Powers are Powerful

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Introduction
In a mathematics lesson for Liberal Arts students that planned to be elementary school teachers, they were reviewing mathematical concepts like *base*, *power*, *exponent* and some of the properties of the exponential functions. As a starting point, I chose a trigger activity that I want to share with the readers.

The discovery
I wrote all the digits and under each one of them I wrote the units digit of their square. The students were asked to articulate their observations.

Base→ Exponent↓	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0	1	2	3	4	5	6	7	8	9
2	0	1	4	9	6	5	6	9	4	1

Observation 1: Without considering the 0, the second row is symmetric with respect to the cell with the number 5.
Observation 2: Some digits disappeared: 2, 3, 7 and 8.
Observation 3: A necessary condition for a perfect square number is that its units digit should be 0, 1, 4, 5, 6 or 9.

These observations do not contradict the fact that every perfect square number has the form $4n$ or $4n+1$ because every natural number has the form $2m$ or $2m+1$ and therefore squares have either the form $4m^2$ or the form $4(m^2 + m) + 1$.

The students were asked to calculate mentally the third power of each one of the ten digits and I proceeded to write the units digit of these numbers. The students were invited again to share their observations.

Base→ Exponent↓	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0	1	2	3	4	5	6	7	8	9
2	0	1	4	9	6	5	6	9	4	1
3	0	1	8	7	4	5	6	3	2	9

Observation 4: In the third row, all the digits reappeared.
Observation 5: The columns under the digits 0, 1, 5 and 6 are constant.
Observation 6: Without considering the digit 0, the sum of digits in cells symmetric with respect to the cell with the number 5 is always 10; the same property appears in the first row.

I asked the students to enlarge the table by writing the units digit of the fourth power of each digit:

Base→ Exponent↓	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0	1	2	3	4	5	6	7	8	9
2	0	1	4	9	6	5	6	9	4	1
3	0	1	8	7	4	5	6	3	2	9
4	0	1	6	1	6	5	6	1	6	1

Observation 7: Some digits disappeared again, even more than those that disappeared in the second row.
Observation 8: A natural number can be the fourth power of a natural number only if its units digit is 0, 1, 5 or 6.

Before I wrote the next row, I asked the students for their expectations. None of them was prepared for the big surprise:

Base→ Exponent↓	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0	1	2	3	4	5	6	7	8	9
2	0	1	4	9	6	5	6	9	4	1
3	0	1	8	7	4	5	6	3	2	9
4	0	1	6	1	6	5	6	1	6	1
5	0	1	2	3	4	5	6	7	8	9

Observation 9: All the digits reappeared.
Observation 10: All the digits reappeared in their original "place". This observation was formulated in mathematical language as follows: the ones digit of the fifth power of any digit is the digit itself.

Two main questions were formulated by the students:
Question 1: What happens with the *other* natural powers of the digits?
Question 2: Is this property true only for the ten digits? Is it true for *any* natural number?

The generalization
The questions were forwarded to the class and the students chose the one they preferred to investigate. After ten minutes of work in small groups or individually, the students' results were shared. One student presented the following reasoning:

> "Since $7^6 = 7^{5+1} = 7^5 7^1$, the units digit of 7^6 is the product of the units digit of 7^5 and 7^1. We found that the units digit of 7^5 is the same as the units digit of 7^1. So the units digit of 7^6 is the units digit of 7^2. I think the same will be true for any other base, not only for 7. So, the sixth row of the table will be like the second one."

Following his steps, other students discovered that the *seventh* row of the table is like the *third* one and the *eighth* is like the *fourth*. Instead of enlarging the table each time by adding rows, a student suggested renaming the first column - the one that presented the exponents - and we got:

Base→ Exponent↓	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1; 5; …	0	1	2	3	4	5	6	7	8	9
2; 6; …	0	1	4	9	6	5	6	9	4	1
3; 7; …	0	1	8	7	4	5	6	3	2	9
4; 8; …	0	1	6	1	6	5	6	1	6	1

From this array, the students discovered that they were in front of a *periodic phenomenon* and the length of its period was 4.

To verify whether this property – the answer to Question 1 - was understood by all the students, I asked them to calculate the following:

(a) the units digit of 5^{19}
(b) the units digit of 9^{13}
(c) the units digit of 9^{14}
(d) the units digit of 2^{23}
(e) the units digit of 3^{64}

To solve exercise (a) it was enough to remember that the units digit of all the natural powers of the number 5 is 5.
To solve exercises (b) and (c) the students needed to see that the units digit of *every odd* power of 9 is 9 and the units digit of *every even* power of 9 is 1.
To solve exercise (d) the students could count rows in the table or could use the periodicity discovered before and calculate: $2^{23} = 2^{20+3} = 2^{20} \cdot 2^3$. Then since 20 is a multiple of 4 (the period of the phenomenon) the units digit of 2^{23} is the same as the units digit of 2^3, which according to the table (column of 2 and third row) is 8.
To solve exercise (e) the students could have used the same strategy: since the exponent 64 is a multiple of 4, the units digit of 3^{64} is the same as the units digit of 3^4, which according to the table (column of 3 and fourth row) is 1.

The approach chosen to investigate Question 2 was trial and error:
A student shared his calculations:
$13^2 = 169$ --- units digit is 9 (like the units digit of 3^2)
$13^3 = 2197$ --- units digit is 7 (like the units digit of 3^3)
$13^4 = 28561$ --- units digit is 1 (like the units digit of 3^4)
$13^5 = 371293$ --- units digit is 3 (like the units digit of 3^5)
The students identified that the fifth power of the number 13 has the same units digit as its first power and conjectured:

For every natural number n, the units digit of n^5 is the same as the units digit of n.

Put differently, the student suggested that for every natural number n, $n^5 - n$ is a multiple of 10 .

The justification

Since the students were eager to know whether this was a true statement, I had their attention guaranteed. First I asked what they expected the units digit of 87^5 would be, and according to their conjecture it was 7. Then I proceed to verify their claim.
Since 87 = 80+7, I wrote $87^5 = (80+7)^5$ and reminded the students of the Binomial Theorem. We could write that

$$87^5 = (80+7)^5 = \binom{5}{0}80^5 \cdot 7^0 + \binom{5}{1}80^4 \cdot 7^1 + \binom{5}{2}80^3 \cdot 7^2 + \binom{5}{3}80^2 \cdot 7^3 + \binom{5}{4}80^1 \cdot 7^4 + \binom{5}{5}80^0 \cdot 7^5$$

The first five addends are multiples of 80, thus their sum is a multiple of 80 making 0 the units digit of their sum, so that the units digit of the number 87^5 is like the units digit of 7^5 .The students recognized that the same reasoning may be applied for any natural number and they were convinced that their conjecture was correct.

Since the occasion was inviting, I wanted to present other approaches to the proof of the recently discovered theorem. The following are the two other proofs presented.

We needed to prove that for every natural n, the number $n^5 - n$ is a multiple of 10. By factoring $n^5 - n$ we got that $n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = (n - 1)n(n + 1)(n^2 + 1)$

The last expression told us that for every natural number n, the number $n^5 - n$ is a multiple of the product of three consecutive numbers. Thus, it is a multiple of 6. In order to show that the number $n^5 - n$ is divisible by 10, it remained to be proved that the number is a multiple of 5.

Proof 1:
There are five possibilities for the natural number n. We needed to show that in each case, the number $(n-1)n(n+1)(n^2+1)$ is a multiple of 5 and it was done by showing that one of its factors is a multiple of 5.
a) If $n = 5k$ then obviously $(n-1)n(n+1)(n^2+1)$ is a multiple of 5.
b) If $n = 5k + 1$ then $(n-1)$ is a multiple of 5,
c) If $n = 5k + 2$ then (n^2+1) is a multiple of 5,
d) If $n = 5k + 3$ then (n^2+1) is a multiple of 5
e) If $n = 5k + 4$ then $(n+1)$ is a multiple of 5.
We concluded then that for every natural number n, $n^5 - n$ is a multiple of 10.

Proof 2:
Since the product of five consecutive natural numbers is a multiple of 5, we tried to write the number $n^5 - n$ with the help of such a product:

$$(n-1)n(n+1)(n^2+1) = (n-1)n(n+1)(n-2)(n+2) + ? = (n-1)n(n+1)(n^2-4) + ?$$

From here we discovered that $? = 5(n-1)n(n+1)$.
Then we had that $n^5 - n = (n-1)n(n+1)(n-2)(n+2) + 5(n-1)n(n+1)$ and, since it is the sum of two multiples of 5, $n^5 - n$ is a multiple of 5 for every natural number n.

Concluding remarks

In this activity, I see exemplified some of Polya's Ten Commandments for Teachers:

- ✓ Be interested in your subject.
- ✓ Know your subject.
- ✓ Try to read the faces of your students; try to see their expectations and difficulties; put yourself in their place.
- ✓ Realize that the best way to learn anything is to discover it by yourself.
- ✓ Give your students not only information, but also know-how, mental attitudes, the habit of habitual work.
- ✓ Let them learn guessing.
- ✓ Let them learn proving.
- ✓ Look out for such features of the problem at hand as may be useful in solving the problems to come – try to disclose the general pattern that lies behind the present concrete situation.
- ✓ Do not give away your whole secret at once – let the students guess before you tell it – let them find out by themselves as much as is feasible.
- ✓ Suggest; do not force information down their throats.

In fact, I hold the Chinese proverb

"Give a man a fish and you feed him for a day.
Teach a man to fish, and you feed him for a lifetime"

which may be considered an Oriental version of Polya's 5^{th} commandment, as one of my teaching axioms. I believe that in *every* opportunity, mathematics teachers need to keep these commandments in mind; sometimes we need to change the priority and the emphasis but having the students guess, experiment, test their own conjectures and then prove is one of the secrets of a powerful and successful mathematics lesson.