The Beauty of Cyclic Numbers

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All rational numbers can be written as a decimal expansion. Sometimes this decimal expansion terminates, for example $\frac{1}{8}$ which can be written as $0{,}125$. However, sometimes the decimal expansion takes the form of a repeating pattern, for example $\frac{2}{11}$ which can be written as $0{,}1818181818\ldots$, or simply as $0{,}\overline{18}$ where the bar indicates the repeating digits. We will refer to such repeating units as *cyclic numbers*.

To begin with, let's restrict ourselves to rational numbers of the form $\frac{1}{b}$ (i.e. unit fractions) and ask the following question. Under what circumstances will such unit fractions terminate? Table 1 provides a summary of the first 40 unit fractions using bars to indicate repeating digits.

TABLE 1: Decimal expansions of the first 40 unit fractions

b	$\frac{1}{b}$	b	$\frac{1}{b}$
1	1	21	$0{,}\overline{047619}$
2	$0{,}5$	22	$0{,}0\overline{45}$
3	$0{,}\overline{3}$	23	$0{,}\overline{0434782608695652173913}$
4	$0{,}25$	24	$0{,}041\overline{6}$
5	$0{,}2$	25	$0{,}04$
6	$0{,}1\overline{6}$	26	$0{,}0\overline{384615}$
7	$0{,}\overline{142857}$	27	$0{,}\overline{037}$
8	$0{,}125$	28	$0{,}03\overline{571428}$
9	$0{,}\overline{1}$	29	$0{,}\overline{0344827586206896551724137931}$
10	$0{,}1$	30	$0{,}0\overline{3}$
11	$0{,}\overline{09}$	31	$0{,}\overline{032258064516129}$
12	$0{,}08\overline{3}$	32	$0{,}03125$
13	$0{,}\overline{076923}$	33	$0{,}\overline{03}$
14	$0{,}0\overline{714285}$	34	$0{,}0\overline{2941176470588235}$
15	$0{,}0\overline{6}$	35	$0{,}0\overline{285714}$
16	$0{,}0625$	36	$0{,}02\overline{7}$
17	$0{,}\overline{0588235294117647}$	37	$0{,}\overline{027}$
18	$0{,}0\overline{5}$	38	$0{,}0\overline{263157894736842105}$
19	$0{,}\overline{052631578947368421}$	39	$0{,}\overline{025641}$
20	$0{,}05$	40	$0{,}025$

Table 1 reveals that of the first 40 unit fractions the following have terminating decimal expansions:

$$\frac{1}{1};\frac{1}{2};\frac{1}{4};\frac{1}{5};\frac{1}{8};\frac{1}{10};\frac{1}{16};\frac{1}{20};\frac{1}{25};\frac{1}{32};\frac{1}{40}$$

Ignoring $\frac{1}{1}$ which is a trivial case, what is it about the other denominators that results in the decimal expansion terminating? If we write the denominators in terms of their prime factors then an interesting pattern emerges:

$$\frac{1}{2^1};\frac{1}{2^2};\frac{1}{5^1};\frac{1}{2^3};\frac{1}{2^1\times5^1};\frac{1}{2^4};\frac{1}{2^4\times5^1};\frac{1}{5^2};\frac{1}{2^5};\frac{1}{2^3\times5^1}$$

The prime factorisation reveals that these terminating unit fractions all have denominators whose factors are only 2 and/or 5. The next two unit fractions with this property are $\frac{1}{50}$ and $\frac{1}{64}$ which can be written in prime factorised form as $\frac{1}{2^1\times5^2}$ and $\frac{1}{2^6}$ respectively. As expected these fractions result in terminating decimal expansions, $0{,}02$ and $0{,}015625$ respectively. To understand the significance of our observation let us first consider fractions of the form $\frac{a}{10}$, $\frac{a}{100}$, $\frac{a}{1000}$, $\frac{a}{10000}$ etc., in other words fractions where the denominator is a power of 10. It should be self-evident why fractions of this form result in a terminating decimal expansion. Now, any fraction whose denominator only has factors of 2 and/or 5 can readily be written as an equivalent fraction of the form $\frac{a}{10^n}$, from which it follows that the decimal expansion must terminate. By way of example:

$$\frac{1}{64}=\frac{1}{2^6}=\frac{1}{2^6}\times\frac{5^6}{5^6}=\frac{5^6}{10^6}=\frac{15625}{1000000}=0{,}015625$$

So this explains under what conditions fractions have terminating decimal expansions and accounts for the observations made from Table 1. But what other patterns does Table 1 reveal? One may perhaps notice that there is an interesting relationship between the expansions for $\frac{1}{7}$; $\frac{1}{14}$; $\frac{1}{21}$; $\frac{1}{28}$ and $\frac{1}{35}$. With the exception of $\frac{1}{21}$ the other four fractions contain exactly the same six digits in the repeating unit, and these six digits seem to appear in a cyclic form, differing only in the starting point of the pattern.

$$\frac{1}{7} = 0{,}142857142857142857142857\ldots$$
$$\frac{1}{14} = 0{,}0714285714285714287142 85\ldots$$
$$\frac{1}{28} = 0{,}0357142857142857142857 1428\ldots$$
$$\frac{1}{35} = 0{,}0285714285714285714285714\ldots$$

What a pity that $\frac{1}{21}$ doesn't fit the pattern as well! However, notice what happens when one takes the repeating unit of $\frac{1}{21}$ and multiplies it by three: $047619\times3=142857$. This of course makes sense since $\frac{1}{21}\times3=\frac{1}{7}$. There certainly seems to be something quite interesting about the cyclic number 142857. With a bit of fiddling around one might for instance notice the following intriguing result:

$$142857\times1=142857$$
$$142857\times2=285714$$
$$142857\times3=428571$$
$$142857\times4=571428$$
$$142857\times5=714285$$
$$142857\times6=857142$$

This gives all six permutations of the cyclic number 142857. At this point it would be a natural question to think about what the result would be if we multiplied 142857 by 7. The answer to this, namely a string of 9s, lies at the heart of why such numbers are cyclic.

From the observed pattern in the multiples of 142857, it should be no surprise that the decimal expansions of the multiples of $\frac{1}{7}$ follow the same pattern:

$$\frac{1}{7} = 0,\overline{142857} \qquad \frac{4}{7} = 0,\overline{571428}$$
$$\frac{2}{7} = 0,\overline{285714} \qquad \frac{5}{7} = 0,\overline{714285}$$
$$\frac{3}{7} = 0,\overline{428571} \qquad \frac{6}{7} = 0,\overline{857142}$$

At this point it should also become apparent why the product 142857×7 resulted in a string of 9s, since $\frac{1}{7} \times 7 = 1$ and 1 is of course equivalent to $0,999999$ ..., i.e. $0,\overline{9}$.

There is another fascinating relationship between the digits of the cyclic number 142857. Notice what happens when the first three digits of the repeating unit are added to the last three digits: $142 + 857 = 999$. Also, notice what happens when we add successive pairs of digits: $14 + 28 + 57 = 99$. What about adding the individual digits of the number 142857 ? Clearly the answer can't be 9, which would have been incredibly pleasing. Even so, the answer is rather interesting since $1 + 4 + 2 + 8 + 5 + 7 = 27$, and if we add the digits of this answer we end up with $2 + 7 = 9$, which is rather pleasing after all! So, adding the digits in groups of three resulted in a string of three 9s, adding the digits in groups of two resulted in a string of two 9s, and adding the individual digits resulted, after a bit of tweaking, in a single 9.

Is this an isolated oddity or are there other numbers where this phenomenon occurs? Consulting Table 1 reveals that after $\frac{1}{7}$ the next unit fractions having cyclic decimal expansions are $\frac{1}{13} = 0,\overline{076923}$ and $\frac{1}{14} = 0,0\overline{714285}$. Let's investigate the first of these:

$$\frac{1}{13} = 0,\overline{076923} \qquad 076 + 923 = 999$$
$$07 + 69 + 23 = 99$$
$$0 + 7 + 6 + 9 + 2 + 3 = 27 \quad \rightarrow \quad 2 + 7 = 9$$

The result is identical to that obtained in the case of $\frac{1}{7}$. Let's continue our investigation with $\frac{1}{14}$. We've previously ascertained that the decimal expansion for $\frac{1}{14}$ is a permutation on the cyclic number 142857, but how will this particular permutation hold up under our partitioning treatment?

$$\frac{1}{14} = 0,0\overline{714285} \qquad 714 + 285 = 999$$
$$71 + 42 + 85 = 198 \quad \rightarrow \quad 1 + 98 = 99$$
$$7 + 1 + 4 + 2 + 8 + 5 = 27 \quad \rightarrow \quad 2 + 7 = 9$$

Adding the digits in groups of three resulted in a string of three 9s. Adding the digits in groups of two resulted in a 3-digit number rather than a 2-digit number. However, by adding the leading digit, a 1 in this case, to the units place of the number remaining after the leading digit has been removed, the result is a string of two 9s. A similar treatment when adding the individual digits resulted in a single 9.

What about cyclic numbers that have a much longer cycle? Table 1 suggests that $\frac{1}{29}$, with a 28-digit cycle, might be an interesting case to consider. We can partition the digits into equally sized groups in a number of ways, viz. two groups of fourteen, four groups of seven, seven groups of four, fourteen groups of two, or 28 individual digits:

$\frac{1}{29} = 0,\overline{0344827586206896551724137931}$

$03448275862068 + 96551724137931 = 99999999999999$

$0344827 + 5862068 + 9655172 + 4137931 = 19999998 \quad \rightarrow \quad 1 + 9999998 = 9999999$

$03 + 44 + 82 + 75 + 86 + 20 + 68 + 96 + 55 + 17 + 24 + 13 + 79 + 31 = 693 \quad \rightarrow \quad 6 + 93 = 99$

$0 + 3 + 4 + 4 + 8 + 2 + 7 + 5 + 8 + 6 + 2 + 0 + 6 + 8 + 9 + 6 + 5 + 5 + 1 + 7 + 2 + 4 + 1 + 3 + 7 + 9 + 3 + 1 = 126 \quad \rightarrow \quad 1 + 2 + 6 = 9$

Tweaking the answers where necessary, our partitioning treatment once again produces a matching string of 9s for each partitioning! If we try to construct a rule for this procedure, perhaps we could word it something like this:

> *For a cyclic number partitioned into k groups of t digits, where $k \times t$ is the number of digits in the cycle, then the sum of the k groups of t digits will result in a string of t 9s. If, in the event of the sum containing more than t digits, the additional leading digits (i.e. the digits on the left of the sum) should be added to the t-digit number on the right of the leading digits.*

Trying to formulate such a description is an interesting exercise in itself! See if this formulation works for some of the other cyclic numbers in Table 1.

A fun way of representing cyclic numbers graphically is by using a "clock face". The clock face we are going to use has the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9 evenly distributed around a circle as shown in Figure 1. Let's take the repeating pattern in the decimal expansion of $\frac{1}{7}$ as an example, i.e. the cyclic number 142857. To represent this number on the clock face, start at 1 and draw a straight line to the next digit in the pattern, i.e. 4, and keep going until you get to the last digit, i.e. 7. Now complete the path by drawing a straight line back to the starting point so that the cycle can repeat itself. The result of this visual representation is the interesting shape shown in Figure 2. Since we have previously established that the decimal expansions of the multiples of $\frac{1}{7}$ all follow the same pattern, the shape shown in Figure 2 is also representative of the decimal expansions of $\frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}$ and $\frac{6}{7}$.

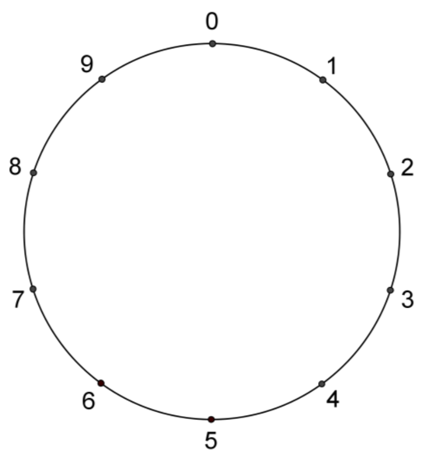

FIGURE 1: Blank clock face

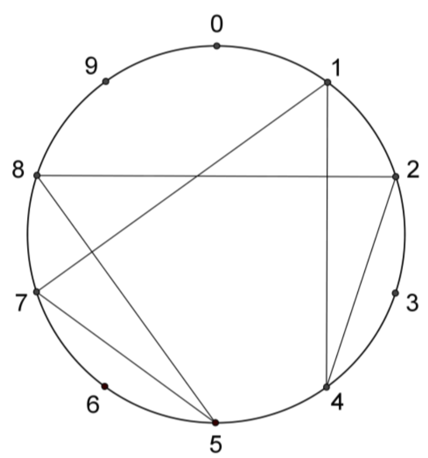

FIGURE 2: Clock face for $\frac{1}{7}$

The clock face representation of the cyclic number 142857 is aesthetically pleasing, the basis of this aesthetic appeal being the symmetry of the image. Is there anything particularly interesting about this symmetry? If one draws a line of symmetry through the shape, then one might notice that the numbers $0, 1, 2, 3, 4$ have as their counterparts $9, 8, 7, 6, 5$ respectively on the opposite side of the line of symmetry. And each pair, 0 and 9, 1 and 8, 2 and 7, 3 and 6, 4 and 5, adds to 9. The digit 9 rears its head again!

What about other cyclic numbers? Is a similar symmetry revealed in their clock face representations? Figure 3 and Figure 4 show the clock face representations for the cyclic numbers appearing in the decimal expansions of $\frac{1}{17}$ and $\frac{1}{19}$ respectively. The line of symmetry is shown in each case.

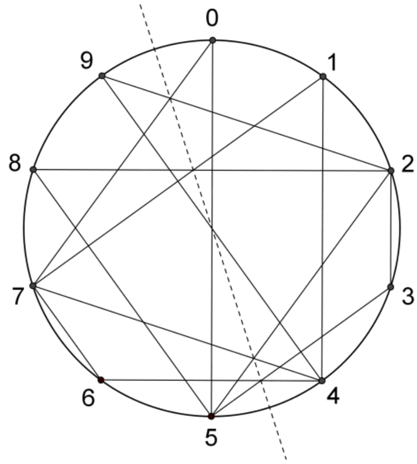

FIGURE 3: Clock face for $\frac{1}{17}$

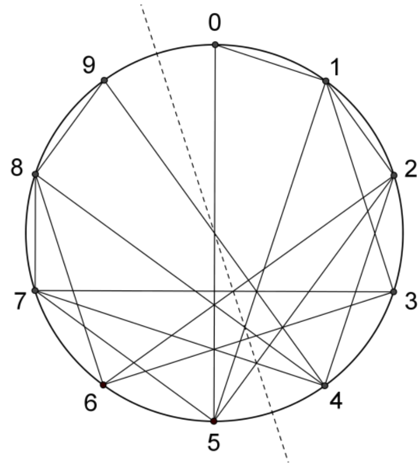

FIGURE 4: Clock face for $\frac{1}{19}$

The cyclic pattern appearing in the decimal expansion of $\frac{1}{17}$ is the same for all multiples of $\frac{1}{17}$ up to $\frac{16}{17}$. In other words, Figure 3 is representative of the decimal expansions of $\frac{1}{17}$ up to $\frac{16}{17}$. Similarly, Figure 4 is representative of the decimal expansions of $\frac{1}{19}$ up to $\frac{18}{19}$. However, in the case of $\frac{1}{13}$ and its various multiples, there are two different cyclic patterns in the decimal expansions. For $\frac{1}{13}, \frac{3}{13}, \frac{4}{13}, \frac{9}{13}, \frac{10}{13}$ and $\frac{12}{13}$ the repeating pattern is 076923 while for $\frac{2}{13}, \frac{5}{13}, \frac{6}{13}, \frac{7}{13}, \frac{8}{13}$ and $\frac{11}{13}$ the repeating pattern is 153846. The reason for the division into these two specific groups is an issue in itself, but have you noticed that each grouping contains three pairs of fractions whose sum equals 1? Figure 5 and Figure 6 show the clock face representations for the two cyclic numbers 076923 and 153846 respectively.

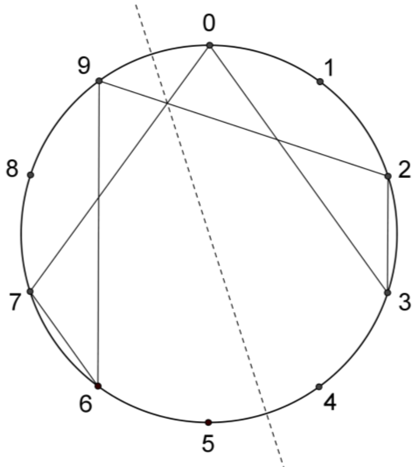

FIGURE 5: Clock face for $\frac{1}{13}, \frac{3}{13}, \frac{4}{13}, \frac{9}{13}, \frac{10}{13}$ and $\frac{12}{13}$

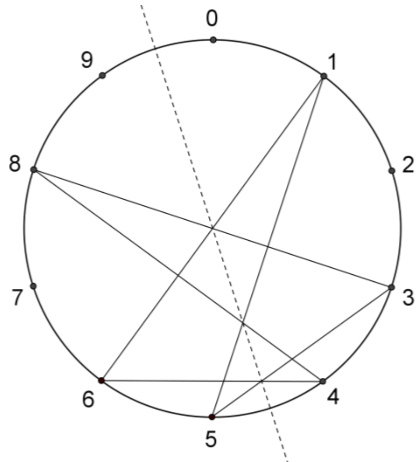

FIGURE 6: Clock face for $\frac{2}{13}, \frac{5}{13}, \frac{6}{13}, \frac{7}{13}, \frac{8}{13}$ and $\frac{11}{13}$

The digits of the cyclic pattern can also be represented in a circle diagram. Let's take the cyclic pattern in the decimal expansions of the multiples of $\frac{1}{7}$. We've previously seen that the multiples of $\frac{1}{7}$ follow the same pattern, the only difference being the starting point in the cycle (Figure 7). A question that naturally arises relates to the significance of each fraction's starting point in the cyclic pattern. Distributing the cyclic pattern evenly around a circle – and associating each digit of the cyclic pattern with the fraction that begins with that particular digit in its decimal expansion – reveals a number of interesting things. Firstly, note that the diametrically opposite fractions outside the circle add up to 1. Secondly, notice that the diametrically opposite digits inside the circle add up to 9. Finally, which of course relates to our first observation, the result of adding up the decimal expansion for diametrically opposite fractions is a string of 9s. The digit 9 once again rears its head!

$$\frac{1}{7} = 0,\overline{142857}$$
$$\frac{2}{7} = 0,\overline{285714}$$
$$\frac{3}{7} = 0,\overline{428571}$$
$$\frac{4}{7} = 0,\overline{571428}$$
$$\frac{5}{7} = 0,\overline{714285}$$
$$\frac{6}{7} = 0,\overline{857142}$$

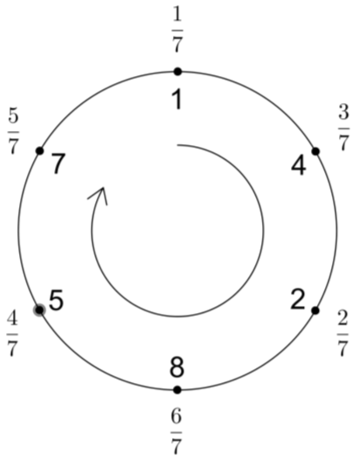

FIGURE 7: Circle diagram for multiples of $\frac{1}{7}$

CONCLUDING COMMENTS

Cyclic numbers stemming from the decimal expansions of certain fractions provide a fascinating context for mathematical exploration and investigation. The purpose of this article was not to engage too heavily with the theory behind cyclic numbers as this has been done extensively elsewhere through the use of the algorithmic process of long division, number theory, modular arithmetic, and group theory. Rather, what we hope we have accomplished in this article is to show how a simple idea can be developed into an ever expanding investigation resulting in intriguing discoveries which hopefully spark a mathematical desire to explore both further and deeper.

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