity and accuracy to the interpretation of their data (Tavakol and Dennick, 2011). In this study, the Cronbach
alpha was calculated for the 25-item test and found to be 0.78 which is viable since an acceptable value must lie between 0.70 and 0.90 (Mutodi and Ngirande, 2014).

To observe content validity, the test were constructed and structured in such a way that they were clearly articulated and directed. All statements were formulated to eliminate the possibility of misinterpretations. This was followed by a pre-tested administered to 32 students who were excluded from the participants in the main study. The identified amendments were made to ensure simplicity and clarity of some questions, making it fully understandable to the participants (Masitsa, 2011).

DATA ANALYSIS

A one-tailed paired data t-test at the 0.05 significance level was used to test whether there is a significance difference in performance between pre-test and the post-test performance. A significant change that is equal to or lesser than the level .05 would indicate that there was reason to reject the null hypothesis with at least a 95% confidence level. To find whether there is a relationship between the uses of concrete teaching and learning materials and conceptual understanding and performance (understanding and problem-solving skills) in directed numbers, Pearson Product Moment correlations were used.

RESULTS

Forty (40) students participated in this quasi-experimental study. The students were not taught how to add, subtract, multiply and divide directed numbers using manipulatives prior to taking the pre-test. The students were taught how to carry operations on directed numbers using manipulatives. The post-test was similar to the pre-test but not identical. Students were given as much time as needed to take both test. Both tests were scored based on the number of correct answers attained with 100% being a perfect score. The changes between the pre-test and post-test scores were calculated for this group of students to determine the improvement factor.

Descriptive statistics

<Insert Table 1 about here>

The mean score for the pre-test was 25.20% while the mean score for the post test was 46.68%. The mean score increased by 21.35% while the standard deviation decreased by 3.145. The t-test score for the difference between the two means was 12.36. The standard deviation for the pre-test was 10.994 which indicate that there was little variability before the students were taught using concrete materials. The post-test had a higher variability compared to the pre-test, suggesting that that instruction had an impact on students’
performance. Such preliminary findings can only be confirmed by conducting a hypothesis test.

The results in table 2 above indicate that the performance of 90% of the students increased while 7.5% of the students showed reduced performance. There was no change in performance for only 1 student (2.5%).

**Inferential statistics**

A t-test was conducted to compare the differences between the two means. Results are shown in Table 1 above. Results from table 1 indicate the mean score for the pre-test was 33.67% (M_{pre-test}= 25, 20%, N_{pre-test} =40, sd_{pre-test} = 10.994), which was significantly lower than the mean score of M_{post-test}=46.68%, N_{post-test} =40, sd_{post-test} = 14.139) obtained from the post-test. The null hypotheses stated that the use of manipulatives does not create a significant increase in students’ mathematic test scores at a .05 level of significance. A t-test for the difference between the two means was conducted. The results for the test are shown in table 3 (df =39, t = 12.648, p=0.00). Therefore, the null hypothesis was rejected since the p-value is less than 0.05. Hence we conclude that there is a significant difference in performance between pre-test and post-test.

The results are further confirmed by the confidence interval test. When a 95% confidence interval is constructed, all values in the interval {18.073; 24.887} are considered plausible values for the parameter being estimated. Values outside the interval are rejected as relatively implausible. In this test the hypothesized value (t = 12.64) falls outside the confidence interval so we can reject H_0 at the 5% significance level (α = .05). We conclude that the use of manipulatives in teaching directed numbers has an effect on understanding and retention.

Pearson Correlation analysis was used to determine the strength and direction of the relationships. Correlation is an approach to the analysis of relationships between two quantitative variables (i.e. manipulatives and mathematics performance). Pearson's correlation coefficient (r) is a measure of the strength and direction of the relationship between the two variables. Table 4 shows that there is a positive relationship between post-test performance and the use of manipulatives mathematics performance (r = 0.673, p=0.000). This means that the use of manipulatives has a significant effect on performance.
The results are compatible with findings from a research conducted by Blair and Schwartz (2012) which revealed that the use of manipulatives in teaching directed numbers has a positive effect on performance. The results of the study also yields a statistically significant positive correlations with change in performance due to the use of manipulatives and performance ($r = 0.553, p = 0.000$). This means that the more a student is exposed to manipulatives, the greater the performance.

**DISCUSSION**

The purpose of this study was to examine the effectiveness of manipulatives on students’ understanding of directed numbers. It was also hypothesized that the use of manipulatives in mathematics yields a statistically significant increase in student performance. Due to the use of manipulatives; the participants recorded an increase in their performance, showed more interest and enjoyed the lessons. The students not only learned but also had the ability to construct their own knowledge and develop a fundamental understanding through hands-on approach. Students grew more confidence and interest and had an opportunity to sharpen in their mathematics skills. This form of knowledge ensures that students will take more ownership in their learning. It is this act of manipulation that allows for connections to be made through the different experiences (French, 2007).

Moyer (2001) supports this by stating that it is the active manipulation of materials that ‘allows learners to develop a repertoire of images that can be used in the mental manipulation of abstract concepts’ (p: 176). In support of this view, National Council of Teachers of Mathematics (2000) showed that the use of manipulatives can also provide images that help pupils contextualise mathematical ideas and support retention. Delaney (2001) suggests the need for teachers to plan activities that encourage children to develop mental images that can be internalised and used to process mathematics when the resource is not physically present.

**CONCLUSIONS, RECOMMENDATIONS AND IMPLICATIONS FOR TEACHING**

Manipulatives can be effective aids to put across a mathematical idea and to students’ understanding. But their effectiveness depends upon what the teacher is trying to achieve. To draw maximum benefit from students’ use of manipulatives, teachers need to continually align their activities with the question, “What do I want my students to understand?” Manipulatives have an important role to play in allowing teachers to model or demonstrate representations of a mathematical concept or idea, and in supporting children’s developing mathematical understanding and thinking. Their effectiveness as teaching and learning resource depends on teacher’s understanding of how the particular representation helps the learner to develop mental imagery. The process of abstracting mathematical ideas from
Manipulatives is the most difficult for many students. Teachers need to assist learners to abstract the mathematical ideas from the materials.

Manipulatives are important when teaching for the conceptual understanding. However, teachers must bear in mind that that manipulatives should be used selectively as a means of facilitating the ongoing transition from the concrete (physical and visual) stage in student learning, to the more abstract knowledge and deep understanding. Other factors such the learner’s mental age and background knowledge should be factored in when selecting the materials for teaching a mathematical concept. Manipulatives and models also afford learners greater access to language and mathematical terminology. A physical representation of a mathematical idea or solution might provide learners with greater confidence in his or her solution. New vocabulary in a new topic is easier to learn when used in the context of manipulatives. Working with manipulatives deepens understanding of concepts and relationships, makes skills practice meaningful, and leads to retention and application of information in new problem-solving situations. In turn, the valuable time spent on concrete lessons has the sustained, long-term effect of building student confidence and deepening mathematics understanding.

Apart from the improved performance and retention observed after using manipulatives to teach directed numbers other benefits such as improved communication, less mathematical anxiety in a mathematics classroom were noted. Exploring manipulatives, especially self-directed exploration, provides an exciting classroom environment and promotes in students a positive attitude toward learning. Finally manipulatives make learning fun and enjoyable.

REFERENCES


List of Tables

Table 1: Descriptive data for test results

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<td>+</td>
<td>_</td>
<td>+</td>
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<td>+</td>
<td>+</td>
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Performance Summary

<table>
<thead>
<tr>
<th>Group</th>
<th>Number</th>
<th>Mean</th>
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</tr>
</thead>
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<tr>
<td>Post-test</td>
<td>40</td>
<td>46.68</td>
<td>14.139</td>
<td>******</td>
</tr>
<tr>
<td>Pre-test</td>
<td>40</td>
<td>25.20</td>
<td>10.994</td>
<td>******</td>
</tr>
<tr>
<td>Post-test - Pre-test(d_i)</td>
<td>40</td>
<td>21.35</td>
<td>12.879</td>
<td>12.36</td>
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</table>

Table 2: Performance

<table>
<thead>
<tr>
<th>Performance</th>
<th>Frequency</th>
<th>Percentages</th>
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<tr>
<td>Decreased</td>
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<td>7.5</td>
</tr>
<tr>
<td>Increased</td>
<td>36</td>
<td>90</td>
</tr>
<tr>
<td>No Change</td>
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<td>2.5</td>
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<td><strong>Total</strong></td>
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Table 3: Testing for the means

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<tr>
<th>Variable</th>
<th>T</th>
<th>Df</th>
<th>SE</th>
<th>Sig</th>
<th>Mean Diff</th>
<th>95% confidence Interval</th>
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</thead>
<tbody>
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<td>Performance</td>
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<td>39</td>
<td>1.7383</td>
<td>0.000</td>
<td>21.48</td>
<td>Lower 18.073 Upper 24.887</td>
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<td>Effect of manipulatives</td>
<td>Performance</td>
<td>Pre-test</td>
<td>Post-test</td>
<td></td>
<td></td>
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<tr>
<td>--------------------------</td>
<td>-------------------------</td>
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<td>----------</td>
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<td></td>
<td></td>
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<tr>
<td>Effect of manipulatives</td>
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<td>.553**</td>
<td>.307</td>
<td>.673**</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>Sig. (2-tailed)</td>
<td></td>
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<td></td>
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<tr>
<td></td>
<td>.000</td>
<td>.000</td>
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<td>.000</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>N</td>
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<td>40</td>
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</tr>
<tr>
<td>Performance</td>
<td>Pearson Correlation</td>
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<td>.414**</td>
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</tr>
<tr>
<td></td>
<td>Sig. (2-tailed)</td>
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<tr>
<td>Pre-test</td>
<td>Pearson Correlation</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Post-test</td>
<td>Pearson Correlation</td>
<td>.673**</td>
<td>.414**</td>
<td>.497**</td>
<td>1</td>
<td></td>
</tr>
<tr>
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<td>Sig. (2-tailed)</td>
<td>.000</td>
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<td>40</td>
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</tbody>
</table>

**. Correlation is significant at the 0.01 level (2-tailed).
ON THE SOCIOLOGY OF KNOWLEDGE AND ENSURING LEARNING FOR ALL IN SOUTH AFRICAN PRIMARY MATHS EDUCATION

Peter Pausigere
South African Numeracy Chair, Rhodes University

This paper focuses on South Africa’s recent primary maths education curriculum restructuring in relation to the classification of knowledge. It discusses how the current educational changes at the primary level embody strongly classified maths knowledge. Sociologically, such knowledge orientations resonate with middle class learners. Analysing local primary maths policy documents and drawing from Bernstein’s central thesis about the social class basis of education and using the concept of classification, the study explains how strongly classified intra-connected primary maths knowledge intends to provide access to specialised and powerful knowledge to children of different social classes.

INTRODUCTION

This article focuses on the social class knowledge assumptions in the recently restructured South African primary maths education’s Curriculum and Assessment Policy Statements (CAPS). It investigates the social class ideologies dominant in the local primary maths knowledge curriculum and explains how such class-based interests can be interrupted to enable learning for all children. Drawing from Bernstein’s (1975, 1990, 2000) sociology of education theory and specifically from his central arguments about the social class nature of education and using the concept of knowledge classification, the paper explains how middle class ideologies and interests are foregrounded in the local primary maths knowledge curriculum, as revealed in CAPS.

The study discusses possibilities for interrupting middle class cultural reproduction tendencies and thus enables increased knowledge access to working class learners in South African primary maths classes. Whilst generally social class inequalities have been reported in South African primary maths education (Fleisch, 2008; Hoadley; 2007), evidence from the 2012 to 2014 Annual National Assessment (ANA) results and their analysis in relation to social class (poverty quintile) indicates that 40% of the learners in fee-paying middle class schools achieve considerably better marks than 60% of the disadvantaged learners in public schools (DBE, 2012; 2013; 2014). Large scale national research from the 2001 and 2008 Department of Education systemic evaluations, findings from TIMSS 1999 and 2003, the OECD education policy review and the National Planning Commission diagnostic overview cite educational inequality as a critical issue (Graven, 2014). Official statistics also confirm that two thirds of local children live in poverty stricken households, with South Africa having been noted as one of the countries with the highest and extreme levels of social and economic
inequalities in the World for so long (Spreen and Valley, 2014; Graven, 2014). Thus marginal groups’ learners (lower ability students and working-class children) are given differential access to knowledge and are ironically regarded as ‘failed mathematicians’ (Muller and Taylor, 2000 in Hoadley, 2007, p. 682). This evidence prompts this study to investigate how local primary maths knowledge may be offering unequal chances of success for children of different social classes. Thus this study interrogates two key research questions:

*What are the social class assumptions in the South African primary maths knowledge curriculum?*

*How can these dominant social class group knowledge ideologies be revoked in formal education experiences?*

These two research questions and the educational knowledge code category of *knowledge classification* and how it relates to social class help structure the analysis and ensuing discussion in this paper. Informed by Bernstein’s sociological theoretical perspective, policy document analysis and key primary maths education literature the study explains how strongly classified and connected primary maths knowledge ensures access to powerful and specialised forms of knowledge for children from different social class groups.

**THEORETICAL FRAMEWORK: THE SOCIOLOGY OF KNOWLEDGE**

This study draws on Bernstein’s (1975, 1990, 2000) theory about the sociological nature of knowledge, which analyses how middle class culture and ideologies are reproduced through education. Bernstein’s concept of classification will be used to interrogate and understand knowledge structures and the social class assumptions within the South African primary maths education as revealed in curriculum policy documents. Elsewhere I have (Pausigere and Graven, 2013) examined official teacher identities promoted in curriculum documents using a Bernsteinian lens. Bernstein’s (1990, 2000) structuralists approach postulate that school knowledge and official pedagogic practices privileges middle class interests and ideologies, which marginalises working class children. The argument that curriculum has a middle class social base and origin is also noted in education literature (Maton and Muller, 2006; Spreen and Vally, 2014; Cooper and Dunne, 1998) and is illustrated locally in Hoadley’s (2007) primary maths education empirical study. It is important to explain Bernstein’s concept of classification as this will help illuminate how middle class socialisation articulates and relates with school knowledge.
The classification of knowledge

Classification refers to the relationship between subject categories or contents (Bernstein, 2000). With strong classification, areas of knowledge and subject contents are well insulated into traditional subjects and this gives rise to a collection code (Bernstein, 1975). Weak classification refers to an integrated curriculum with blurred boundaries or reduced insulation between contents (Bernstein, 1975). According to Bernstein, integrated codes had a stronger ideological life in the late 60s and early 70s in Great Britain with collection codes prevalent in Britain before the 1960s. Locally an integrated code was evident in the 1997 Outcomes Based Education curriculum reforms, with the recent educational changes indicating strong classification of collection codes.

Besides the relations between subject contents, another key aspect of classification is the strength of the boundary between educational and everyday knowledge. Thus in his later work Bernstein (2000) distinguished between horizontal (everyday knowledge) and vertical (school knowledge) discourses, with the latter consisting of specialised symbolic structures of explicit knowledge. Such content derives from the parent discipline and its internal determinative logic. Furthermore and importantly for this study school discourse has a ‘verticality dimension’ which is characterised by hierarchical integration and subsumption of knowledge (Bernstein, 2000; Maton and Muller, 2006; Muller, 2007). The hierarchical integrative aspect of educational knowledge enables greater combinatorial power within knowledge forms (Muller, 2007). As the subject of maths is within the modalities of horizontal knowledge structures of vertical discourses the study will explain how hierarchical intra-subject connections can benefit primary maths education and the equality cause. Some of the most influential primary maths studies have pointed that connected primary maths concepts and knowledge supports conceptual understanding (Kilpatrick et al, 2001) and enables learners to be numerate (Askew et al, 1997).

The sociology of knowledge classification

Both strong and weak classification respectively carries old middle and new middle\(^{20}\) class knowledge-orientation perspectives (Bernstein, 1975). The middle class

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\(^{20}\) The new middle class consists of people who work in the service provision industry such as teachers, police officers, nurses, lawyers etc. and old middle class is made of employees who work in the production and distribution of goods and services for example retail proprietors, business managers, commercial farmers, technicians. The working class is made up of skilled and unskilled manual workers such as farm labourers, miners, fishery workers, shop floor workers, bricklayers, security guards,
sociological account of knowledge is important for this study and how working class children can access primary maths knowledge. However the middle class schooling narrative has resulted in one of the oldest education dilemmas for the working class learners. They are sentiments that weak classification legitimatisate the inclusion of the working class culture at school and provides opportunities for knowledge acquisition (Bernstein, 1975; 2000). Furthermore weak knowledge classification aligns with both the progressive agenda aimed at empowering learners and education’s social justice imperatives (Hoadley, 2007; Spreen and Vally, 2014).

However weakly classified knowledge might lead to the perpetuation of inequalities through limited access to specialised and powerful forms of knowledge (Bernstein, 2000; Hoadley, 2007; Muller and Maton, 2006; Muller, 2007). Worse still weakly classified knowledge creates contextual and ‘orientation to meaning’ ambiguities which lead marginal classes’ children to misrecognise school knowledge (Hoadley, 2007, p. 680; Bernstein, 2000). The need for higher order cognitive competences is found in the working class parents’ preferences for strong classification of the ‘visible pedagogy of the collection code at the primary level’ (Bernstein, 1975, p. 127). Taking note of the sociological theoretical perspectives and empirical research in primary maths and science the study will argue that strong classified and intra-connected primary maths knowledge provides epistemological access for all learners.

RESEARCH METHODOLOGY

The research methodology used in this study is educational policy sociology research. This methodological approach has its roots and origins in policy document analysis and the modern sociology of education (Ball, 1997). This study thus analyses South African primary maths policy documents using Bernstein sociological framework. The key local primary maths education policy documents analysed in these study are the CAPS’ Foundation and Intermediate phases’ primary maths subject guidelines, the Foundations for learning campaign and ANA reports. Curriculum policy documents reveal the officially valued ‘knowledge and skills’ which embed national ideologies and social class assumptions. As Bernstein’s work generally focuses on the social class nature of education it provides the study with conceptual, descriptive and analytical tools to examine the interrelationship between social class and primary maths knowledge in the South African education policy context.

The analysis, synthesise and presentation of information obtained from policy documents was theory-driven and informed by the research questions. The coding and domestic workers etc.
exploration of primary maths education policy documents data was guided by Bernstein’s theoretical perspective about the sociology of knowledge and the classification concept. Thus the two research questions underpinning this study and the sociologically reinterpreted primary maths policy documents help structure the analysis and ensuing discussion in this paper.

DISCUSSION

The Sociology Of South African Primary Maths Knowledge

This section of the paper discusses the research findings of this study interpreted through Bernstein’s thesis about the social class nature of knowledge and his key concept of classification. Thus in the purview of the research questions, sociological theoretical underpinnings and pertinent literature I explain how the recent CAPS curriculum changes at the primary level are orientated towards strongly classified primary maths knowledge which has a middle class social origin. The ensuing discussion argues for strongly classified and intra-connected primary maths knowledge which provides access to disciplinary knowledge and enhances conceptual understanding to learners of different social classes.

The South African primary maths’ knowledge strong classification sociological assumptions

South Africa’s primary maths policies emphasise the need for the acquisition of key mathematics conceptual knowledge. Across foundation and intermediate phases the local primary maths consists of the key content area of numbers; geometry (space and shape); functions, patterns and algebra; measurement and statistics (DBE, 2011a; DBE, 2011b). These five content areas relate with some of the key branches of mathematics, for example arithmetics (number theory), geometry and algebra and sub-divisions within the mathematics discipline such as measure theory and statistics. It has been theoretically noted that content-rich subjects within horizontal knowledge structures (maths, science, logic, and physics) derive their contents from the parent discipline (Bernstein, 2000; Muller, 2007). Similarly the recontextualised South African primary school maths derives its elementary-level refocused contents from the disciplines and sub-disciplines of mathematics. Recontextualising from a horizontal knowledge structure parent discipline of the strongest grammars serves to indicate the strong classification within the local primary maths knowledge. The old middle class prefers strong classification for it was ‘domesticated through strong classification...of the family and public schools’ and this leads to ‘cultural reproduction’ (Bernstein, 1975, p.121). Thus the key contents of the local primary maths indicate strong classification
which has been sociologically explained as embedding middle class ideological interests.

Strong classification is also indicated in the policy documents specification of the mathematical competences to be acquired from each content area. The acquisition of key foundational mathematical skills is highly prioritised in the recently introduced primary maths education policies. The main content area of ‘numbers, operations and relations’ which makes 60% and 50% of the foundation- and intermediate-phase mathematics content ensures that learners are ‘numerate’ and thus acquire ‘secure number sense and operational fluency’ (DBE, 2011, p. 8; DBE, 2012; DBE, 2013). Mental mathematics is also highlighted as promoting the development of number sense and number concepts through it’s emphasise on ‘number bonds and multiplication table facts’ (DBE, 2011a: 8; DBE, 2012; DBE, 2013; DOE, 2008). The content areas of data handling (statistics) and measurement provide contextualised opportunities for the use, understanding and development of numbers (DBE, 2011a; DBE, 2011b). The local primary mathematics curriculum documents, also ensures that learners engage in problem-solving activities, thereby enabling for the understanding of higher order mathematical concepts (DBE, 2011a; DBE, 2011b). The content areas of geometry and algebra are noted as essential for providing a foundation for spatial understanding and developing formal algebraic work (DBE, 2011a; DBE, 2011b). The South African primary mathematics education broadly focuses on improving the learners’ number sense, operational fluency and the development and understanding of numbers and higher order mathematical concepts. The development of such fundamental mathematical skills have been identified in influential and international primary mathematics studies as being central for developing learners’ mathematical proficiency (Kilpatrick et al, 2001; Askew et al, 1997). Furthermore the local primary maths documents highlight the importance of geometry and algebra which help in laying the basic skills for understanding proofs, abstract mathematical concepts and formalised algebra in secondary school. Theoretically an orientation to key critical concepts and skills of primary school mathematics results in learners to attain and acquire ‘states of knowledge’ (Bernstein, 1975, p. 92). Such a focus upon the basic and fundamental skills of primary school mathematics results in strong classification. From a knowledge social-class base perspective the higher status skills codes modality favours the dominant class with poor working class learners experiencing difficulties in recognising such key concepts (Maton and Muller, 2007). Similarly the mathematical competences and skills emphasised in local primary maths classes represent the ideological experiences of the middle class families.

The conceptual progression within the South African primary maths curriculum also shows strong knowledge classification (Pausigere and Graven, 2013). The
foregrounded core primary maths content and highlighted key mathematical skills have marked progression across the primary grades. This conceptual progression is divorced from relational ideas, everyday knowledge or the notion of relevance characterising integrated codes of weak classification. Thus number ranges increases, the introduction of different kind of numbers and the need to develop more efficient calculation strategies underpins the disciplinary progression in the main content area (DBE, 2011a; 2011b). The sequential conceptual development in algebra occurs ‘in the range and complexity of relationships between numbers in the patterns’ across the phases (DBE, 2011a, p. 18; DBE, 2011b). In the space and shape content area knowledge progression is achieved by focusing on new properties and features of shapes and objects, incrementally within grades. Conceptual development in the area of measurement is achieved through the introduction of new forms of measuring and new measuring units across the grades. Progression in data handling across the primary grades is attained by working with new forms of and new analytical tools for representing and reporting data (DBE, 2011a; 2011b). Knowledge progression in the primary level is underpinned by the key mathematics content areas and foregrounds the critical numeracy skills. It has been noted that conceptual progression in collection codes proceeds from surface to deep knowledge structures (Bernstein, 1975), in fact “true” knowledge is characterised by knowledge progression’ (Muller, 2007, p.70). Locally this is revealed in the sequential development of disciplinary content and skills in primary maths education subject guidelines. Such marked sequential knowledge progression is characteristic of strong classification which leaves ‘the elite having access to the deep structures’ of knowledge, thus ‘access to realising of new realities’ (Bernstein, 1975, p. 92). The conceptual progression within the local primary maths curriculum shows strong knowledge classification whose assumptions resonate with the middle class.

South Africa’s primary maths policies emphasise on key mathematics conceptual knowledge, the acquisition of critical numeracy skills underpinned by sequential conceptual development and progression reflect strong knowledge classification. Sociologically this strong classification and structuring of local primary maths knowledge embed middle class assumptions. The next and final part of the paper answers the second research question and in the process explains how strongly classified intra-connected primary maths knowledge provides access to disciplinary knowledge and enhances conceptual understanding to children of different social classes.
Primary maths’ knowledge that provides access to children of different social classes – strong classification

Besides having a middle class bias, strong classified knowledge is still the sine qua non for the South African primary maths equality cause. As local primary maths policies foregrounds key mathematical conceptual knowledge and critical numeracy skills – the transmission of ‘basic competences’ of the ‘collection code at the primary level’ are preferred by working class parents for that is the core, ‘impersonal’ and intellectual function of schools (Bernstein, 1975, p. 127-128). Sociologists of education disdain relegating subordinate social groups to lower status forms of educational knowledge, as these perpetuate social class inequalities and creates contextual and orientation to meaning ambiguities which lead working class children not to recognise horizontal knowledge structures (Maton and Muller, 2006; Bernstein, 2000; Hoadley, 2007). Thus there has been consensus that strongly classified vertical discourses provide epistemological access to ‘powerful’ and ‘specialised’ forms of knowledge to poor children (Muller, 2007, p. 80; Bernstein, 2000, p. 157). The need for exposing working class children to ‘specialised knowledge of (primary) mathematics’ is also illustrated in Hoadley’s (2007, p. 702) empirical study. Similarly this study argues that the strong classification evident in the emphasis on key primary mathematical competences marked by sequential progression provides knowledge access to children of all social classes in South Africa. However the strong classification requirement only, is not enough – it’s one part of the effort to redress knowledge social class inequalities.

Intra-connected primary maths knowledge

The argument for the need for intra-connected primary maths knowledge emanates from sociological knowledge perspectives (Bernstein, 2000; Maton and Muller, 2006; Muller, 2007), empirical research in primary maths (Askew et al, 1997; Kilpatrick et al, 2001) and primary science studies (Morais, Neves and Pires, 2004). The lack of emphasises on illustrations of connections between different mathematical ideas in local primary maths subject guidelines also instigated the need to explore such opportunities. Drawing from these studies and policy document analysis the paper explains the case for connected mathematical concepts for providing knowledge access for all learners.

Sociological the conceptual integration of knowledge within horizontal knowledge structures is called ‘verticality’ (Muller, 2007). The principle of verticality explores ‘relations within knowledge forms’ and focuses on the development of knowledge structures through ‘the integration and subsumption of knowledge into more overarching… propositions’ (Muller, 2007, p. 70; Maton and Muller, 2006, p. 25). Bernstein (2000, p. 161) clarifies that such integration is at the ‘level of meanings’ and not between ‘segments or contexts as in horizontal discourse’. For illustrative purposes
I relate the proposition about the integrative nature of knowledge to one of the disciplines of mathematics. For example the area of arithmetics is made up of the following sub disciplines; numbers, the four basic arithmetic operations, decimal arithmetics, compound unit arithmetics and elementary statistics. These different categories of arithmetics show connections and relations within number theory. Thus these sub-branches are integrated and subsumed in the more overarching and generalising mathematical discipline of arithmetics. This relates to what was stated by Bernstein (2000) that integration within domains is at the level of meaning and procedures-hierarchically rather than segmental or contextual connections. Such within discipline connections are a key aspect of the structure and growth of horizontal knowledge structures.

Similarly the importance of primary maths connections in enabling conceptual understanding and supporting learners in being numerate is noted in the two most influential studies in primary maths education (Kilpatrick et al, 2001; Askew et al, 1997). The significance of linking mathematical concepts and ideas is also reiterated in a local primary maths study (Askew, Hamsa and Matthews, 2012). However these studies do not explain how primary maths conceptual connections enable successful learning and understanding for children from different social classes. The importance of conceptual connection in enhancing social class equality is similarly noted in primary maths science studies (Morais, Neves and Pires, 2004). Based on longitudinal empirical research in primary maths sciences, Morais et al (2004) posits that strong inter-disciplinary connections overcome the effect of children’s social background. With both maths and science being regarded as horizontal knowledge structures with strongest grammars, I believe their knowledge structure similarities provide opportunities for this study to argue that intra-disciplinary primary maths connection can enable learning for all children.

Locally such connections can be promoted, for example, between natural numbers, integers, fractions, decimal, place value and problem solving; in the four mathematical operations of addition and subtraction, multiplication and division; or between the mathematical concepts of angles, fractions, percentages, ratios and pie charts illustrations. Within discipline connections expand and extend knowledge ‘repertoires’ and ‘reservoirs’ of working class children giving them the ability to ‘gaze’ and ‘recognise’ alternatives within the overarching knowledge principle (Bernstein, 2000). It broadens knowledge options from which learners can choose familiar alternatives to understand key concepts. The South Africa primary maths curriculum policy guidelines must make clear conceptual connections as has been the case in Britain’s primary education – and this can provide epistemological lens for a range of alternatives for disadvantaged learners as well as a source for generating and promoting ‘coherent’ maths lessons (Askew et al, 2012).
 Whilst having argued for the need for strongly classified intra-connected primary knowledge in local primary maths classes, the most important factor in enabling successful learning for children from different social background are competent teachers (Morais et al, 2004; Muller, 2007). It is the acumen of knowledgeable practitioners who can provide access and induct learners to strongly classified and intra-disciplinary connected primary maths knowledge. Such epistemological opportunities ensure learners’ conceptual understanding and mathematical proficiency to children from different social classes and most importantly it exposes them to powerful and specialised forms of knowledge.

CONCLUSION

Whilst this paper has focused on the social class base of local primary maths knowledge, further research should explore the social class assumptions within local primary maths pedagogic practices. There is also need for empirical research to show how strongly classified intra-connected primary maths knowledge overcomes the effect of children’s social background. Whilst the strong classification in local primary maths education reflects the interests of the middle class – such knowledge orientations are not detrimental for the working class children and can interrupt education social class reproduction tendencies. Informed by sociological theoretical perspectives and relevant empirical research the study also calls for intra-connected primary maths knowledge to revoke middle class ideological interests, ensure knowledge access to working class learners and narrow inequalities in South African primary maths class. The opportunities for within-discipline connections must be encouraged and made explicit in the policy curriculum documents. The primary maths epistemological positions advocated herein (and their sociological reifications) are not applicable to literacy or life skills which have different knowledge structures. Finally the call for strongly classified intra-connected primary maths knowledge require competent teachers who themselves ‘stand on the shoulders of giants’ (Muller, 2007, p. 79) and can ensure the same for learners from different social groups, thus interrupting cultural reproduction.

Acknowledgement: Thanks to Mellony Graven for her critique and feedback. This work is supported by the FirstRand Foundation (with the RMB), Anglo American Chairman’s Fund, the Department of Science and Technology and the National Research Foundation.

REFERENCES


Formative assessment – the process of eliciting and interpreting evidence about what students have learned and then using this information to make instructional decisions – is viewed by many as an essential aspect of effective instruction. In fact, *Principles to Action: Ensuring Mathematical Success for All* (NCTM, 2014) identifies *eliciting and using evidence of student thinking* as one of eight non-negotiable teaching practices critical for successful implementation of ambitious standards. According to Leahy, Lyon, Thompson and Wiliam (2005, p.19), “in a classroom that uses assessment to support learning, the divide between instruction and assessment blurs. Everything students do—such as conversing in groups, completing seatwork, answering and asking questions, working on projects, handing in homework assignments, even sitting silently and looking confused—is a potential source of information about how much they understand”.

Drawing on an episode of mathematics teaching in a middle grades classroom in the U.S., this session will illustrate some methods for collecting evidence of student thinking during a lesson and discuss ways in which these data can be used to inform short-term and long-term instructional decisions.
AN ANALYSIS OF LEARNERS’ WAYS OF WORKING TO SOLVE TRIGONOMETRIC PROBLEMS IN THE HIGH-STAKES NATIONAL SENIOR CERTIFICATE (NSC) MATHEMATICS

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School of Science and Mathematics Education, University of the Western Cape

Examination scripts of examinees who wrote November 2012 National Senior Certificate Paper 2 were analysed to unearth the ways of working they employed to solve trigonometry problems. The analysis is underpinned by the role the context of high-stakes mathematics examination exerts on examinees’ endeavours to solve the examination problems. It was found that strategies such as target-setting, reversal to address encountered resistances and reflexivity on immediate prior work structure the solution-seeking process. Suggestions are made regarding the use of such analyses for the teaching and learning of Mathematics.

INTRODUCTION

The Department of Basic Education (DBE) provides various kinds of feedback on the National Senior Certificate Mathematics examination every year. One of the reports is the diagnostic report of learners’ responses to items on the examination (see for example, Department of Basic Education, 2015). The purpose of the diagnostic report is to provide valuable information that could be used to improve teaching and learning and ultimately improve achievement results. In the diagnostic report mistakes, errors, misconceptions and other issues as forthcoming from the work of examinees are identified and pointed out. For example, in the diagnostic report for the November 2014 Mathematics examination for the first question on trigonometry, question 5, the diagnostic report (DBE, 2015, p 127) states

Common errors and misconceptions

(a) Candidates experienced difficulty in selecting and applying the correct trigonometric ratio in a right-angled triangle. They did not understand the instruction: ‘Show, by calculation, that \( x = 60^\circ \).’

(b) Many candidates did not make the relationship between the sides and angles of the triangles. Hence they did not know when to use the sine and cosine formulae. It was very disappointing that some candidates used the sine ratio in the cosine formula:

\[
AD^2 = AP^2 + DP^2 - 2(AP)(DP)\sin \angle PDA. \]

This formula is given in the information sheet. Some candidates did not understand that PA bisected
Formative assessment – the process of eliciting and interpreting evidence about what students have learned and then using this information to make instructional decisions -- is viewed by many as an essential aspect of effective instruction. In fact, *Principles to Action: Ensuring Mathematical Success for All* (NCTM, 2014) identifies *eliciting and using evidence of student thinking* as one of eight non-negotiable teaching practices critical for successful implementation of ambitious standards. According to Leahey, Lyon, Thompson and Wiliam (2005, p.19), “in a classroom that uses assessment to support learning, the divide between instruction and assessment blurs. Everything students do—such as conversing in groups, completing seatwork, answering and asking questions, working on projects, handing in homework assignments, even sitting silently and looking confused—is a potential source of information about how much they understand”.

Drawing on an episode of mathematics teaching in a middle grades classroom in the U.S., this session will illustrate some methods for collecting evidence of student thinking during a lesson and discuss ways in which these data can be used to inform short-term and long-term instructional decisions.
A Brief Literature Review On The Teaching And Learning Of Trigonometry

Trigonometry requires a number of sophisticated mathematical concepts such as, the association of numbers with sides of a triangle representing length and the measure of the angle; ideas of exponential functions of a triangle and symbolic notation (Raj and Nega 2011). Real world problems involving trigonometry is common in engineering, physics, construction and design. For learners to pursue these fields of study they must develop a background of trigonometric knowledge in school.

Trigonometry is an area of mathematics that students struggle with (Gür, 2009). Orhun (2002) studied the difficulties faced by learners when using trigonometry for solving problems. S/he found that the learners did not develop the concept of trigonometry and that they made some mistakes. Gür (2009) suggests three generalize misconceptions related to trigonometry: Firstly, many misconceptions are related to a concept that produces a mathematical object and symbol. For example: sine is a concept and a symbol of trigonometry. Secondly, misconceptions are related to processes. For example: as in representing the results of calculation of $\sin 30^\circ$ and the value of $\sin x$. Lastly, many misconceptions are related to precepts that are the ability to think of mathematical operations and objects. For example: $\sin x$ are both a function and a value (Gür, 2009).

Brown (2006, p. 228) studied students’ understanding of sine and cosine. She came up with a framework called, trigonometric connection. Her study indicated that many students had an incomplete or fragmented understanding of the three major ways to view sine and cosine: as coordinates of a point on the unit circle, as a horizontal and vertical distance that are graphical entailments of those coordinates and ratios of sides of a reference triangle. Delice (2002) identified levels for measurement of students, knowledge and defining students’ skills when working with trigonometry. The research showed that students have misconceptions and learning complexities, which are attributed to the fact that before learning the trigonometric concepts, the students learn some concepts, pre-requisite concepts, incorrectly or defectively.

From a South African mathematics curriculum perspective, De Villiers and Jugmohan’s (2012) research found that learners have little understanding of underlying principles of trigonometry. Furthermore, Chauke (2013) points out that the grade 12 mathematics students’ in Gauteng province had difficulty with trigonometric functions in the end-of-year examination.

Thomson (2008) aptly sums up that trigonometry is notoriously difficult for middle-school and secondary school students in the United State of America (USA). He claims
that the difficulty lies in an incoherence of foundational meanings developed in the lower grades through to grade 10.

Although the research reported above is concerned primarily with the difficulties learners experience with trigonometry, it does not address how examinees actually go about to produce the responses that they produce in high-stakes Mathematics examination settings. This particular “research gap” is the cornerstone of the research reported in this article.

Since, as alluded to in the aforementioned sentence, responses to high-stakes Mathematics items is produced in a particular setting or context the next section describes the theoretical antecedents of import used in this study.

Underlying Theoretical Constructs

According to Lave (1993) knowledge and learning will be found distributed throughout the complex structure of person-acting-in-setting. Knorr Cetina (1999) offers the notion that knowledge should be seen as something that is being produced in a particular context. In line with these notions, the responses to items in a high-stakes Mathematics examination are produced within this high-stakes Mathematics examination context.

The high-stakes examination context for school examinations is a particular context. It comprises the examinees, invigilators, the examination question paper with a formula sheet — the problem text with the questions and the answer book within which the work of the examinee is recorded. Another component of the context is that learners have access to calculators and normally the invigilators have some device to indicate to examinees the time that have elapsed and the amount of time they have available for the completion of the time-restricted examination. Crucially present in the context is the examinees. Since the examination is at the end of the schooling career of the learners who reached grade 12, they are present with their entire school mathematics history. Thus the school mathematically-historicised examinees are present as agents and they bring to bear their entire experience of school mathematical work to seek solutions to the problems posed in the examination.

The theoretical constructs of importance for this work emanate from ethnomethodology and the social study of scientific work. “Ethnomethodology is the study of the ways in which ordinary people use commonsense knowledge, procedures, and considerations to gain an understanding of, the world navigate in everyday situations” Garfinkel (1967,
According to Lynch (1993) the word ethnomethodology means literally “folk investigation of the principles or procedures of a practice”. Ethnomethodology analyses the methods or procedures members of society use for conducting the different affairs that they accomplish in their ordinary daily actions. It is concerned with what Garfinkel (1967, p. 1) has calls “practical sociological action and reasoning” with the objective to extract social facts from the practical social action. In terms of the research reported in this article, this sociological action and reasoning is the production of learners’ responses to questions in high-stakes mathematical examinations. A further perspective that is drawn on is sociology of scientific practices (Lynch, 1993).

Various studies related to the production of Mathematics were conducted within the two perspectives referred to above. A seminal study is that of Livingston (1986) who demonstrated how a proof is developed through the interaction of the ongoing actions of the prover, his/her written recordings on say a chalkboard or on paper of the progress being made and the final rendition of the finished product. Of importance for the research being reported here is Livingston’s (1986) differentiation between the “lived work” and “the account” of a proof. The account of the proof is what is recorded in textbooks or research papers whilst the “lived work” is that which are characterised as rough work.

In addition to Livingston’s (1986, 2006) extensive work on proofs in Mathematics, research into the ways of working in Mathematics from ethnomethodological and social study of science stances are also reported by Pickering (1995), Merz and Knorr-Cetina (1997), Julie (2003), Greiffenhagen (2008) and Roth (2012). These studies cover a diverse range of contexts within which mathematical work is done. Pickering (1995) studied historical documents on Hamilton’s construction of quaternions to account how disciplinary agency drives the meaning-making process through the “the dialectics of resistance and accommodation” (p. 22). Merz & Knorr-Cetina (1997) focussed on theoretical physics done at the Theory Division at the European Laboratory for Particle Physics (CERN). They view theoretical physics as a “thinking science” akin to Mathematics. The development of a mathematical model by practising teachers was the focus of Julie’s (2003) research. Greiffenhagen (2008) studied the production of Mathematics by a lecturer during actual teaching and Roth (2012) followed a track similar to Livingston (1986) by considering the development of the theorem “the sum of the internal angles of a triangle is 180°”. The constructs used to depict the ways of working are diverse and Livingston (2006) calls for “a vocabulary that provides descriptive insight into the lived, perceived details and reasoning of mathematical discovery work.” (p. 63). Some of the extant vocabulary emanating from these studies and, where needed, postulate new ones that can be usefully employed to depict examinees’ ways of working in high-stakes examinations.
The data corpus for this study was the answer books of 250 learners who wrote the November 2012 NSC Mathematics 2nd paper. These learners were from schools whose teachers participate in a professional development project focusing on the development of teaching Mathematics (Julie, 2013). As stated above the focus is on trigonometry questions and the examinees’ work on these questions is the selected data. To initiate the analysis 10% of the questions papers were selected and the responses focused on in this article are those which through inspection show clearly a combination of the “lived” work and the account of the work as represented by the final answers presented. Indicators of the “lived” work are sections of work scratched out and new attempts being made to obtain a response for the examination item. Only two such items are focused on in this article due to lack of space. As will become clear work of this nature is detailed and Greiffenhagen (2008), for example, uses an entire page to lay bare a summary of the way of working for column addition of 23, 45, 49, 11, 87, 35, 14, 23 and 19.

**A Description Of The Examinees’ Ways Of Working As Results**

In this section two examples of the examinees’ ways of working from the ethnomethodological perspective and the accompanying social study of science are presented. The constructs embedded in these perspectives are used and where the extant constructs were not found adequate for describing the ways of working, new ones are postulated. Crucial in this description is the role the elements of the high-stakes examination context described above plays in the process of the examinees struggle to find responses to the items being used.

*Example 1:*

The examination question and an examinee’s response is given in figures 1, 2(a) and 2(b).

![Figure 1: Question 8.2.2 (DBE, November 2012)](image-url)

**Figure 1: Question 8.2.2 (DBE, November 2012)**
In Figure 2(a) the problem statement is copied in line 1 after the question number, 8.2.1. A scan of the entire script (figures 2(a) & (b) indicates that the examinee has set the target to reach through simplification of the left-hand side. This quest led to something akin “I must get rid of the 1 in the numerator.” This “get rid of” is a common colloquial form used in school mathematics. We infer that the mathematically historicised-self in terms of this common colloquial form exerts its agency which renders the 3rd line to be written with the 1 being omitted. The cancellations done in line 3 is the familiar misconcept of cancelling by regarding the numerators and denominators of an algebraic fraction as monomials when they are not. This ‘method’ is applied to the 2’s of \( \sin 2x \) and \( \cos 2x \). By now declaring

\[
\cos x = \sin x
\]

and ostensibly by substitution the examinee arrives at the intended target. For some reason, to which outside readers have no access, this is deemed not
correct, the work is scratched out and “error” written indicating that the way of pursuit is abandoned.

The next attempt starts at line 11, where the problem, 8.2.1, is again written. In line 13 the “get rid of 1” is again applied and in line 15 the same cancellation as was done before is executed rendering \( \frac{\cos x - \sin x}{\sin x - \cos x} \). This is deleted and “error” written next to it. This is a case of reflexivity which is described as “In the course of our ordinary activities... we are building up...the meaning, the order, and the rationality of what we are doing.” The “rationality” here is of the form “this road has been travelled before and did not lead to the desired target”. This spurs the selection of a different path of pursuit by replacing \( \cos 2x \) by \( \cos^2 x \) in line 19. Part of the context is the formula sheet that is part of the question paper and the formulae given for double angles for the cosine function is given as \( \cos 2a = \frac{\cos^2 a - \sin^2 a}{1 - 2\sin^2 a} + \frac{2\cos^2 a - 1}{\sin^2 a - \cos^2 a} \). We infer that this object of the context was used but the second part, \( \frac{2\cos^2 a - 1}{\sin^2 a - \cos^2 a} \), was omitted due the pursuit of trying to obtain the set objective “work towards the target”. In the same vein and using the supplied formulae \( \sin 2x \) is replaced by \( \sin^2 x \), the other term of \( \cos 2a = \frac{\cos^2 a - \sin^2 a}{1 - 2\sin^2 a} + \frac{2\cos^2 a - 1}{\sin^2 a - \cos^2 a} \). The same erroneous cancellations are performed and a further attempt is made to obtain \( \frac{\sin x}{\cos x} \). It appears that there is a search for a relationship between \( \cos x \) and \( \sin x \) by recording \( \cos x - \sin x \). The entire effort is abandoned by scratching out all the work in Figure 2(a) and writing “cancelled”.

The new path followed in Figure 2(b) starts the same way as the two previous attempts by first recording the question. \( 1 - \cos 2x \) is replaced by \( \cos^2 x \) and \( \sin 2x \) by \( \sin^2 x \). The quest to obtain \( \frac{\sin x}{\cos x} \) is still exerting agency and \( \cos^2 x \) and \( \sin^2 x \) are cancelled because their cancellation will render the target which is sought.

Example 2:
The question pertinent to this example is given in Figure 3 below.
It is observable that the examinee first copies the formula for the area of a triangle from the formula sheet that forms part of the question paper. In terms of the mathematically historicised-self the experience of the examinee with area and trigonometry is linked to the area of the triangle. The figure re-inforces this because it is very near to the figure that is being used when the area of a triangle is being dealt with in textbooks and during teaching. In the 2nd line, the data for the sides are substituted. Lack of access to any further information, the best plausible explanation is that 2 was multiplied by 3 and the division by 2 was carried out. However, the \( \frac{1}{2} \) was retained probably due to the rush of the moment—there are a total of 13 questions and the linear fashion with which most examinees navigate examinations meant that this question was attempted near the end of the expired time.
In line 4 \( \frac{3}{2} \sin 90^\circ \), 90° was substituted for \( \theta \) with 90° most probably drawn from the restriction \( 0^\circ < \theta \leq 90^\circ \) given in the problem text. In lines 5 and 6 the simplifications are carried out to reach \( \frac{3}{2} \) as the final answer indicated by the arrow (\( \rightarrow \)), a popular indicator used in school mathematics to signify “this is my final answer”.

What is noticeable is that a line with an arrowhead is drawn across the last 3 lines from the lower left side to the right side of line 4. Plausibly the examinee worked solely with the answer text after line 4 and upon the declaration of the answer \( \frac{3}{2} \), there was a return to the problem text which indicates that the target had to be \( 6 \sin \theta \). This meant that what Pickering (1995) defines as a resistance was encountered. This causes the examinee to make a reversal defined by Julie (2015) as working a problem through to its conclusion and when encountering resistance reverting to some prior point and starting a new attempt from that point onwards. This new attempt starts at line 3. A probable explanation is that the examinee realised that the figure consists of two congruent triangles, a property of parallelograms which forms part of examinee’s entire corpus of mathematical experiences—the mathematically historicised examinee. With this U-turn the examinee arrived at the correct answer although nowhere on the answer text any indication of the congruency is provided.

**DISCUSSION AND CONCLUSION**

The teasing out of the examinees responses to the items of high-stakes mathematics examinations render the textures of their solution-seeking as they navigate the terrain to reach defensible, from their perspectives, answers to the examination questions. The strategies and tactics are highly driven by the context within which the high-stakes examination is situated. In the above examples target setting and working towards the identified target; the primacy of the mathematically historicised-self; reflexivity on previously solution-seeking endeavours in the immediate context; resistance encountering and reversal to seek alternate solution avenues characterised the ways of working. Analyses such as these are more in-depth than those underpinned by the “errors and misconceptions” paradigm. The reported form of analyses complements diagnosis emanating from the “errors and misconceptions” paradigm and provides information which makes the ways of working of the examinees visible. This visible-making of ways of working is a fairly common practice in a community of mathematicians. It is not strange to find that mathematicians discuss with one another their ways of working in their endeavours to solve problems they are engaged with. They will discuss their false starts, their blockages, strategies they used to get out of
dead-ends and hints they got from others that advanced their thinking to resolve dilemmas they face.

It is our contention that the visible-making of the ways of working of examinees in high-stakes examination can contribute towards enhancing future examinees dealing with high-stakes examination questions in the examination setting. One way to achieve such is to provide these future examinees with the produced work of past examinees and for them to analyse this in a manner similar to what was done above. This will obviously be accompanied by exemplification of such analyses. For example, learners can be provided with one of the examples used above with the analysis provided in learner-understandable language. They can then be provided with a near-similar example, drawn from the ‘real’ examination scripts, asked to do a similar analysis and then find the given solution. Activities such as this from the “errors and misconceptions” paradigm have found their way into textbooks and other learning resources. For example, learners are asked to discuss and evaluate whether \( \frac{(5a)^{-2}}{5a^{-3}} \) is simplified correctly as follows:

\[
\frac{(5a)^{-2}}{5a^{-3}} = \frac{(5a)^{-2} \cdot a^3}{5} = \frac{5a}{5} = a
\]

Mason (2000, p 10) asserts that exposing learners to a “confusion [committing common errors], with the expectation that [learners] have been awakened to [them]…will make them more alert in the future.” This can also be achieved by engaging learners in the unravelling of ways of working of responses to high-stakes examination questions.

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Orhun, N. (2002). Solution of verbal problems using concept of least common multiplier (LCM) and greatest common divisor (GCD) in Primary School


This article examines geometrical concepts that can be introduced to learners through observing building structures. The study employed a qualitative interpretive paradigm. Data was collected by means of observing pictures picked from internet websites. Due to limitations the focus was much on geometrical concepts, areas and volumes of three dimensional shapes. Realistic mathematics education approach was used as a theoretical framework underpinning the study to analyse data collected. Findings reveal that geometrical concepts such as three dimensional shapes can be identified and explored to develop formulae of various areas and volumes of prisms, cylinders, pyramids and cones. Different types of angles, triangles, quadrilaterals can be taught also through observing building structures. This study suggests the use of linking mathematics with building structures as a strategy which can lead to a better understanding of mathematics.

INTRODUCTION AND BACKGROUND

It is of great interest to any person irrespective of gender, race or whether educated, illiterate, poor or rich to have a strong beautiful house. One key aspect that leads people not to achieve their dreams is that they do not know what to do in order to achieve their goals. Many students perceive mathematics as a difficult subject which sometimes does not make sense to them. They also tend to think that mathematics is not applicable to solving real-life problems. It is an interest of all stakeholders to search strategies that can be used to improve the mathematics offerings presented to school-goers in South Africa (Julie, 2004). This study explores mathematics involved and its application in building and construction of houses. The study observed ‘built and construction’ in order to look at the kind of mathematics that is applied to build houses.

In South Africa there is no much emphasis on the use of mathematics in real-life situation (Julie, et al., 2011). It is the goal of this study ‘mathematics in build and construction’ that Realistic Mathematics Education should be realised as a practical approach to school mathematics, in particular geometry as proposed by Julie, 2004. Siyepu and Mtonjeni (2014) rearticulated Freudenthal (1991) definition of a Realistic Mathematics Education (RME) approach as a mathematical approach using a theory which emphasises that the teaching and learning of mathematics should be connected to reality as well as made a human activity (p.213).
Zulkardi (2010) summarises five characteristics of Realistic Mathematics Education approach as follows:

(1) use real-life contexts as a starting point for learning; (2) use models as a bridge between abstract and real, that help students learn mathematics at different levels of abstractions; (3) use student’s own production or strategy as a result of their doing mathematics; (4) interaction is essential for learning mathematics between teacher and students, students and students; and (5) connection to among strands, to other disciplines, and to meaningful problems in the real world. (p.13)

This study was triggered by the questions raised to explain the conference theme specifically “what kinds of learner support materials are being used to assist teachers and learners”. In this study the question was adapted to what kinds of learner support materials are being used to assist teachers to enhance understanding of mathematics? The study proposes the use of built and construction as a strategy of linking mathematics in particular geometry with real life context.

**Study in context**

The researcher used building structures as a source of data for the study. The use of building material in the building of a house requires basic understanding of mathematics. Figure 1 below shows the beginning of a house construction. In order to build strong rigid house people needs bricklayers who know how to make a firm foundation.

![Figure 1: Two rows on top of a shallow foundation in building construction.](image-url)
Foundation in the building of a house

The picture above shows shallow foundation. Shallow foundation is expected to be about 1m in depth. Deep foundations can be made at depth of 20-65 m. Shallow foundations are used for small, light buildings, while deep ones are for large, heavy buildings. This involves measurement and accuracy. Is it not possible that disasters such as falling of the synagogue in Nigeria (TB Joshua’s temple) were based on insufficient knowledge of the architectures that planned and designed the house? If that may be the case, investigations may not only focus in political misunderstandings and lack of appropriate skills in architectures. The lack of basic skills in mathematics includes insufficient mathematical knowledge that should be applied to build strong houses.

In the building of a foundation one needs to know how to hand mix concrete so it can deliver maximum strength and durability. Mixing bags of concrete is not a complicated job, but to get the most strength from the concrete, one has to recognise when it has the right amount of water mixed in it. Too little water and the particles in the mix would not stick together. Too much water weakens the concrete. Mixing water with the cement, sand, and concrete will form a paste that will bind the materials together until the mix hardens. The strength properties of the concrete are inversely proportional to the water/cement ratio.

Basically this means the more water you use to mix the concrete (very fluid) the weaker the concrete mix. The less water you use to mix the concrete (somewhat dry but workable) the stronger the concrete mix. Accurate concrete mixing ratios can be achieved by measuring the dry materials using buckets or some other kind of measuring device. By measuring the mixing ratios you will have a consistent concrete mix throughout your entire project.

The application of mathematics comes in as an essential aspect in the whole process. Bricklayers should be able to apply mathematics in building a foundation through understanding the sizes of cement bags, sand, rock and litres of water to be used. This involves calculation of how many cement bags needed to complete the foundation and how much concrete and litres of water should be bought to finish the foundation.
Building the wall of a house

Building the wall of a house needs a plan first. The plan gives a layout what should be done to start the house from the beginning. This entails bricks, cement, water and sand. All these materials need calculations and accuracy in the building process. This study reports mathematics involves in the building process. Bricks normally are in a rectangular form. In order to build an accurate house, bricks have to be congruent. Congruent means equal in all respects. Components of bricks give many aspects of mathematics, namely, edges, vertices, faces and cylindrical holes. From a rectangle facet, properties of a rectangle can be taught, namely, all angles are right angles, diagonals are equal, and opposite sides equal. As the wall build up bricks have to take up a pattern in an overlapping manner. The overlapping patterns can help teachers to introduce concepts such as tessellation. Tessellation is a pattern made of identical shapes: the shapes must fit together without any gaps, the shapes should not overlap.

Inserting door frames and window frames

Once a foundation is complete, bricklayers start placing door and window frames in their appropriate positions. Most of door-frames and window-frames are in a rectangular form. Inserting door-frames and window-frames needs accuracy and it needs to be inserted in the middle of the brick wall, at the same they have to be upright. Door frames consist of a rectangle. Once a door frame slants, that can affect the wall as
well as angles in joints of doors. Angles can become obtuse and acute instead of keeping right angles.

Frame of a roof

Household roofs are normally made of rafters and nails. There are many different kinds of roofs depending from region to region. This study focuses in a gable as common roofing in modern houses that are built in sub-urban areas in South Africa. There are special terms that used in building of gable roofing such as strut; ridge beam, side post, king post, and principal post. Figure 3 below shows a picture of a gable roof frame.

Figure 3 shows the kind of a roof frame normally used in households.

From the diagram above there are different types of angles such as acute, right, straight, obtuse and reflex that may be used in the classroom for learners to identify these angles based on their characteristics. The diagram can be used to introduce special kinds of angles, namely, adjacent, complementary and supplementary. Different types of quadrilaterals such as trapezium, parallelogram can also be introduced. Same applies in concepts such as diagonals and triangles can be introduced as well. **Strut** is a diagonal member supporting the principal rafter, joining it to the junction of the king post and the ridge beam. **Ridge beam** is a level member at the base of a truss upon which the side posts and the king post rest; these in turn support the principal rafters. **Side post** is a small vertical member supporting the principal rafter. **King post** is a vertical member supporting the tie beam and joining the principal rafters to the ridge beam. **Principal rafter** is a diagonal member of a roof truss; it functions as a rafter.
The use of house buildings as a starting point in introduction of three dimensional objects such as cones; prisms; cylinders and pyramids that can assist students to start from visualisation as the first level of introducing shapes based on van Hiele’s theory. The figure 4 below shows pictures of shapes that can be picked up in built and construction.

![Cone, Cuboid, Cylinder, Triangular Prism](image)

**Figure 4 shows solids of a Cone; Cuboid; Cylinder; Triangular Prism and Square-based Pyramid**

Figure 5 below shows nets of solids as listed above

![Net of a Cone, Net of a Cuboid, Net of a Cylinder, Net of a Triangular Prism](image)

**Figure 5 shows nets of solids of a Cone; Cuboid; Cylinder; Triangular Prism and Square-based Pyramid**

A roof of a rondavel resembles a cone. A cone is a three dimensional shape that has one circular base and one vertex. Full details were explained in Siyepu (2012, p. 337). A
A cuboid is a rectangular prism with six flat sides and all angles are right angles. A cuboid consists of vertices, edges and faces see figure 6 below.

Figure 6 above shows vertices, edges and faces of a cuboid

A shape of a face brick as shown in figure 7 below resembles a hollowed cuboid. Face bricks should be used in classroom interaction to allow learners to explore vertices, edges and faces of a rectangular prism. The purpose of using concrete material in teaching and learning of mathematics is to make sense and meaning in mathematical concepts that are explored in classroom discussions. Nets of cuboid should be used to allow to learners to investigate and make conjectures on how mathematicians reach that formula for the total surface area of a cuboid which is \[ A = 2lh + 2bh + 2lb \].

A cylinder is a closed solid that has two parallel circular bases connected by a curved surface. The net of a cylinder is used to derive a conjecture on how to reach that the formula for the total surface of a cylinder which is \[ A = 2\pi r^2 + 2\pi rh \]. Teaching and learning of the area of a cylinder should be based on the understanding of the area of a circle. To be exact, learners should understand the area of a circle first as a prerequisite of grasping the area of a cylinder. The three diagrams below show strategies that should be used to develop a conjecture on how to find out that the area of a circle which is \[ \pi r^2 \]. As shown below this start by cutting a circle into equal sectors (12 in this example). Then just divide one of the sectors into two equal parts, we now have thirteen sectors-number them 1 to 13 as shown in the 2\textsuperscript{nd} diagram below. Then rearrange the 13 sectors as in the 3\textsuperscript{rd} diagram below:
Figure 7 shows development of a conjecture for the area of a circle.

Looking at the 3rd diagram, the height is the circle’s radius. Then the other two longer sides represent the circumference of a circle of which one side is half of the circumference. Then we can conclude from that, that the area of a circle is $\pi r^2$. Look at the figure 8 below:

Figure 8 shows divided circle that resembles a rectangle.

Then the total surface area of a cylinder can deduced from the net given in figure 9 below:

Figure 9 shows a picture of a cylinder.

Clearly the length of the rectangle = Circumference of the base = $2\pi r^2$. Then the width of the rectangle = Height of the cylinder = $h$; Then we can conclude that the

Total Surface Area of a Cylinder = Area of the Lid + Area of the base + Curved Surface Area

$$= \pi r^2 + \pi r^2 + 2\pi rh$$
$$= 2\pi r^2 + 2\pi rh$$
$$= 2\pi r(r + h)$$
How to find the volume of a cylinder

Although a cylinder is technically not a prism, it shares many of the properties of a prism. Like prisms, the volume is found by multiplying the area of one end of the cylinder (base) by its height. Since the end (base) of a cylinder is a circle, the area of that circle is given by the formula $Area = \pi r^2$. Multiplying by the height $h$ we get

$Volume = \pi r^2 h$

How to find the total surface area of a triangular prism

The total surface area of a triangular prism can be deduced from the net shown below

Looking at the figure above there are two congruent triangles. The area of each triangle is half base times height. That implies the total surface area of these two triangles is base times height. Then the area of the whole net is area of the two triangles plus area of the three rectangles when all rectangles are congruent. In symbols it can be written as:

$$A = \frac{1}{2}bh + \frac{1}{2}bh + bl + bl + bl$$

$$A = bh + 3bl$$

$$A = h(h + 3l)$$
**How to find the volume of a triangular prism**

Volume of a triangular prism can be deduced as a half of the volume of a cuboid, whereby the volume of a cuboid is \( V = l \times b \times h \). Then volume of a triangular prism is \( V = \frac{l \times b \times h}{2} \).

**How to find the total surface area of a square based pyramid**

A pyramid is a three-dimensional figure made up of a base and triangular faces that meet at the vertex, \( V \), which is also called the apex of the pyramid. A square based pyramid has four congruent triangular faces and a square base.

![Figure 11 shows a net of a square based pyramid and its solid](image)

of a square based pyramid and its solid

The total surface area of a pyramid is the sum of the areas of its faces including its base. It can be deduced from the net of a square based pyramid above that the total surface of a square based pyramid is

\[
A = \frac{1}{2}bh + \frac{1}{2}bh + \frac{1}{2}bh + \frac{1}{2}bh + b^2
\]

\[
A = 2bh + b^2
\]

\[
A = b(2h + b)
\]

From the area of a square based pyramid, it can be deduced that the volume of a squared pyramid is \( V = \frac{1}{3}s^2h \).
IMPLICATIONS OF THE STUDY

The use of building material to introduce geometry can help learners to visualise shapes easily. For instances many learners enter schooling right from grade R knowing cement bricks, purlins and rafters but they do not associate these with mathematical concepts such as cuboid. Different types of angles, triangles and quadrilaterals can be taught. Discussion of the study dwell much on how to use shapes in building material to teach geometrical concepts such as prisms, cones, cylinders and pyramids. This study emphasises the approach that may be adopted by teachers of mathematics to develop conjectures on how geometrical formulae of areas and volumes of three dimensional shapes can be developed. The study also link mathematics with real-life context. Probably this can make sense to learners as to why do they study geometry.

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IMPROVING THE KNOWLEDGE OF EDUCATORS: A CASE OF MATHEMATICAL PROFESSIONAL DEVELOPMENT

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This paper is derived from a twelve-week professional development workshop for FET (Grade 10-12) mathematics educators at a Continuing Professional Development Institute. The aim was to help improve their ability to teach mathematics concepts in Curriculum and Assessment Policy Statements (CAPS). Data consist of a pre/post-assessment of educators’ knowledge of the cognitive demands of mathematical tasks and written examination from the professional development sessions. Results of the study revealed that the professional development workshop did improve their knowledge of new FET challenging mathematical tasks and confidence in teaching different mathematics topics such as Calculus and Data Handling (Statistics). Implications and recommendations for professional development in mathematics are provided.

INTRODUCTION AND BACKGROUND

In South Africa, as in other countries, teachers are expected to have a “deep understanding” of mathematical content knowledge and also be aware and competent in use of a range of alternative and appropriate teaching methods with an emphasis on meaning. According to Ball and Wilson (1990) mathematics education majors have not been exposed to enough alternative teaching methods to be capable of teaching mathematics with an emphasis on meaning. Ball and Wilson (1990) go further and mention that pre-service secondary mathematics teachers often lack sufficient mathematical understanding to teach the subject effectively. According to Faulkner and Cain (2013), there is qualitative evidence that strong teacher understanding of mathematics has an impact on appropriate classroom discussions, as well as quantitative evidence that it has an impact on learners achievements. As results, the Limpopo Department of Education (LDoE) through its teacher development institute implemented educator Continuing Professional Development (CPD) programme as an initiative to enhance the skills of mathematics educators in order to improve the achievement of the learners. The basis for the programme was that there was great concern that a sizeable number of mathematics educators had inadequate skills and knowledge especially in the new curriculum. The focus of the programme was to: develop conceptual understanding of mathematics; deepen mathematics knowledge relating to the CAPS; expose participants to various subject teaching methodologies.

This study investigates whether a twelve-week professional development workshop targeted for Further Education and Training (FET: Grade 10-12) educators can help
strengthen the participants’ knowledge and skills in mathematics particularly in Data Handling (Statistics) and Calculus topics. Calculus and Data handling forms a major part of the Grade 12 South African Mathematics curriculum and they are new to most educators (Department of Education, 2007). In this regard, Calculus forms about 40% of Algebra, as seen in the Grade 12 paper one national examinations set by the Department of Basic Education in 2014. This forms about 20% of the overall assessment.

LITERATURE REVIEW AND CONCEPTUAL FRAMEWORK

The following is a brief review of literature in the field and the concepts that are used to frame the research.

Teacher Professional Development

The lack of adequate teacher professional development opportunities for some teachers is a barrier to learners’ academic achievement in mathematics. In “Learning without Limits: An agenda for the Office of Post-secondary Education” (U.S. Department of Education, 2000), it is reported that experienced teachers do not have adequate opportunities to improve their knowledge and skills, and that in-service training opportunities for teachers are “second rate” (2000:32). The report cites the following problems regarding the in-service training of teachers: In-service training remains largely short-term and non-collaborative; In-service training is often unrelated to the teachers’ needs and the challenges faced by their learners; Teachers are offered in-service training opportunities that last for a few hours (less than eight). In this regard, Education reformers are paying considerable attention to the role that effective professional development can play in improving the teaching of mathematics. There appears to be consensus on the features of high-quality professional development, among others: It should focus on deepening subject matter knowledge specifically for teaching, including understanding how students learn and the specific difficulties they may encounter in mastering key concepts; It has sufficient time for significant learning; It focuses on what teachers are being asked to do and builds on what teachers already know and are able to do; Educators are actively engaged, rather than just listening to a lecture or watching a demonstration; Educators learn together, enabling them to support each other in using what they have learned (NAE, 2009, p. 6).

Teacher-Confidence in Calculus and Data Handling

Confidence is a source for teachers to accept and test different instructional strategies (Toh, 2009). Nonetheless, many mathematics teachers are not proficient or confident enough when teaching certain mathematics principles, let alone in the new
curriculum topics such as Data Handling and Calculus. According to Lubinski (1994) knowledge of the content and pedagogy in conjunction with learners’ thinking, allows a teacher to design blueprints for worthwhile mathematics tasks. In this respect it is reasonable to expect that teachers will feel successful when their learners perform well in mathematics. It should also be expected that teachers would feel frustrated and unsuccessful when the learners perform badly.

**Teachers’ Quality and Content Knowledge in Mathematics**

Teacher quality is a key determinant of learners’ achievement and having a strong content knowledge is crucial for effective teaching (Toh, 2009). There is clear evidence on the relationship between teachers’ mathematical content knowledge and their ability to teach well in classrooms (Ball, Thames & Phelps, 2008). Schmidt (1999: 81) also observes that: *What teachers teach and how they teach it are affected by their subject matter belief and preferred pedagogical approaches, things that are consequences of their training and experiences.*

According to Mohr (2006) and Schulman (1987), the content of mathematics includes (1) content knowledge, that is, knowledge of the concepts, procedures, and problem-solving processes within the area of mathematics they are teaching, and (2) pedagogical knowledge which is the capacity of a teacher to transform the subject knowledge that he or she possesses into forms that are pedagogically powerful and yet adaptable to the variations in ability and background presented by the learners. The opinion of the author is that secondary mathematics educators should have strong mathematical knowledge, a positive attitude towards mathematics and teaching, as well as an alignment with proper pedagogical beliefs particularly in Data handling and Calculus.

**Teacher Competence in Calculus and Data handling**

Research has indicated that teaching and learning of Calculus and data handling can be challenging as it involves abstract and complex ideas (Gordon, 2004; Zachariades et al., 2007). In teaching Calculus and Data handling teachers focus more on procedures than on understanding of underlying concepts(Zachariades et al.;2007). Gordon (2004) and Axtell (2006) concluded that calculus and data handling curriculum should be improved by focusing on the conceptual understanding, balanced with the use of graphical, numerical, algebraic and verbal representations. Researchers (Begg and Edward, 1999; Chick and Pierce , 2008) agree that many teachers did not get adequate training in teaching Data handling and Calculus. The situation is more challenging for our South African teachers, few of whom have had suitable training in statistics and Calculus in their professional training.
RESEARCH METHODOLOGY

The participants for this study were formed by the entire cohort of secondary mathematics FET (Grade 10-12) educators who attended the workshop (N = 50). The study was divided into two components: Analysis of educator content tests. At the start of each new section/topic, the educators wrote pre and post tests on every topic. These tests were developed and administered and scored by the trainers and then captured and analysed. Secondly, Interviews and Questionnaires for Educators which were carried out after the training have been completed. The interviews took place in order to assess the success of the workshop from the trainees’ point of view. The aim of the questionnaire was to determine if participants’ expectations of the CPD workshop were met and to determine whether the objectives of the training had been understood. The structure of the questionnaires used differed from section to section: direct stating of information about the subject, ticking the box that relays one’s opinion (e.g agree, disagree, no opinion) and general comments. This ensured gathering of relevant information, directness of responses while leaving room for full expression of the educators’ opinions and thoughts. However, this paper reports only on pre and post test educator achievements.

FINDINGS AND DISCUSSION

The educators were assessed on five pre tests – one test for each topic that was facilitated and two post tests, which comprised a selection of the same items that were used in the pre-tests. The key challenge with analysing the test data were the fluid nature of the participants i.e., there is a difference between the number of educators who participated in the pre- and post-tests and across the different topics. For example, while 36 educators participated in the baseline test, 45 participated in the post test for calculus. This made the analyses difficult. Further, we could not tell when these educators entered and left the workshop and to how much of the workshop they were exposed. To deal with this, further analysis was conducted according to whether teachers wrote BOTH pre and test topics. In other words, we did not consider the contributions of educators who ONLY appeared either in post or pre test. This reduction in sample size takes place throughout the analysis of the rest of the workshop. It should be noted that the number of non-FET teachers was few (n=4) and was not statistically significant. However, for illustrative purposes, the analysis was also further

21 The analysis focused on the same items that teachers worked on. The items were matched by topic and skill and the total scores were recalculated and used as percentage scores.
disaggregated by whether the educator was an FET or non-FET educator. The non-FET results however need to be read with caution as the sample size is so small.

Calculus topics

The educators’ total mean score in calculus was 37% in the pre-test which increased to 61% in the post-test. This increase of 24 percentage points from the pre-test, which is a statistically significant difference, shows that the total mean score was an improvement rate of 62% and suggests that the intervention had an impact on educators’ calculus content knowledge. Also positive is the maximum score attained on this topic was 100% in the post-test as opposed to the 88% in the pre-test. However, a worrying finding is that there was an educator who still scored zero in the post test (see Table 1). In relation to the subtopics, there was improvement in all three subtopics assessed. In particular, educators did well in problems involving derivatives as shown by the mean of 50% in the pre-test and 76% in the post-test, but they struggled with differentiation where the lowest score was attained in both the pre and post tests (i.e., 13% and 40% respectively). Although educators scored the lowest in differentiation, this was the subtopic where the improvement rate was the highest (i.e., an improvement of 60% over the pre-test was attained). This shows that the educators’ content knowledge is weak in establishing derivatives using rules of differentiation but their knowledge of establishing derivatives from first principles is strong.

Table 1: Educator performance in calculus and its subtopics

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Derivative</th>
<th>Differentiation</th>
<th>Graph</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pre</td>
<td>Post</td>
<td>Pre</td>
<td>Post</td>
</tr>
<tr>
<td>Mean</td>
<td>50%</td>
<td>76%</td>
<td>13%</td>
<td>40%</td>
</tr>
<tr>
<td>Min</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Max</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>

22 Although there was data for 45 educators were tested on this topic, because of excluding data where only pre- or post-test data was available, the sample size was reduced to 33.

23 To calculate the improvement rate, the formula is: post score – pre score x 100 / pre test score
As can be seen from Table 2, the performance on the pre-tests by the few non-FET educators was notably lower than the FET educators but they were able to achieve higher post test scores. This finding, although unexpected, does suggest that non-FET educators benefited greatly from the programme in terms of enhancing their knowledge of calculus.

Table 2: Educator performance in calculus and its subtopics by FET and Non FET group

<table>
<thead>
<tr>
<th>Group</th>
<th>Statistics</th>
<th>Derivative</th>
<th>Differentiation</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Pre</td>
<td>Post</td>
<td>Pre</td>
</tr>
<tr>
<td>Non-FET (n=4)</td>
<td>Mean</td>
<td>45%</td>
<td>85%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>Min</td>
<td>20%</td>
<td>60%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>Max</td>
<td>60%</td>
<td>100%</td>
<td>0%</td>
</tr>
<tr>
<td>FET (n=29)</td>
<td>Mean</td>
<td>50%</td>
<td>74%</td>
<td>15%</td>
</tr>
<tr>
<td></td>
<td>Min</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>Max</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Data handling

Overall, educator performance increased by a wide margin of 43 percentage points, which is equal to a remarkable improvement rate of 300% on this topic. The biggest increase was on data representation, moving from a pre score of 14% to 64%, an increase of 50 percentage points (Table 3).

Table 3: Educator performance in data handling and its subtopics

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Univariate data</th>
<th>Data representation</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pre</td>
<td>Post</td>
<td>Pre</td>
</tr>
<tr>
<td>Mean</td>
<td>14%</td>
<td>48%</td>
<td>14%</td>
</tr>
<tr>
<td>Min</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Max</td>
<td>100%</td>
<td>100%</td>
<td>82%</td>
</tr>
</tbody>
</table>

24 Although there was data for a maximum of 46 educators were tested on this topic, because of excluding data where only pre- or post-test data was available, the sample size was reduced to 34.
We found it interesting that very little difference was observed between FET and non-FET educators at both the pre and post test levels.

Table 4: Educator performance in data handling and its subtopics by FET and Non FET group

<table>
<thead>
<tr>
<th>Group</th>
<th>Statistics</th>
<th>Univariate data</th>
<th>Data representation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Pre</td>
<td>Post</td>
</tr>
<tr>
<td>Non-FET (n=4)</td>
<td>Mean</td>
<td>13%</td>
<td>48%</td>
</tr>
<tr>
<td></td>
<td>Min</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>Max</td>
<td>50%</td>
<td>70%</td>
</tr>
<tr>
<td>FET (n=30)</td>
<td>Mean</td>
<td>14%</td>
<td>48%</td>
</tr>
<tr>
<td></td>
<td>Min</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>Max</td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Table 5: Educator performance across the two Maths topics

<table>
<thead>
<tr>
<th></th>
<th>Pre test</th>
<th>Post test</th>
<th>Percentage point difference</th>
<th>Improvement rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculus</td>
<td>37%</td>
<td>61%</td>
<td>+24%</td>
<td>+65%</td>
</tr>
<tr>
<td>Data handling</td>
<td>14%</td>
<td>56%</td>
<td>+42%</td>
<td>+300%</td>
</tr>
</tbody>
</table>

Educator Opinion And Preparedness

Educators were asked to rate the importance and their preparedness in terms of effective Mathematics teaching according to the 13 areas listed in Table 6.

Table 6: Areas for effective Mathematics Teaching

| 1) | Provide concrete experience before abstract concepts. |
| 2) | Develop students’ conceptual understanding of Mathematics. |
3) Take students’ prior understanding into account when planning curriculum and instruction.
4) Make connections between Mathematics and other subjects.
5) Have students work in cooperative learning groups.
6) Have students participate in appropriate hands-on activities.
7) Engage students in inquiry-oriented activities.
8) Use calculators.
9) Use computers.
10) Engage students in applications of Mathematics in a variety of contexts.
11) Use performance-based assessment.
12) Use portfolios.
13) Use informal questioning to assess student understanding.

The areas that were rated the highest by educators in terms of importance and their preparedness include developing students’ conceptual understanding of mathematics, having students work in groups and the use of calculators (Figure 1). Educators rated the use of portfolios and computers as amongst the least important. This is an interesting observation since educators that have limited access to technology deem it not important for effective mathematics teaching.

**Figure 4: Importance of effective Mathematics teaching vs Preparedness according to educators**
CONCLUSION AND IMPLICATIONS OF THE RESEARCH

Findings reveal that progress has been made through the course of study and a heightened confidence in the participants’ ability to transfer what has been learned into the classroom environment is evident. Thus, professional development is the key for initiating and securing change in math instruction. This study investigates whether a twelve-week professional development workshop targeted for Further Education and Training (FET: Grade 10-12) educators can help strengthen the participants’ knowledge and skills in mathematics particularly in Data Handling (Statistics) and Calculus topics. The findings of this paper, especially on the items that the educators have not performed well in both the pre and post test, identifies the area of school mathematics that the educators have not picked up even after the workshop. This adds to the knowledge of research on secondary mathematics educators’ content knowledge. Hopefully this can spur further interest in this area of research.

REFERENCES


Today’s learners acquire, assimilate and share information differently. The internet and social media in particular have revolutionised information exchange methodologies thus it is necessary and sufficient that the maths education value exchange adapt for meaning learning to take place. The one way model of educator to learner information transmission is fast becoming redundant with the plethora of sources that are available at speed and on-demand. Interactive media now caters for different learning styles and can be customised for pace and repeatability. The role of the educator has to be reviewed amidst the information explosion. Google, FaceBook and other leading IT companies have taken advantage of technological advancement and have turned Industrial Age business models upside down through the creation of platforms that derive value from co-creation and consumption of knowledge. The lowered barriers to entry in information exchange have led to democratisation of knowledge. The network effects of social are cross-cutting across all spheres of life including maths education. The aims of the talk, From Pipeline to Platform Thinking for Mathematical Success are to give:

- An exposition of the shift in learning as enabled by technology.
- Possible proactive steps and remedies in our current system.
- A snapshot of the future of learning as extrapolated from recurring trends.

The educator should be able to take advantage of abundant technology to optimise impact in learners’ lives. The role of all stakeholders in the maths education space has to be reviewed and re-configured if the learner being produced by the system will be relevant in the ICT and data-driven society of the future. Sufficient evidence has been present in the scientific community supporting the positive influence of technology in learning. IST in mathematics education is a no-brainer and all educators ought to embrace it lest they become dinosaurs in the near future.

**REFERENCE**