Methods of Multiplication

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INTRODUCTION

In this article we explore three different visually engaging methods of carrying out long multiplication. The standard algorithmic approach to long multiplication, as typically taught at primary school level, is generally along the lines of that illustrated in Figure 1.

$$\begin{array}{rrrr} & & 5 & 2 \\ & \times & 3 & 4 \\ \hline & 2 & 0 & 8 \\ 1 & 5 & 6 & 0 \\ \hline 1 & 7 & 6 & 8 \end{array}$$

FIGURE 1: Standard algorithm for long multiplication.

Such an algorithmic approach is of course succinct and effective in the sense that once the process has been mastered it can be carried out swiftly and efficiently. However, one wonders to what extent students really understand the algorithm. For many students it is no doubt a rote method that is mindlessly followed without any real conceptual understanding of the underlying mathematical processes.

One way to engage students with these underlying processes is to expose them to alternative methods of long multiplication. Three such alternative approaches are explored here – (i) drawing lines, (ii) using a lattice, and (iii) subdividing a rectangle. While these three methods might initially seem unrelated, they all have at their heart the same underlying process as the standard algorithm for long multiplication. For ease of comparison, each of the three methods is described using the same 2-digit numbers used in Figure 1, i.e. 52×34.

METHOD 1 – DRAWING LINES

This method (Figure 2) begins by representing each of the 2-digit numbers by means of line segments. The first number, 52, is represented by 5 line segments followed by 2 line segments drawn obliquely from left to right. The second number, 34, is represented by 3 line segments followed by 4 line segments, and is drawn obliquely (with an opposite slant to the first number) from left to right such that it overlaps the first number. The various points of intersection are then tallied and subdivided into three regions.

The region on the far right contains 8 points of intersection. The central region contains a total of 26 points of intersection – 20 at the top and 6 at the bottom. From the total tally of 26, the digit 6 is retained in the central region, and the 2 is carried to the left-most region. The region on the far left contains 15 points of intersection which, with the 2 carried from the central region, gives a total of 17. These three numbers – i.e. 17, 6, and 8 – when read from left to right provide the final product. We thus have $52 \times 34 = 1768$. This sequential process is illustrated in Figure 2.

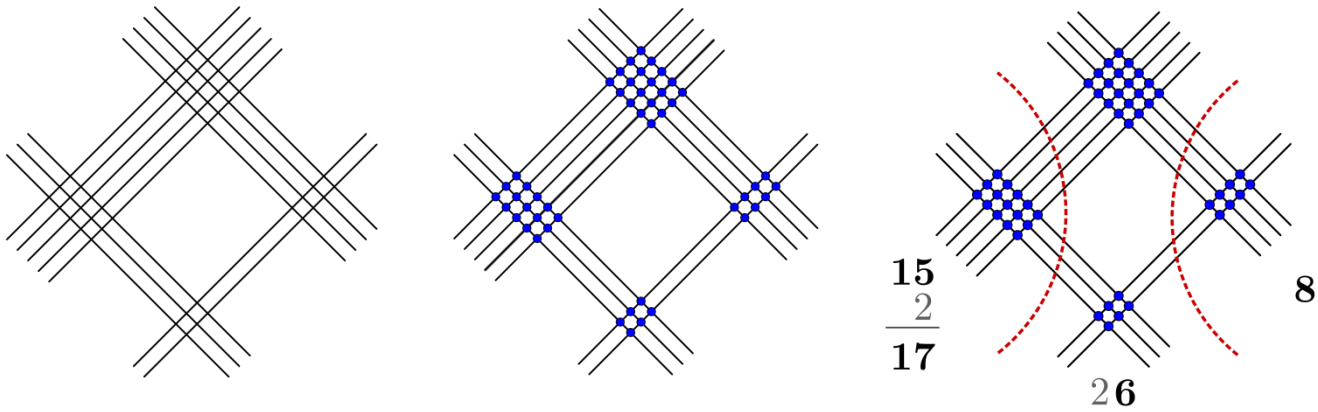

FIGURE 2: Multiplication by drawing lines.

METHOD 2 – USING A LATTICE

The lattice method (Figure 3) begins by drawing a square lattice – a 2×2 lattice in this particular case – with forward slanting diagonals. The two numbers to be multiplied are then written on the outside of the lattice – one number above the lattice running left to right, and the other number to the right of the lattice running top to bottom. Four individual products are then entered into the lattice. The product of 2 and 3 is 6, and this is entered into the lattice in the top right square as 06. The product of 2 and 4 is 8, and this is entered into the lattice in the bottom right square as 08. The other two products, $5 \times 3 = 15$ and $5 \times 4 = 20$ are entered as 15 and 20 in the top left and bottom left squares respectively. The numbers running obliquely between the diagonals are then summed. The first sum is simply 8. The second is 6+0+0 giving a total of 6. The third is 0+5+2 giving a total of 7. The final sum is simply 1. Each of these sums is recorded outside the lattice as indicated. Reading in an L-shape from top left to bottom right gives the answer to the product, namely $52 \times 34 = 1768$. This sequential process is illustrated in Figure 3. Note that if any diagonal had given a 2-digit sum the "units" digit would be recorded while the "tens" digit would be carried to the next diagonal.

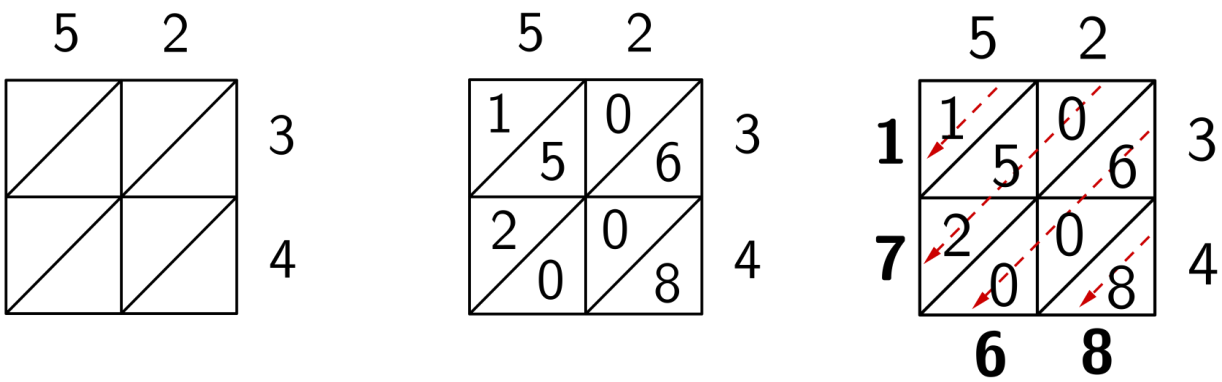

FIGURE 3: Multiplication using the lattice method.

METHOD 3 – SUBDIVIDING A RECTANGLE

In this method (Figure 4), the two numbers being multiplied – in this case 52 and 34 – are represented by the two adjacent sides of a rectangle (not necessarily drawn to scale). Each of the sides is then sub-divided. Since in this case each side is a 2-digit number, the sub-division is into tens and units. 52 is thus subdivided into 50 and 2 while 34 is subdivided into 30 and 4. These subdivisions split the rectangle into four component parts, each component itself being a rectangle. The area of each of these smaller rectangles is then calculated. The top left rectangle has dimensions 50×30 and hence an area of 1500. The top right rectangle has dimensions 50×4 and hence an area of 200. The bottom left rectangle has dimensions 2×30 and hence an area of 60, while the bottom right rectangle has dimensions 2×4 and hence an area of 8. Adding these four individual areas gives the total area of the original large rectangle and thus the answer to the product. Thus: $52 \times 34 = 1500 + 200 + 60 + 8 = 1768$.

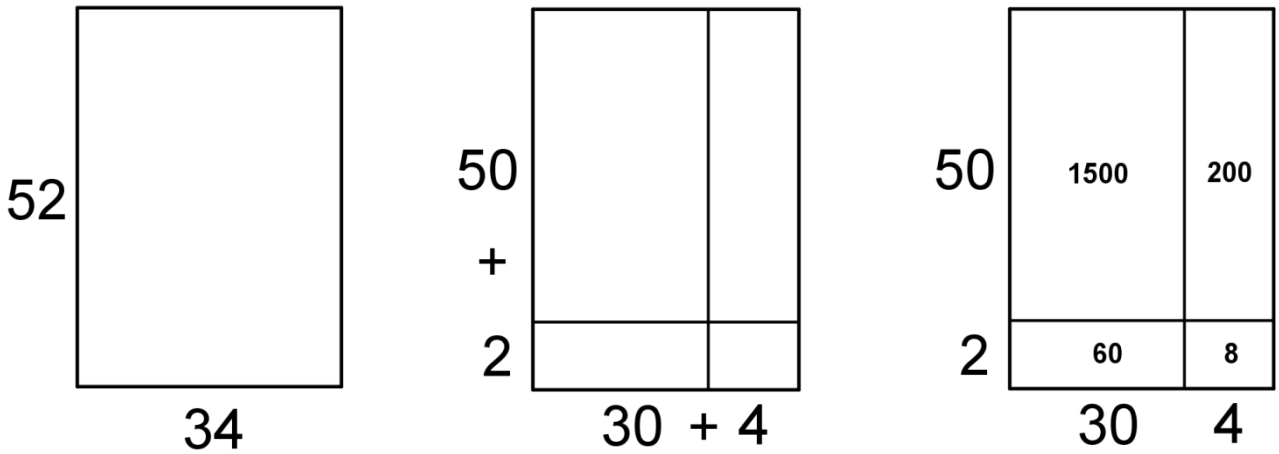

FIGURE 4: Multiplication by subdivision of a rectangle.

COMPARING THE THREE METHODS

As mentioned earlier, while these different methods might at first seem unrelated, they all share as their conceptual basis the same underlying process as the standard algorithm for long multiplication. The standard approach to long multiplication is essentially based on the distributive law, although this aspect of the process is partly obscured by the algorithm itself which overlaps some of the individual steps. An expanded version of the standard algorithm is shown in Figure 5. To multiply 52 by 34 we begin by multiplying the two units digits together, i.e. 4×2, to get 8. The units digit of the second number is then multiplied by the tens digit of the first number, i.e. 4×5, to get 20. This is then written next to the 8, i.e. it is transposed one column to the left since it is actually $4 \times 50 = 200$. The 200 and the 8 thus overlap to give 208. A place holder zero is then put at the far right of the next row. The tens digit of the second number is then multiplied by the units digit of the first number, i.e. 3×2, to get 6. The 6 is written next to the place holder zero since the calculation actually being carried out is $30 \times 2 = 60$. The tens digit of the second number is then multiplied by the tens digit of the first number, i.e. 3×5, to get 15. This is then written next to the 60, i.e. it is shifted two columns to the left since it is actually $30 \times 50 = 1500$. The 60 and the 1500 thus overlap to give 1560. The two combined sums are then added to give the final product.

$$
\begin{array}{r}
5\ 2 \\
\times\ 3\ 4 \\
\hline
2\ 0\ 8 \\
1\ 5\ 6\ 0 \\
\hline
1\ 7\ 6\ 8
\end{array}
\quad \Longrightarrow \quad
\begin{array}{r}
5\ 2 \\
\times\ 3\ 4 \\
\hline
8 \\
2\ 0\ 0 \\
6\ 0 \\
1\ 5\ 0\ 0 \\
\hline
1\ 7\ 6\ 8
\end{array}
$$

FIGURE 5: An expanded form of the standard algorithm for long multiplication.

Once we appreciate that the standard algorithm for long multiplication is essentially based on the distributive law, the connection between the standard algorithm and the other three methods starts to become clear. We can illustrate the product of 52 and 34 using the distributive law as follows:

$$
\begin{aligned}
52 \times 34 &= (50 + 2)(30 + 4) \\
&= 50 \times 30 \ + \ 50 \times 4 \ + \ 2 \times 30 \ + \ 2 \times 4 \\
&= 1500 \ + \ 200 \ + \ 60 \ + \ 8 \\
&= 1768
\end{aligned}
$$

Figure 6 compares the three alternative methods. The separate products of 1500, 200, 60 and 8 are clearly apparent in both Method 1 (drawing lines) and Method 3 (subdividing a rectangle). In the case of Method 1 the lines intersect 15 times in the left-most region (the hundreds column, thus 1500), 20 times at the top, and 6 times at the bottom, of the central region (the tens column, thus 200 and 60 respectively), and 8 times on the far right (the units column). In the case of Method 3, the individual products are generated by the dimensions of the four smaller rectangles.

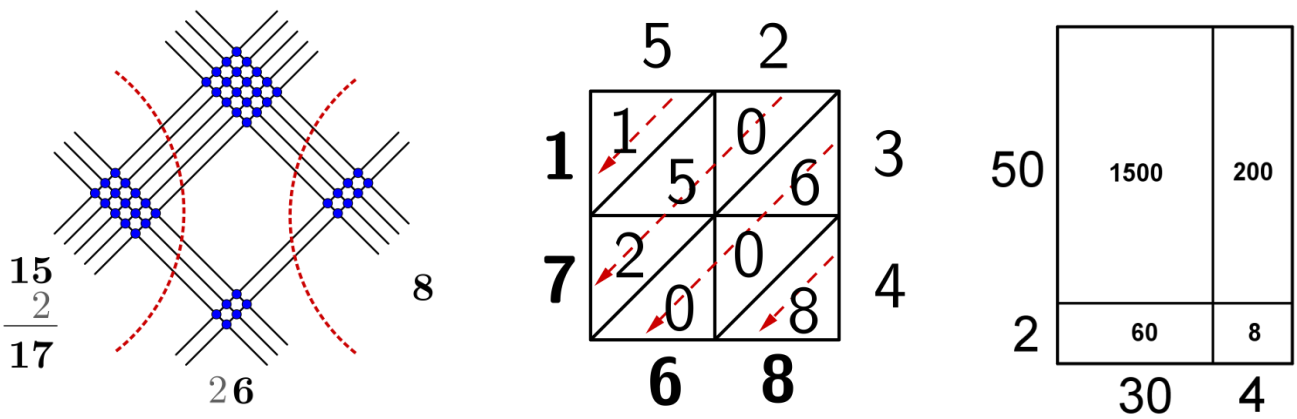

FIGURE 6: Comparing the three methods.

Understanding Method 2 (using a lattice) is a little trickier, but perhaps the simplest approach is to rotate the lattice 45° counter-clockwise (Figure 7) so that the regions between the diagonals become columns – units, tens, hundreds and thousands (working right to left). In much the same way that the standard algorithm for long multiplication automatically offsets the various digits into their correct columns, the diagonals in the lattice method perform a similar function. Looking at Figure 7 we can see three horizontal rows. The middle row, i.e. 1508, is the overlap of 1500 (i.e. 30×50) and 8 (i.e. 2×4). The top row, i.e. 06, is actually 060 ($30 \times 2 = 60$) since the 6 is in the tens column. The bottom row, i.e. 20, is actually 200 ($4 \times 50 = 200$) since the 2 is in the hundreds column. The algorithmic equivalence of the lattice method and the standard method for long multiplication should now be clear.

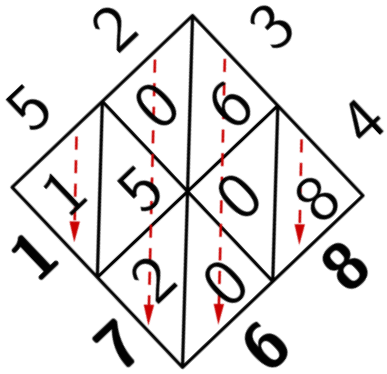

FIGURE 7: The lattice rotated 45° counter-clockwise.

EXPANDING THE PROCESS TO BIGGER NUMBERS

All four methods can be applied to larger numbers. Figure 8 illustrates the standard algorithm as well as the lattice method being applied to the product 235×324, while Figure 9 illustrates the same product for the other two methods.

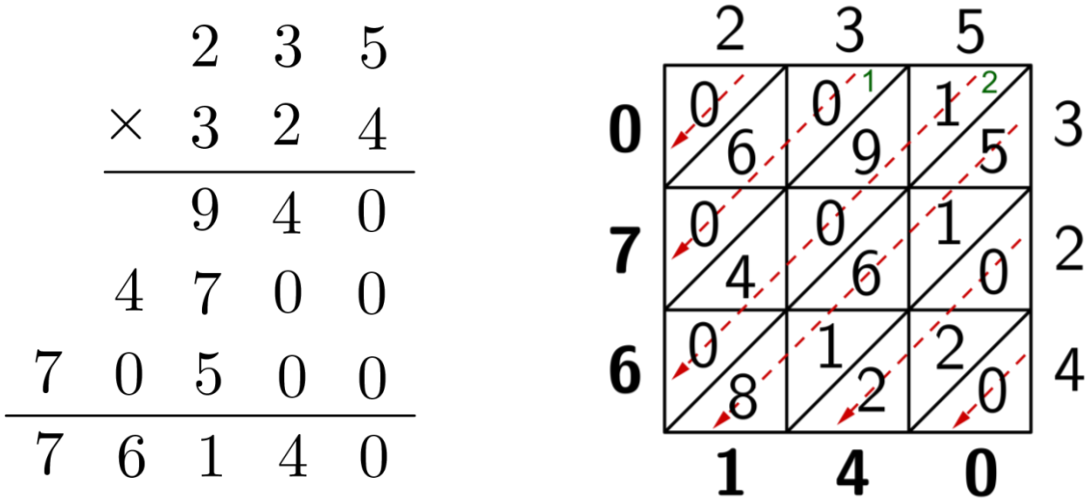

FIGURE 8: 235×324 using standard algorithm and lattice method.

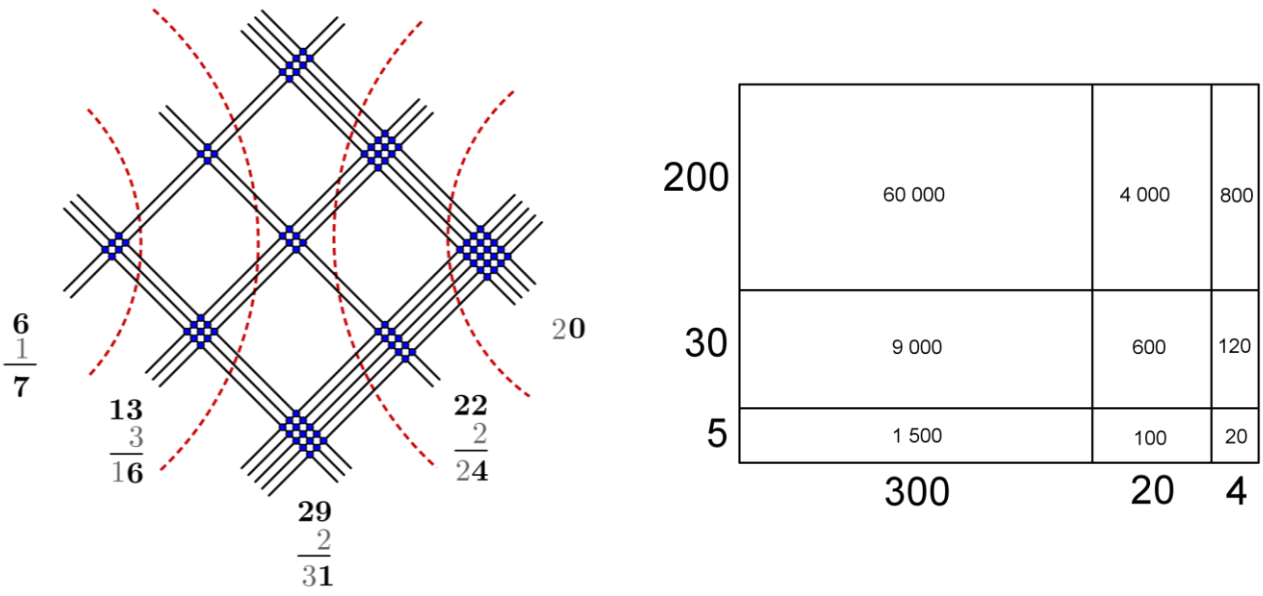

FIGURE 9: 235 × 324 using lines and a subdivided rectangle.

CONCLUDING COMMENTS

The three alternative methods of long multiplication described in this article are both fascinating and intriguing. A quick search on YouTube or Google will reveal a plethora of video clips and web pages illustrating the "magic" of these unconventional methods. However, the beauty in these alternative approaches is not that they somehow "magically" produce the correct answer, but rather that the underlying process at work is one and the same as that which forms the basis of standard long multiplication. The real magic lies in making this connection and exploring the underlying concept to appreciate *why* the various methods work.

An exploration of these alternative methods lends itself rather nicely to a classroom investigation. We tried this with our Grade 9s when covering products of binomial expressions. A YouTube clip was first shown which demonstrated the particular method in action. Pupils were then required to apply the method to a number of different products. This process was repeated for each of the three methods. In addition, we extended Method 3 (subdividing a rectangle) to include the multiplication of algebraic binomials. This proved particularly useful for pupils as a means of conceptualising the middle terms of the product. Feedback from the investigation was extremely positive, with pupils of all ability levels enjoying and really engaging with the activity.