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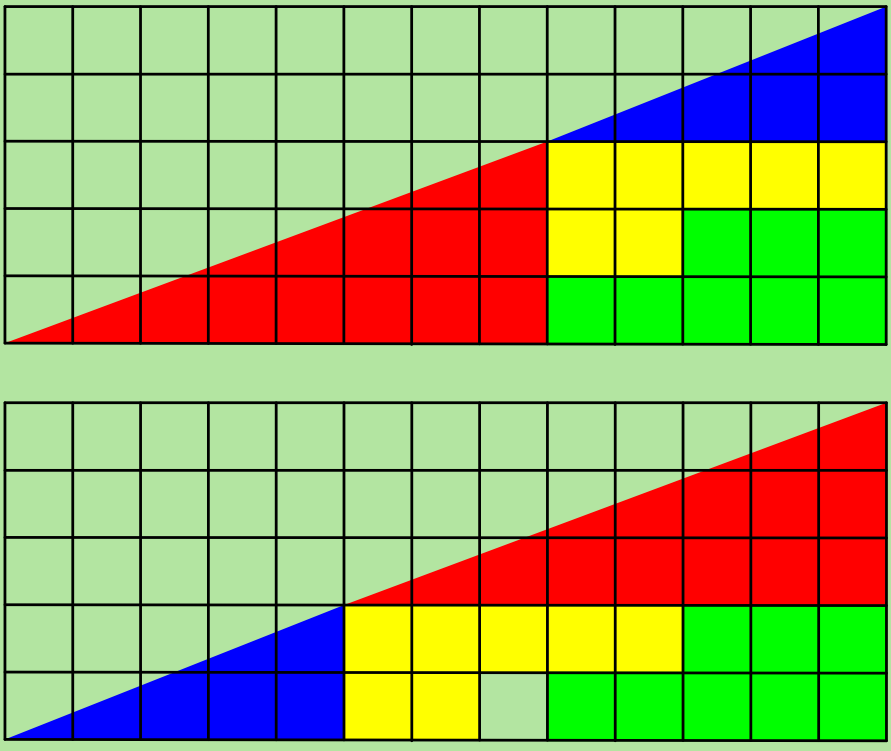

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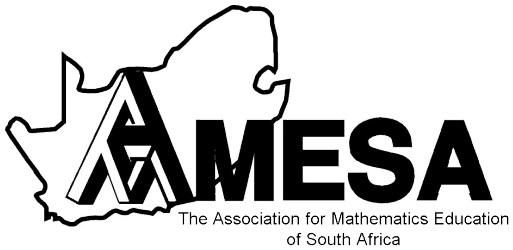
Learning and Teaching Mathematics is a journal of the Association for Mathematics Education of South Africa (AMESA). This journal is aimed at mathematics teachers at primary and secondary school level and it provides a medium for stimulating and challenging ideas, offering innovation and practice in all aspects of mathematics teaching and learning in school. Learning and Teaching Mathematics aims to inform, enlighten, stimulate, challenge, entertain and encourage mathematics educators. Its emphasis is on addressing the challenges that arise in the mathematics classroom. It presents articles that describe or discuss mathematics teaching and learning through the eyes of practising teachers and learners. While this journal 'listens' to research and considers it in the activities, lesson ideas, and teaching strategies that it publishes, it is not a research publication.



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Articles submitted will be reviewed by the editors and members of the Editorial Board. The Board will ensure that the papers make a contribution to our understanding of mathematics learning and teaching, that the mathematics presented is correct, and that the language and layout used is user friendly. Support will be provided by the editors to contributors in relation to meeting the above requirements.

The main criterion of acceptance is that the article should make a contribution to the improvement of school mathematics teaching and learning. See the inner back cover for more information on the submission of materials and articles for publication.

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From the Editor

Dear LTM readers

In the first article of LTM 37, Nadia Theba, Shikha Takker and Craig Pournara present a variety of tasks which could support teachers to identify and address common errors that learners make when simplifying algebraic expressions with brackets. In the second article in this issue, Duncan Samson, Moshe Stupel and Mark Evans share their experiences of how a simple premise involving twelve matchsticks led to some fascinating mathematical exploration. The third article, by James Metz, revisits a classic mathematical illusion and explores ways to create similar visual illusions, while in the fourth article Moshe Stupel and Zino Toare present two activities designed to encourage numerical reasoning and flexibility. Alan Christison then considers the circumstances under which quadratic sequences contain repeated terms.

In the sixth article, Michael de Villiers explores a geometrical scenario that leads to a surprising equilateral triangle, while in the seventh article Duncan Samson and Moshe Stupel present a variety of different proofs for the classic "three squares problem". In the final article of issue 37 of LTM, Michael de Villiers asks "*what if?*" and explores a simple idea of taking a triangle and constructing arbitrary triangles on each of its sides.

We hope you enjoy the diverse array of articles in this issue and remind you that we are always eager to receive submissions. Suggestions to authors, as well as a breakdown of the different types of article you could consider, can be found at the end of this journal. If you have an idea but aren't sure how to structure it into an article, you are welcome to email the editor directly – we'd be happy to engage with you about turning your idea into a printed article.

Duncan Samson

Developing Learners' Understanding of Brackets in Algebraic Expressions

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INTRODUCTION

Brackets play an important role in algebraic expressions. However, their meaning is dependent on their place and use in such expressions. In this article, we highlight the different roles that brackets play in mathematical structures and the potential problems encountered by learners when making sense of brackets in algebraic structures. We begin by describing the different meanings brackets can have. We then highlight three common errors that learners make when simplifying algebraic expressions with brackets and discuss the source of these errors. We also suggest a variety of tasks which could support teachers to identify and address such errors and offer conceptual understanding to learners.

THE MEANING OF BRACKETS IN SCHOOL MATHS

Brackets have different meanings in mathematics, and the meaning is determined by the context. We give three examples to illustrate this:

- Brackets can be used as a grouping tool in expressions involving multiplication. Two familiar formulas that make use of brackets in this way are $T_n = a + (n-1)d$ and $A = P(1+i)^n$.
- Brackets can be used purely as a grouping tool, for example $\cos(\theta - 10°)$. Here there is no multiplication implied by the brackets.
- Brackets are also used in conventional mathematical notation, for example when representing ordered pairs such as $(2\,;3)$, when representing the domain or range of a function, for example $x \in (-\infty;\ \infty)$ or $y \in [-2\,;1)$, or in function notation such as $f(x)$ or $g(2)$. In these contexts there is also no multiplication implied by the brackets.

In this article our focus is on the use of brackets in simplifying algebraic expression at Grades 8 – 10 levels. In this context, brackets have two non-exclusive uses: (a) as a grouping or separation tool and (b) as a signifier of a mathematical operation. By way of example, in the expression $2(x-3)$, brackets group the terms x and -3 signifying that both terms must be multiplied by 2, while in the expression $2x-(x+3)$, brackets separate the grouped terms $(x+3)$ and signify that both terms in brackets must be subtracted from $2x$. In the expression $(x-2)(-4x+1)$, brackets group the pairs x and -2, and $-4x$ and $+1$, signifying that both terms of both brackets need to be multiplied by each other, while in the expression $(x-2)+(-4x+1)$, the grouped terms $(-4x+1)$ must be added to the grouped terms $(x-2)$.

We now consider common errors that learners make with brackets and explore the potential source of such errors.

COMMON ERRORS AND THEIR SOURCES

Based on our research and teaching experience, learners often have difficulties using brackets in algebra because they do not understand the role of brackets in algebraic expressions. The three common errors include: (a) an assumption that brackets imply the action 'to multiply', (b) using multiplication incorrectly, and (c) applying an incorrect order of operations. We discuss each of these errors in turn.

Do brackets always mean multiplication?

A common misconception is that the presence of brackets automatically means 'to multiply'. Learners overgeneralise the meaning of brackets as multiplication in situations where this is not appropriate. For example, in the case of $-5(x+3)$, we multiply -5 with both x and $+3$ to obtain $-5x-15$, but in the case of $5-(x+3)$, both x and $+3$ are subtracted from 5 to produce $-x+2$. Learners often ignore the minus symbol before the bracket, such as in $5-(x+3)$, and use multiplication inappropriately to obtain $-5x-15$.

The source of this difficulty could be linked to prior learning that brackets imply multiplication. For example, when working with numbers and brackets such as $(2)(-3)$, learners are taught that brackets indicate multiplication. The misconception of overgeneralising the meaning of brackets to multiply is then also applied in algebraic expressions involving letters. For example, in grade 9, much emphasis is placed on multiplying two binomials such as $(x-3)(x+2)$. When presented with an expression where the brackets indicate grouping, such as $(x-3)+(x+2)$, learners may focus on the familiar binomial structure and ignore the operation of addition between them, and therefore inappropriately apply multiplication.

The procedure of multiplication

Another difficulty is that multiplication in algebraic expressions is not always executed correctly, especially when structures consist of letters and numbers within brackets. We identified three common types of incorrect multiplication.

Firstly, learners use multiplication incorrectly by *multiplying adjacent terms only*. For example, in $2(x^2-4x+1)$, brackets group three terms x^2-4x+1, meaning that the 2 should be multiplied by all three terms in brackets to obtain $2x^2-8x+2$. A common error is for learners only to multiply the adjacent terms 2 and x^2 to obtain $2x^2-4x+1$. Evidence of this error suggests that learners do not appreciate that they are applying the distributive property to each component of the trinomial.

Secondly, learners use multiplication incorrectly by operating on 'like terms' only. For example, in the product $(x+5)x$, learners may multiply only the like terms to obtain x^2+5. This error could stem from the knowledge that only like terms can be added and subtracted, which is then overgeneralised to the multiplication of unlike terms.

Thirdly, in situations when multiplication is used appropriately, incorrect products are sometimes obtained. For example, in $(x-2)(x+3)$, both terms in the first bracket should be multiplied by both terms in the second bracket to obtain $x^2+3x-2x-6$. There are three different cases of multiplication required for this type of product, i.e. multiplying letters with letters $x \times x = x^2$; multiplying letters with numbers $-2 \times x = -2x$ and $3 \times x = +3x$; and multiplying numbers with numbers $-2 \times 3 = -6$. In each of the three cases, the plus and minus symbols need to be considered as both operations (x subtract 2, and x add 3) and signs (negative 2 and positive 3). When dealing with the different cases, learners may obtain incorrect products. For example, simplifying $(x-2)(x+3)$ to $2x+3x-2x+1$ shows that the two first terms in each bracket and the two last terms have been added instead of being multiplied.

Order of operations

The third type of error is where learners operate on algebraic expressions using an incorrect order of operations. Some examples of this are highlighted below.

Errors often arise when there is more than one possible order of operations, for example in expressions involving multiplication only, where it is not always appropriate/efficient to multiply from left to right. The expression $-2(x+3)x$ involves the multiplication of two monomials and a binomial, and multiplication could be carried out in any order. If a learner multiplies from left to right, they will get $(-2x-6)x$. This step needs to be followed by distributing x into the bracket. Alternatively, the learner could first multiply the two monomials, -2 and x, to obtain $-2x(x+3)$ and then distribute this into the bracket. Questions like this require the learner to pay careful attention to the structure of the expression in order to avoid errors.

In more complex structures, learners often struggle with the order of operations. Consider the expression $2(x-4)^2$. Learners may have difficulty deciding which operation to do first, asking themselves 'what does it mean to "do brackets first" and which part of the expression with brackets must be dealt with first? Should I multiply 2 into the bracket first and then distribute the exponent, or should I square the bracket first and then distribute the 2?' If a learner decides to multiply 2 into the bracket first, they will obtain incorrect answers such as $(2x-8)^2$ because the structure of the expression changes to $(2x-8)(2x-8)$. If learners distribute the exponent into the brackets, incorrect answers such as $2(x^2-16)$ would be obtained. In this case, the learner is unable to see the hidden structure of $(x-4)^2$ as $(x-4)(x-4)$. But even if the learner correctly begins by squaring the binomial, multiplication errors such as $2(x^2-8x-16)$ can still arise. Expressions involving different types of mathematical symbols makes it less clear what to do first, and the mental effort of having to use different operations in a specific order makes it particularly challenging for learners.

Finally, using BODMAS to determine the order of operations in an algebraic expression is not always appropriate. In earlier grades, BODMAS is used when dealing with numbers only in expressions such as:

$$2+(5-4)$$
$$=2+(1)$$
$$=3$$

The numbers in the brackets are subtracted first, and then added to 2. Here it makes sense to "do the brackets first". However, when working with algebraic expressions, the strategy of doing the brackets first may lead to conjoining errors where learners combine unlike terms in order to simplify the bracket. For example, in $3x-(x+2)$ there are two unlike terms in the brackets which cannot be combined, unlike the numeric case of $(5-4)$. Learners who make this error may end up with $3x-(2x)$ if they "do the brackets first".

TASKS TO ADDRESS LEARNER DIFFICULTIES

Learners must be supported to "see the structure" of algebraic expressions before "operating" on them. To 'see' the structure means being able to identify relationships between the various components. The relationships are frequently determined by how these components are arranged. For example, the arrangement of brackets in $2(-x+3)$ signifies multiplication of a monomial and binomial, and in $(2-x)+3$ signifies addition of a binomial and monomial. If learners can identify relationships between the various components of an expression, they are more likely to identify the appropriate order of operations in simplifying the expression.

We now present a series of tasks that could help teachers address learner difficulties similar to the ones previously described. We have made explicit use of the principle of variation in designing these tasks. Systematic variation in a set of examples has been found useful in drawing learners' attention to key issues in the structure of expressions. This means that a variety of expressions, with small but noticeable changes, must be provided so that learners can develop a deeper experience of working with algebraic expressions and hence may recognise their own errors and correct them.

We provide four tasks with a range of expressions. The tasks are independent of one another because each task focuses on drawing attention to specific errors. This means that the tasks do not cover all problem areas that learners experience when simplifying algebraic expressions, and the tasks can be used in any order.

Task 1: "Seeing" the expressions

The purpose of Task 1 is to draw learners' attention to the arrangements of the brackets in expressions. Task 1 includes only positive coefficients/constants and addition. Having minus symbols in these expressions could distract learners from focusing on the role of brackets intended for this task because minus symbols involve different ways of simplifying. Brackets group single terms $(3x)$ or more than one term (compound terms) such as $(x + 3x)$. This means that the brackets separate single or compound terms from other components in the expressions which can then be multiplied or added. Questions related to multiplication and addition should be asked, such as, (a) when adding, must like terms be considered? (b) how do we add like terms? and (c) how does multiplication affect the exponents of variables/unknowns?

TASK 1: Four expressions are given below. Look at these expressions carefully and answer the questions that follow.

(a) $5x(3x)$ (b) $5x + (3x)$ (c) $5(x + 3x)$ (d) $5 + (x + 3x)$

1. What is the same in the four expressions (a) to (d)?
2. What is different in the four expressions?
3. State the number of terms in each expression (a) to (d).
4. Predict which expressions will give the same answer after simplifying.
5. Now simplify each expression. Were your predictions correct?
6. Create one expression each for (a) and (c), that consists of a different arrangement of brackets and components, but which would be equivalent to the respective expression after simplifying.

The first question in Task 1 shows learners that the same numbers (3 and 5), letters (x), plus symbols and brackets can be arranged in various ways to produce different structures. We intentionally use 'plus symbol' in this section because we refer to the arrangement of the symbol rather than the sign or operation in the expressions. Looking more closely, specific arrangements make expressions even more similar. Expressions (a) and (c) are similar because the first terms are placed directly in front of the bracketed terms. Expressions (b) and (d) are similar because there is a plus symbol between the first term and the bracket. In expressions (a) and (b), both have a single term in brackets, while in expressions (c) and (d), both have the same compound term in brackets. Learners need to consider what operation is signified by the brackets when terms and brackets are placed directly adjacent to each other, and what operation is indicated when a plus symbol is placed between the preceding term and brackets.

The second question brings learners' attention to the difference between the expressions, even though the same components are used. Expressions (a) and (b) both have $5x$ and $3x$ but the difference is that (b) has a plus symbol before the brackets. Similarly, (c) and (d) consist of 5 and $(x + 3x)$, but (d) has a plus symbol before the brackets. How does the plus symbol before the brackets in (b) and (d) make the expressions different to (a) and (c) respectively? Further differences between expressions (a) and (c), and (b) and (d), are evident. In (a) and (c) and in (b) and (d), what is different about the terms in brackets? Which are single terms and which are compound terms? How do compound terms affect the operations of addition and multiplication?

The third question encourages learners to think about the positioning of the plus symbols that separate terms. In expression (a) both $5x$ and $3x$ are one term. Expression (b) consists of a plus symbol before the bracket, producing two terms. Both expressions (c) and (d) consists of a plus symbol within the brackets, resulting in two terms within the brackets. But what does it mean to have the two terms within a bracket? Do the brackets group the two terms x and $3x$ as one compound term? Expression (c) has no plus symbol before the brackets, so how many terms are evident? In (d) there is an additional plus symbol before the bracket, so how many terms are in this expression?

The fourth question will encourage learners to first predict results by looking at the expressions carefully. Learners will need to apply their knowledge of adding and multiplying terms. If a learner is able to distinguish which expressions involve multiplying and adding they will then understand that multiplying and adding will not result in the same answer. For example, using multiplication in $5x \times 3x = 15x^2$ affects both the coefficient and the letter, but using addition in $5x + 3x = 8x$, like terms are added where only the coefficient is affected.

Learners will simplify the expressions for the fifth question to check if their predictions were correct. The sixth and final question addresses the issue of noticing the meaning of brackets, as grouping, separating, adding and multiplying; and to identify ways to change the order of terms to obtain equivalent expressions. Components can be used in a variety of ways to obtain equivalent expressions, so there are numerous possible answers.

Task 2: Meanings of brackets

The purpose of Task 2 is for learners to consider brackets as a grouping tool of a single term $(2x)$ or compound terms $(x - 3)$ in relation to minus symbols in expressions. The positioning of the compound terms in relation to the other components influences the operation to be performed. When there is no minus symbol directly between a component and the compound term, then multiplication is implied. If there is a minus symbol between a compound term and another component, then subtraction is required.

TASK 2: Answer the questions for the six expressions given below.

(a) $2x(x - 3)$ (b) $(x - 3)2x$ (c) $(x - 3)(-2x)$

(d) $(x - 3) - 2x$ (e) $2x - (x - 3)$ (f) $-(2x)(x - 3)$

1. Circle the expressions involving multiplication and highlight the expressions involving subtraction.
2. Tick the pair of expressions which will give the same answer.
   (i) a and b (ii) c and d (iii) d and e (iv) a and f
3. Is it possible to remove the brackets from any of the expressions without changing the expression? If so, rewrite the expression without brackets and simplify.

The first question draws learners' attention to the position of the minus symbol in relation to the compound term. If there is a symbol before or after the bracket, thinking about the required operation becomes critical. For example, in (e) what does the minus symbol before the bracket mean, and in (d), what does the symbol after the bracket mean? If there are no minus symbols directly before or after the bracket, is multiplication always the required operation? For example, consider (c) – where is the minus symbol positioned, and will multiplication or subtraction be necessary?

The second question will encourage learners to recognise that certain arrangements of brackets and signs involve the same operation and will have the same answer. This question is similar to Task 1 Question 4, which demonstrates that brackets and symbols can be rearranged in different ways to obtain the same answer. In multiplication, the order of operations does not affect the answer such as in expressions (a) and (b), and (c) and (f). But when subtracting, the order of operating *does* affect the answer, such as in expressions (d) and (e).

The third question brings to attention the role of brackets in grouping, and where such grouping is necessary. When multiplication or subtraction affects both terms in brackets, then brackets will be necessary, such as expressions (a), (b), (c), (e) and (f). However, in (d) the presence of brackets around the compound term does not affect the expression.

Task 3: Identifying operational errors in expressions

In Task 3, learners focus on possible responses to simplifying expressions involving squaring and multiplication by a numeric monomial.

TASK 3: Learner responses to two expressions are given below. Study the expressions and responses and answer the questions that follow.

1. Simplify $(x-4)^2$

   (i) Three incorrect responses are provided below. Highlight the part of the response where the learner has made an error and describe what error was made.

Response 1:	Response 2:	Response 3:
$(x-4)^2$	$(x-4)^2$	$(x-4)^2$
$= x^2 - 4^2$	$= (x-4)(x-4)$	$= (x-4)(x-4)$
$= x^2 - 16$	$= x^2 - 8x - 16$	$= x^2 - 8x + 8$

   (ii) How do you simplify $(x-4)^2$?

2. Simplify $3(x+2)^2$

   (i) Three responses are provided below. Identify the correct response.

Response 1:	Response 2:	Response 3:
$3(x+2)^2$	$3(x+2)^2$	$3(x+2)^2$
$= (3x+6)^2$	$= 3(x+2) + 3(x+2)$	$= 3(x^2 + 4x + 4)$
$= (3x+6)(3x+6)$	$= 3x + 6 + 3x + 6$	$= 3x^2 + 12x + 12$
$= 9x^2 + 36x + 36$	$= 6x + 12$	

   (ii) Consider the two incorrect responses. Explain how the order of operations has been used incorrectly in each incorrect response.

The first question requires learners to identify the incorrect parts of the response when squaring a binomial. Response 1 suggests that $(x-4)^2$ was not recognised as multiplication of two binomials, rather each term in brackets was independently squared. Both responses 2 and 3 show the correct interpretation of the multiplication of two binomials, although errors were made during multiplication.

The second question presents two incorrect and one correct response to simplifying $3(x+2)^2$. Such responses are prevalent and need particular attention. Learners can be encouraged to first think about which operation should be performed. Generally, when using BODMAS with numeric examples, brackets are simplified first as previously discussed. However, with algebraic expressions it is seldom possible to simplify the bracket and so one must deal with the exponent and the monomial in relation to the entire compound term. Response 1 shows that the learner first multiplied 3 into the brackets and then squared the binomial. In response 2, the square on $(x+2)$ is recognised as repeating the bracket. However, the error is that the 3 positioned directly before the brackets is considered as a component of the bracket and is then also inappropriately repeated. Another error in the same expression is that the two brackets are added instead of multiplied.

Task 4: Placement of brackets

The purpose of Task 4 is to make the different meanings of the brackets explicit for the learners and then to combine this knowledge with the use of signs and operations to create equivalent expressions.

TASK 4: Look at the expressions and incomplete learner responses below. Answer the questions that follow.

1. Insert brackets into the expressions so that the given answer is CORRECT. You may not change the answer.

   (i) $2\,x+3+x-5$
   $=2x+6+x-5$
   $=3x+1$

   (ii) $2\,x+4\;\;x-4$
   $=2\,x^2-16$
   $=2x^2-32$

2. The four examples below are incomplete because the signs and operations have not been given. Complete each example by inserting the correct sign or operation.

   (i) $3x-(2+y)$
   $=3x\;\square\;2\;\square\;y$

   (ii) $-3x(2+y)$
   $=\square 6x\;\square\;3xy$

   (iii) $(3x+1)-(x-2)$
   $=\;\square\,3x\;\square\;1\;\square\;x\;\square\;2$
   $=\;\square 2x\;\square\,3$

   (iv) $-(3x+1)(x-2)$
   $=\square(3x^2\square\;6x\;\square\;x\;\square\;2)$
   $=\square\,3x^2\;\square\;5x\;\square\;2$

The first question requires learners to consider the role of brackets as a combination of *grouping* and *separating* that signifies the operations of multiplication, addition and subtraction. The question encourages learners to place brackets in different positions in the given expression to reach the second step of the response. The aim is for learners to be able to compare the given expression with the second step in order to identify what operation has taken place, and to then place the brackets to obtain that operation. In expression (i), learners should realise that $x+3$ has been been multiplied by 2, requiring a bracket around these terms, $(x+3)$. The next part of the given expression $+x-5$ has not been changed in the second step. Therefore, no operation has taken place, and no bracket is necessary.

The second question represents expressions that are similar to what is encountered in school textbooks. When learners encounter a wide variety of expressions involving a combination of operations, they tend to have difficulties because so much in the expression is changing at once. Therefore, we propose a kind of scaffolding towards mixed examples, which can be reorganised to cluster the expressions in different ways, to assist learners in identifying key differences. In this question, learners must consider the relationship between grouped terms in brackets and plus/minus symbols in expressions. If a minus symbol is placed directly in front of a bracket such as in (i), (iii), and (iv), how does it affect the sign of the terms in brackets? If there is a negative term before the bracket such as in expression (ii), what operation must be used? In all instances, the signs of the terms in brackets will change. In expression (ii), multiplication of the numbers and letters is also required. After brackets are no longer present in the expressions, learners then have to use different methods of simplifying expressions by adding and subtracting like terms.

CONCLUDING COMMENTS

The teaching of simplifying algebraic expressions in classrooms tends to focus on one or two methods at a time. For example, a particular lesson might deal only with addition of algebraic expressions using examples such as $(2x + 4) + (x - 3)$, another lesson might deal only with subtraction of expressions such as $(4x + 7) - (2x + 1)$, and a separate lesson might only deal with multiplying two binomials using examples like $(2x + 1)(x - 3)$. This means that learners are taught how to deal with different structures in isolation. However, challenges occur when learners are given a variety of structures with different arrangements of components. Learners are not always able to see and interpret different types of structures when clustered together, resulting in incorrect operations being used. We have provided examples of tasks which attempt to avoid this isolation yet still make use of variance amidst invariance to bring into focus aspects of algebraic simplification that typically lead to errors. The tasks encourage learners to pay attention to the structure of the expressions before operating on them. We encourage teachers to do likewise as they adapt standard exercises from textbooks.

ACKNOWLEDGEMENT

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Building Shapes with Integer Areas Using Twelve Matchsticks

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INTRODUCTION

The following diagram shows a square made from 12 matchsticks. The perimeter of the square is 12 units, and the area of the square is 9 square units.

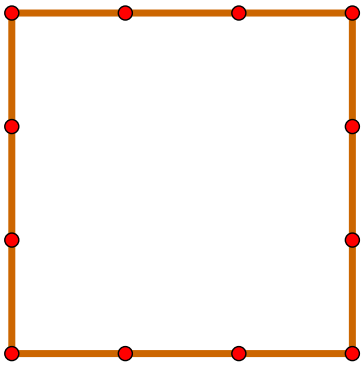

FIGURE 1: The starting configuration with area 9 square units

Without breaking any matchsticks, and without any matchsticks overlapping, is it possible to create a single enclosed region of perimeter of 12 units that has an area of 8 square units? What about 7 square units? What about 6, 5, 4, 3 or 2 square units? Or even 1 square unit?

The above investigation was carried out with different groups of pupils in South Africa and Israel. In this article we share our combined experiences and illustrate how a simple starting premise led to some fascinating mathematical exploration.

GETTING STARTED

Pupils were quick to create enclosed regions of 8 square units and 5 square units by building rectangles with dimensions 4 by 2 and 5 by 1 respectively.

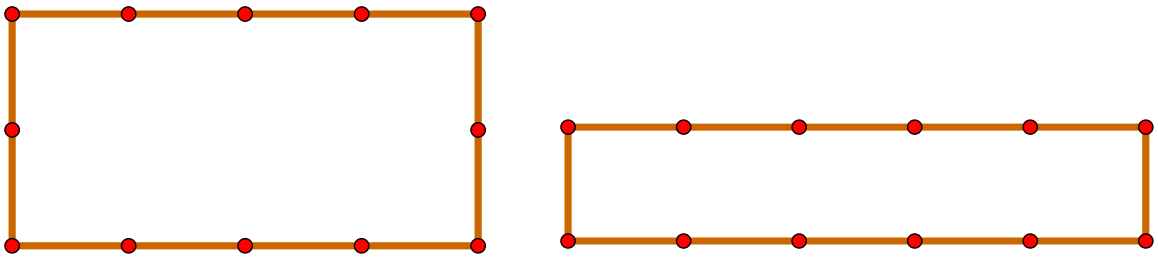

FIGURE 2: Exploring other possible rectangles

After a bit of exploration, most pupils realised that they could keep the perimeter fixed yet decrease the area by modifying either the original 3 by 3 starting configuration or the 4 by 2 rectangle. This modification involved making some of the corners concave rather than convex. A few examples of this are illustrated in Figure 3, showing polygons with areas of 8, 7, 6 and 5 square units.

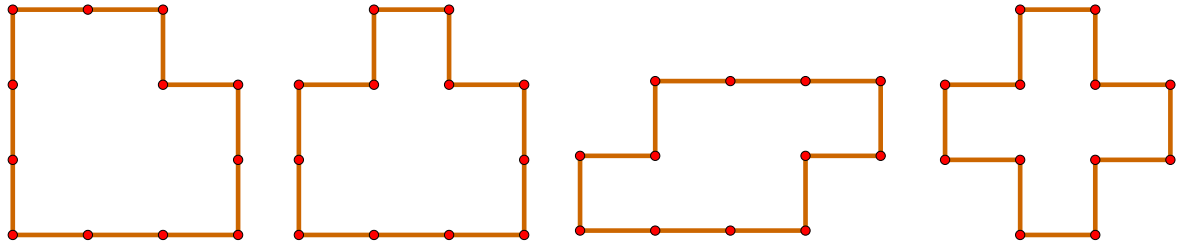

FIGURE 3: Changing the area by making some corners concave

Once the strategy of creating concave polygons was established, many variations on this theme emerged, as illustrated in Figure 4.

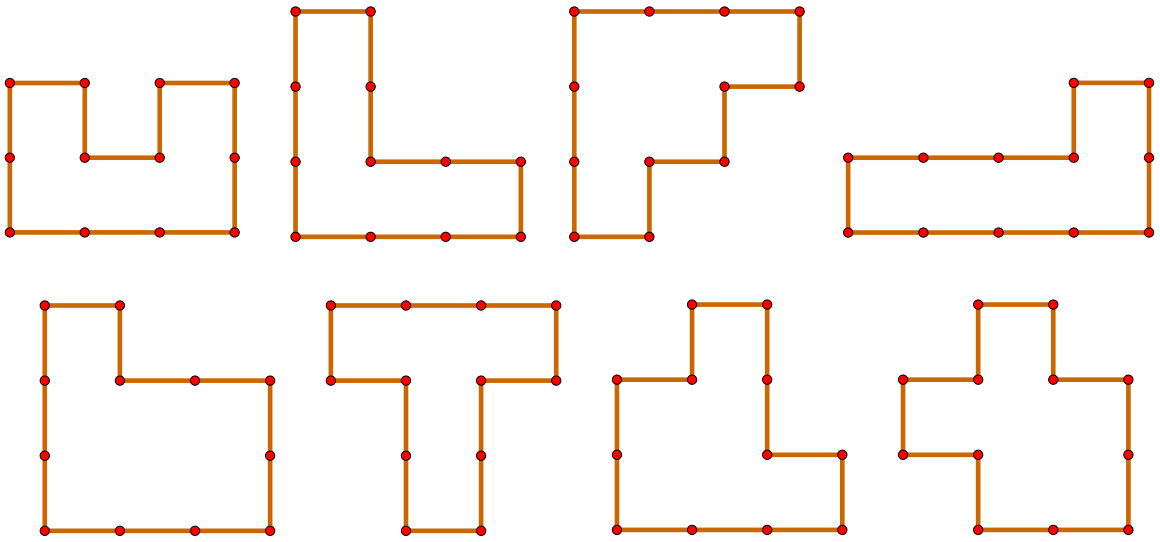

FIGURE 4: Variations based on concave polygons

Although there were many interesting configurations, they all resulted in shapes with areas that had already been created – 5, 6 and 7 square units. In order to create shapes with smaller areas a different approach was necessary.

MOVING BEYOND RIGHT ANGLES

Up until this point, all shapes had been created by placing the matchsticks either horizontally or vertically, i.e. with shapes only involving right angles. Given the requirement that the shapes created had to have integer areas, this seems a reasonable approach. The next leap was for pupils to establish a strategy whereby they could incorporate oblique matchsticks yet still be sure that the polygon created had an integer area. Some pupils discovered this strategy on their own, while for others a bit of prompting and directed questioning was necessary. The strategy is illustrated in Figure 5 where the extra area created by an outward pointing triangle on the right-hand side is exactly cancelled out by an inward pointing triangle on the left-hand side, creating a shape with an area of exactly 4 square units.

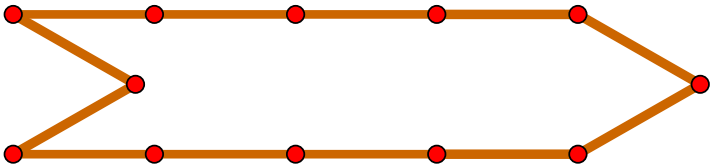

FIGURE 5: A new strategy involving triangles of equal area

Using this strategy, pupils were also able to find other configurations with an area of 4 square units, as shown in Figure 6.

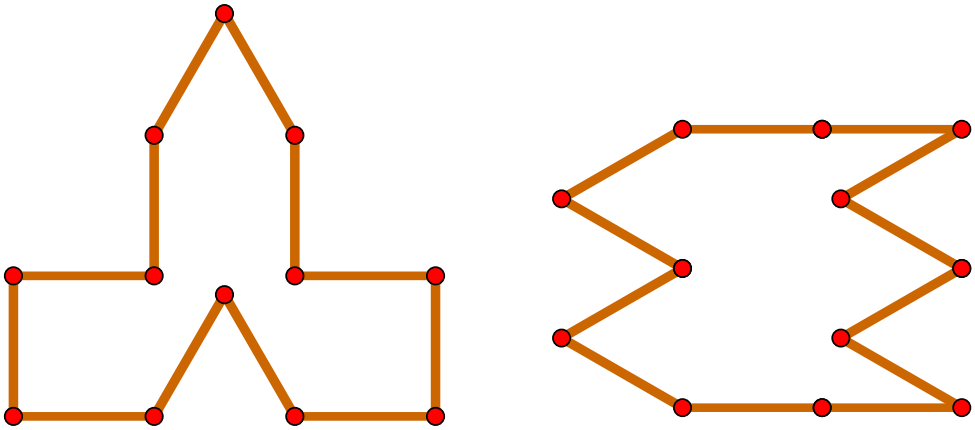

FIGURE 6: Other configurations with an area of exactly 4 square units

Pupils were also able to use this strategy to create polygons with areas of exactly 3 square units (Figure 7) and 2 square units (Figure 8).

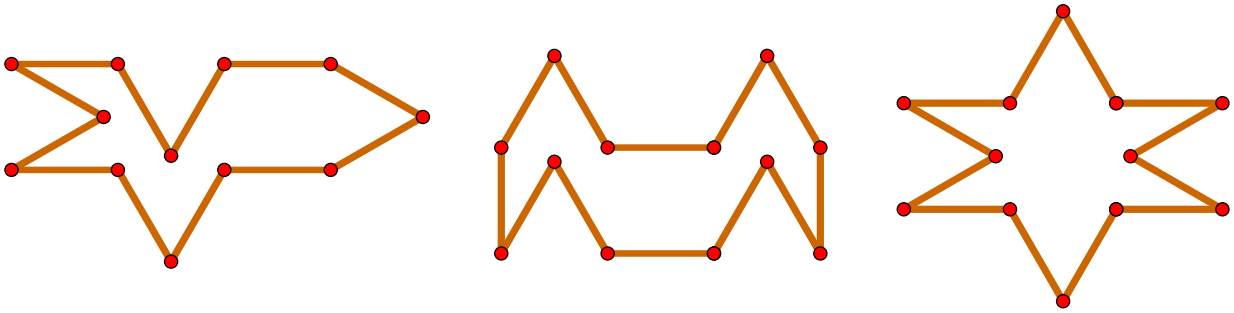

FIGURE 7: Creating polygons with an area of 3 square units

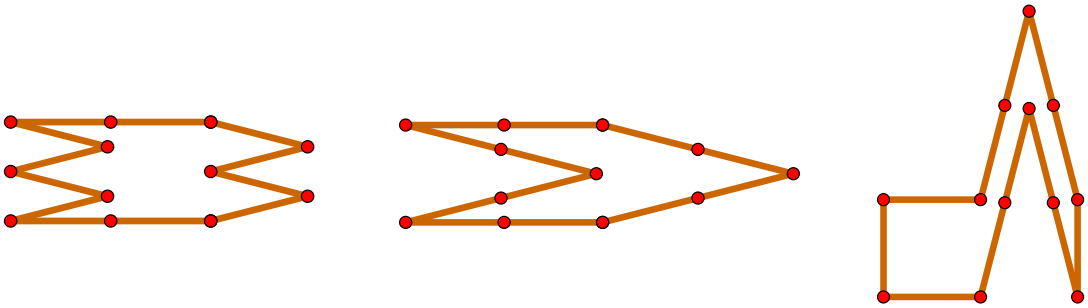

FIGURE 8: Creating polygons with an area of 2 square units

FURTHER POSSIBLE CONFIGURATIONS

The choice of twelve matchsticks was deliberate as it allowed for one further approach to constructing polygons with integer areas. As a prompt, pupils were asked to consider right-angled triangles and well-known Pythagorean triples. This quickly led them to construct 3-4-5 right-angled triangles with the required perimeter of 12 matchsticks. The area of this triangle, namely 6 square units, could then be manipulated down to 5 square units and 3 square units using the earlier strategy of creating concave polygons, as illustrated in Figure 9.

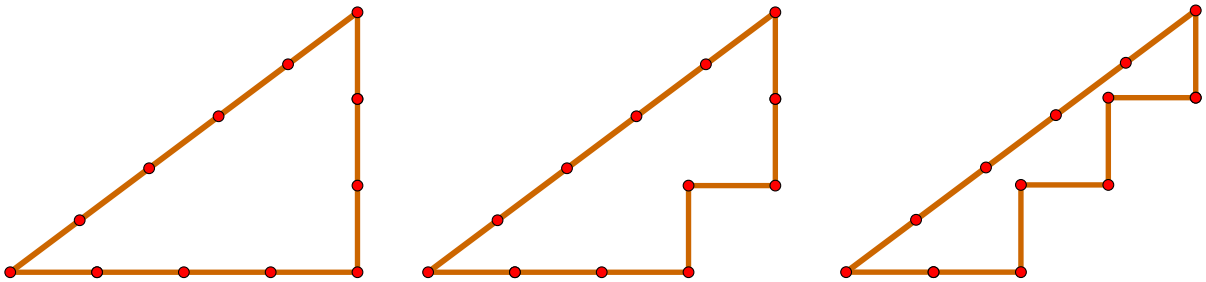

FIGURE 9: Making use of a 3-4-5 right-angled triangle as a starting configuration

Creating a polygon with an area of 1 square unit proved to be a bit of a challenge. However, one group of pupils managed to find a cunning solution by taking the middle configuration of Figure 8 (with an area of 2 square units) and compressing it by a factor of 2, making the perpendicular distance between the horizontal matchsticks half a unit, as illustrated in Figure 10.

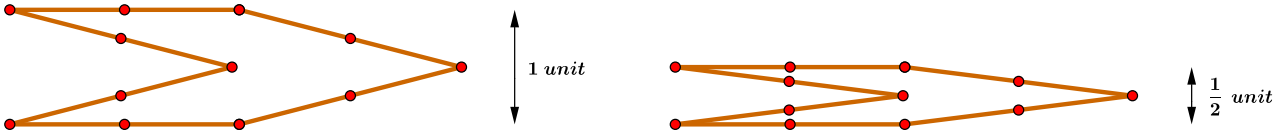

FIGURE 10: Creating a polygon with an area of 1 square unit

Interestingly, no pupils made use of parallelograms (other than squares and rectangles). If they had, they might have chanced upon another way to create a polygon with an area of 1 square unit, as illustrated in Figure 11. This configuration also represents a general approach to obtaining all integer areas from 1 to 5 (as well as all non-integer areas in between) by the simple expedient of appropriately adjusting the perpendicular height of the parallelogram!

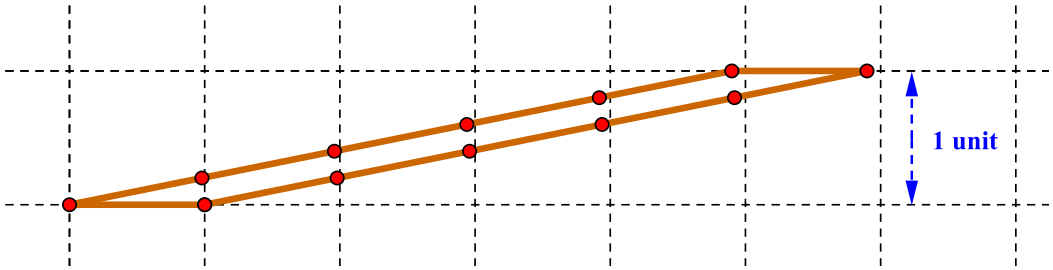

FIGURE 11: A parallelogram with an area of 1 square unit

What's particularly pleasing about these last two examples is that they incorporate a *dynamic* element which naturally lends itself to modelling. The two scenarios could readily be set up in a dynamic geometry software environment and explored further.

PEDAGOGICAL CONSIDERATIONS

One of the pedagogical benefits of these types of activities is that they allow for a differentiated approach. Pupils can work at a level of complexity appropriate to where they are on their own learning curve. The open-ended nature of the task also allows pupils free reign in terms of their creativeness. Although this sometimes resulted in polygons that didn't quite fit the required condition, i.e. having an integer area, this in itself gave rise to some interesting discussions as to *why* the area wasn't an integer value. Figure 12 illustrates a case in point.

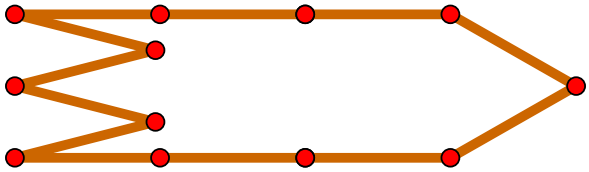

FIGURE 12: A polygon not quite fulfilling the integer area requirement

MAXIMISING THE AREA

For more advanced pupils, a further challenge would be to ignore the requirement that the area needs to be an integer value, and to ask what configuration would give the *greatest* area. Once it is established that this would be a regular polygon – i.e. a dodecagon – they could then work out this maximum area using trigonometry.

In order to calculate the area enclosed by a regular dodecagon with a side length of 1 unit, we can subdivide the area into 12 congruent triangles as illustrated in Figure 13. The perpendicular height of each of these triangles is $h = \frac{1}{2}\tan 75°$. The total area of the dodecagon (the maximum possible area enclosed by twelve matchsticks of unit length) is thus:

$$Area = 12\left[\frac{1}{2} \times 1 \times h\right] = 12\left[\frac{1}{2} \times 1 \times \frac{1}{2}\tan 75°\right] = 3\tan 75° = 6 + 3\sqrt{3} \approx 11{,}2 \text{ units}^2$$

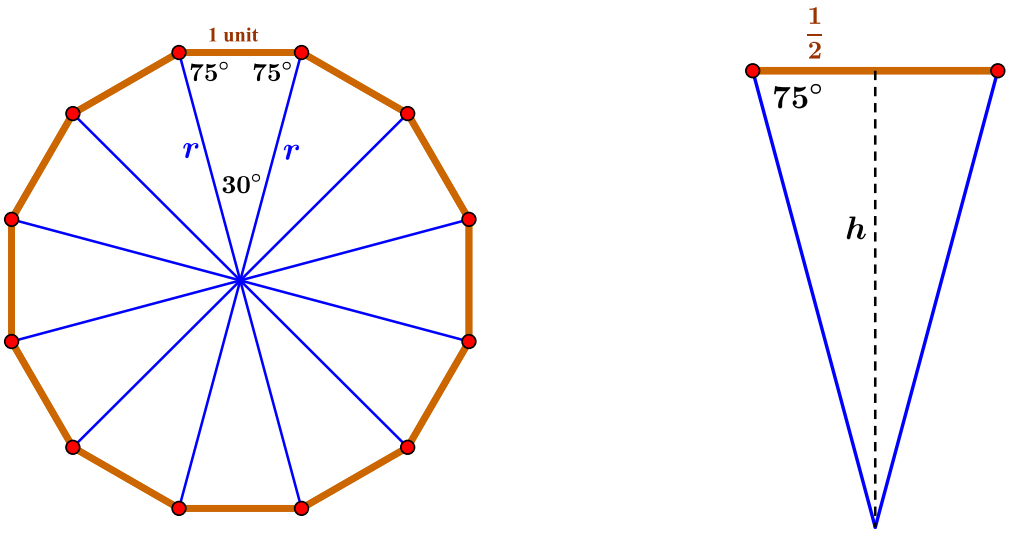

FIGURE 13: Twelve matchsticks enclosing the greatest possible area

A FINAL EXTENSION QUESTION

Inspired by one of the pupils' designs, we decided to explore a stellated hexagon configuration as illustrated in Figure 14.

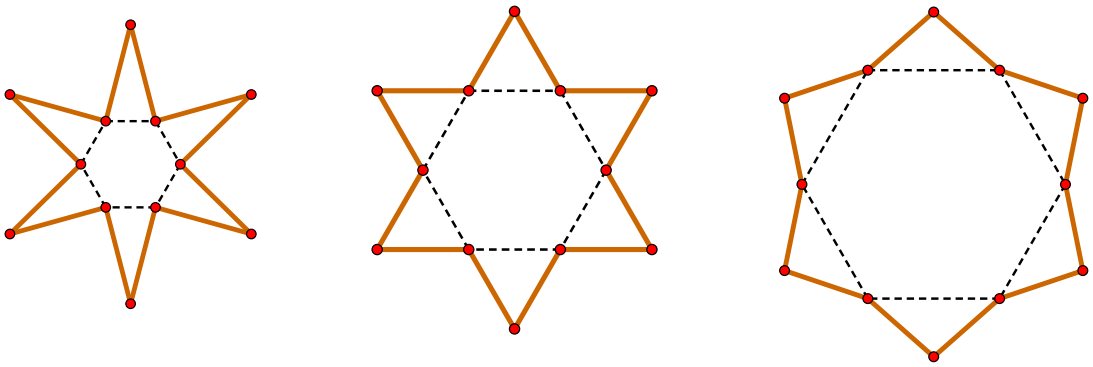

FIGURE 14: Stellated hexagon configurations

The question we asked ourselves was – what would the side length of the inner hexagon need to be to create a stellated shape with an area, for example, of exactly 4 square units? If we let the side length of the inner hexagon be x then it can be shown that the area of the stellated shape is given by:

$$A(x) = \frac{3\sqrt{3}}{2}x^2 + 3x\sqrt{1 - \frac{x^2}{4}}$$

Setting this equal to 4 and trying to solve for x looks like a formidable task, but rather pleasingly the equation collapses to a quartic with an underlying quadratic structure:

$$9x^4 - \left(9 + 12\sqrt{3}\right)x^2 + 16 = 0$$

This can then readily be solved by using the standard quadratic formula and discarding any spurious roots. It is left to the interested reader to explore this further.

CONCLUDING COMMENTS

It was the purpose of this article to share not only an interesting classroom investigation but also to reflect on our combined experiences having explored this activity with different groups of pupils in South Africa and Israel. What started out as a simple prompt soon developed into a meaningful open-ended exploration where pupils not only had the freedom to explore their mathematical creativity but also to collaborate, share findings with fellow classmates and then use this interaction to further refine their own strategies. The open-ended nature of the activity naturally allowed for differentiated learning experiences, and as such was effective not only for pupils of different mathematical abilities and backgrounds but also different age groups. But perhaps most pleasing of all is how such a simple mathematical premise led to such a wealth of mathematical discovery, for pupils and teachers alike!

The "Extra Square" Illusion – and How to Make Others

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INTRODUCTION

Figure 1 shows a well-known mathematical "illusion". The upper diagram seems to show a 5 by 13 right-angled triangle sub-divided into four parts – a red triangle, a blue triangle, a yellow piece comprising 7 unit squares, and a green piece comprising 8 unit squares. When these four parts are re-arranged, as illustrated in the lower diagram, the result seems to be an identical 5 by 13 right-angled triangle, but with a mysterious additional square, seemingly created out of thin air. If one hasn't seen this illusion before, it can be quite perplexing. What exactly is going on?

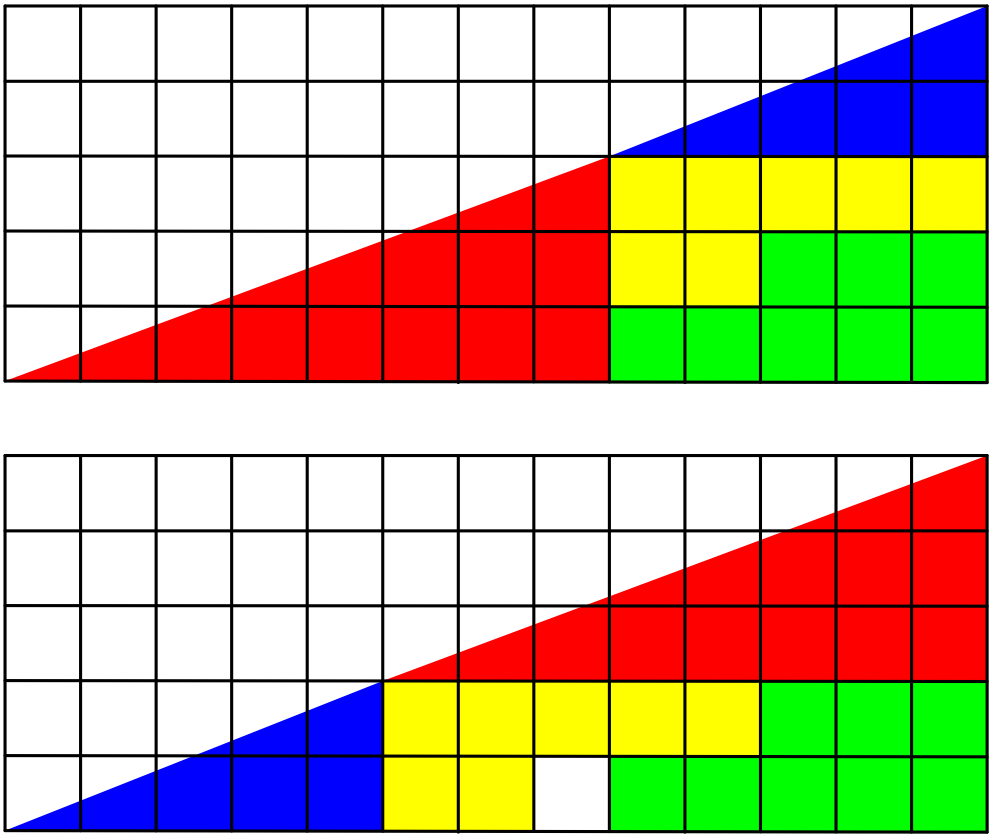

FIGURE 1: The "extra square" illusion

The reason we seem to gain an additional unit square is that each of the two shapes formed from the four component pieces isn't actually a triangle. Although they both appear to be 5 by 13 right-angled triangles, both shapes are in fact quadrilaterals. On closer inspection, notice that the red triangle's hypotenuse has a slope of 3/8 while the blue triangle's hypotenuse has a slope of 2/5. Not only are these two slopes different, but neither is 5/13, which would be the slope of a right-angled triangle with perpendicular sides of 5 units and 13 units.

Figure 2 makes the situation a little clearer. Since the slope of the hypotenuse of the red triangle is 3/8, and that of the blue triangle is 2/5, the result is that the quadrilateral formed is slightly concave. When the pieces are rearranged, the result is a quadrilateral that is slightly convex. The difference in these areas is one square unit, and this is where the "extra square" comes from.

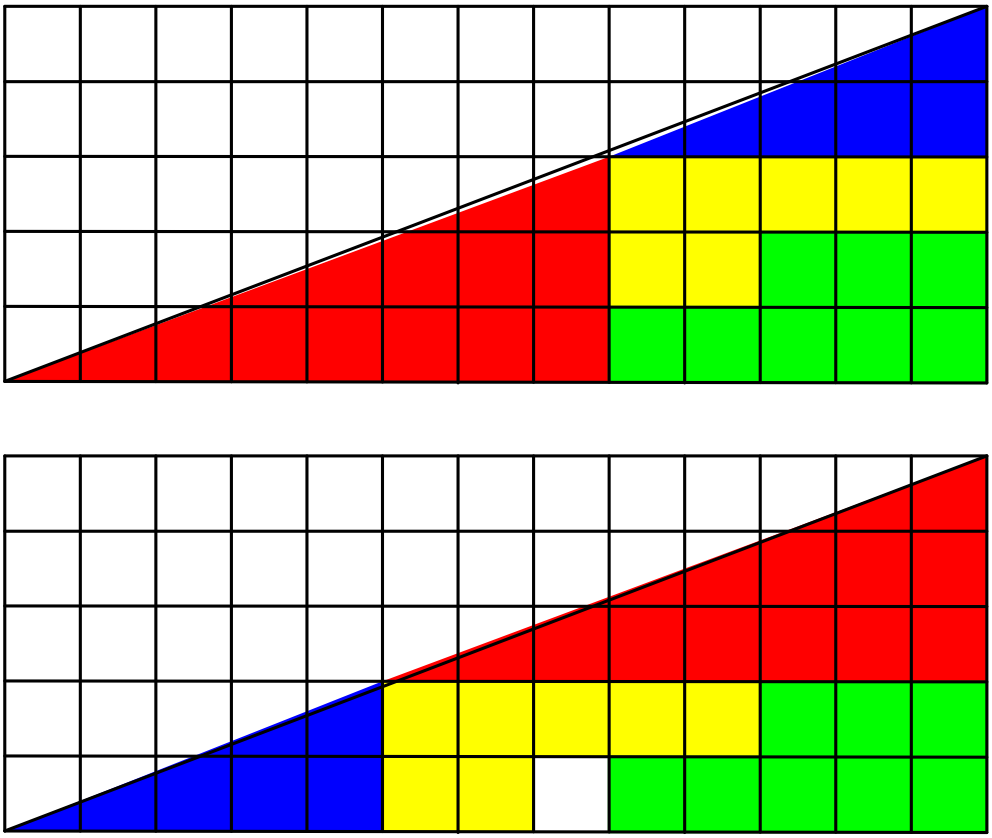

FIGURE 2: Making sense of the illusion

A feature of the two diagrams is that the rectangular region formed by the yellow and green pieces is one unit square larger in the bottom diagram. This is the key to creating similar kinds of illusion puzzles in which the "extra region" is exactly one unit square. The rest of the article takes this structural feature as a starting point to explore how one might create similar visual illusions using pairs of quadrilaterals (that appear to be identical right-angled triangles at first glance) that differ in area by one square unit.

HOW TO CREATE SIMILAR VISUAL "ILLUSIONS"

Begin by choosing two consecutive integers. For convenience, choose integers larger than 5. The smaller of these integers will be the area of the first rectangle (e.g. that created by the yellow and green pieces in the top diagram of Figure 2), while the larger integer will be the area of the transformed rectangle incorporating the extra square. For each integer, list the factor pairs and draw corresponding rectangles with the smaller of the two factors being the width, and the larger being the length. If the two factors are the same, then the corresponding rectangle will of course be a square. Select a rectangle made from the factors of the smaller of the two integers and a rectangle made from the factors of the larger of the two integers. For each pair of rectangles attempt the following: subdivide the first rectangle into two pieces such that these two pieces cover the second rectangle leaving one square uncovered. Record the pairs for which this is possible.

By way of example, imagine we chose the two integers 8 and 9. Let us begin by considering the factor pairs of 2 & 4 (for the smaller integer) and 3 & 3 (for the larger integer). Figure 3 illustrates a number of possible partitions with the extra unit square being located in different places in the transformed rectangle – the centre of the square or one of the four corners.

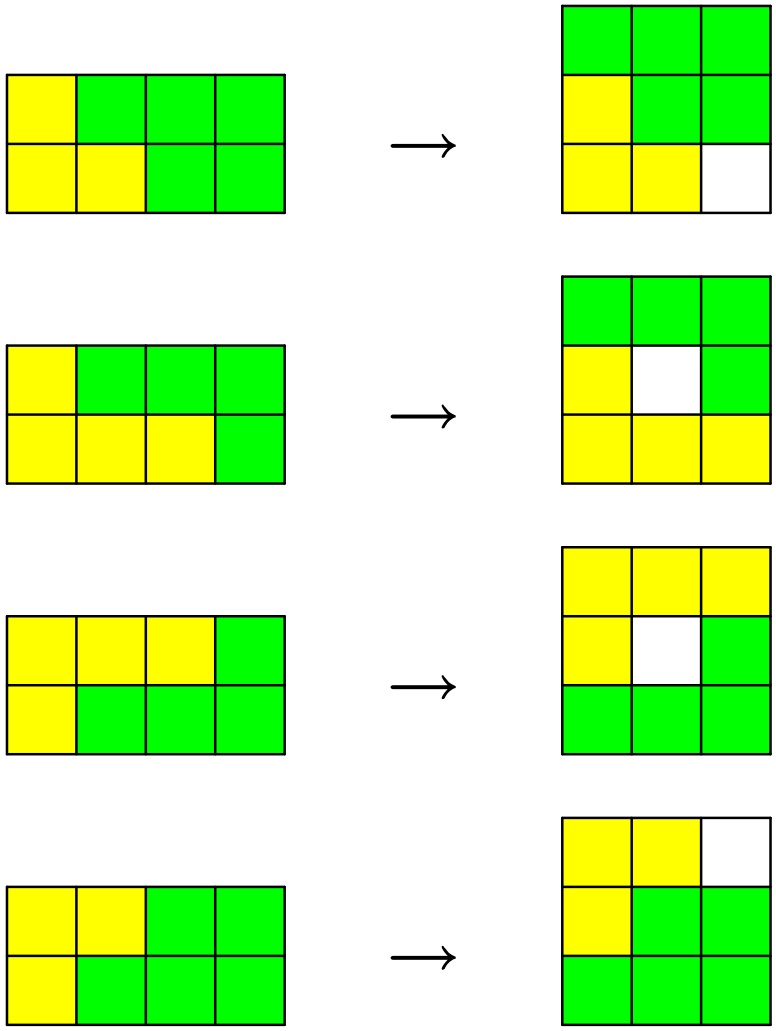

FIGURE 3: Possible partitions for the factor pairs 2 & 4 and 3 & 3

Another combination of factor pairs for the chosen integers (8 and 9) is 2 & 4 (as before) and 1 & 9. For these factor pairs there is only one possible partition, as illustrated in Figure 4. The extra square could of course be positioned at either end rather than the middle.

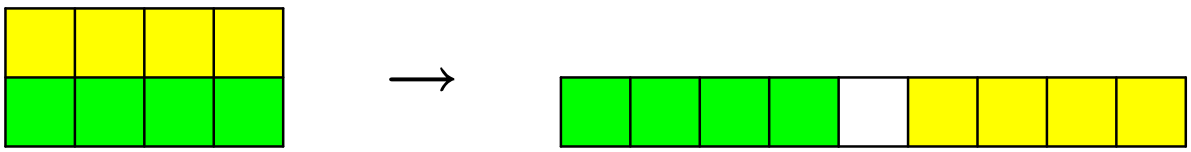

FIGURE 4: A partition for the factor pairs 2 & 4 and 1 & 9

Another combination of factor pairs for the chosen integers (8 and 9) is 1 & 8 and 1 & 9. For these factor pairs there is also only one possible partition, as illustrated in Figure 5. Once again, the extra square could be positioned at either end rather than the middle.

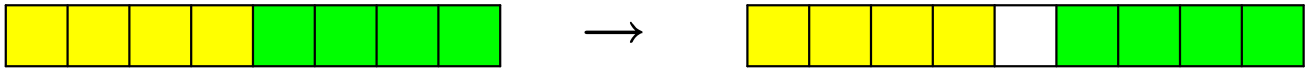

FIGURE 5: A partition for the factor pairs 1 & 8 and 1 & 9

The final factor pairs we need to consider are 1 & 8 for the smaller integer and 1 & 9 for the larger integer. However, the 1 by 8 rectangle cannot be suitably partitioned into two pieces to cover the 1 by 9 rectangle as required, so this pairing cannot be used.

So, for the chosen integers (8 and 9) we have three possible pairings: (a) 2 & 4 and 3 & 3, (b) 2 & 4 and 1 & 9, and (c) 1 & 8 and 1 & 9. Each different partition for each of these three pairings can be used to create an "extra square" illusion similar to the original shown in Figure 1.

CREATING THE PUZZLE PIECES FROM THE CHOSEN FACTOR PAIRS

To illustrate the process let us take the first configuration of the factor pairs 2 & 4 and 3 & 3 as shown in Figure 3. By considering the placing of the triangles in the original and transformed configurations (Figure 6), we see that one triangle will need to have perpendicular sides of 2 units and 3 units while the other will need perpendicular sides of 3 units and 4 units.

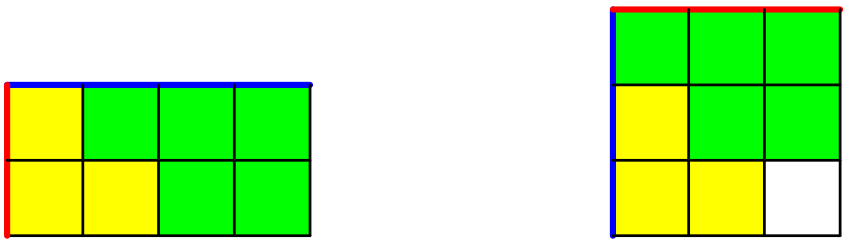

FIGURE 6: Determining the sizes of the two triangular pieces

Figure 7 shows the final result, i.e. all four puzzle pieces in their original and transformed configurations creating a new "extra square" illusion.

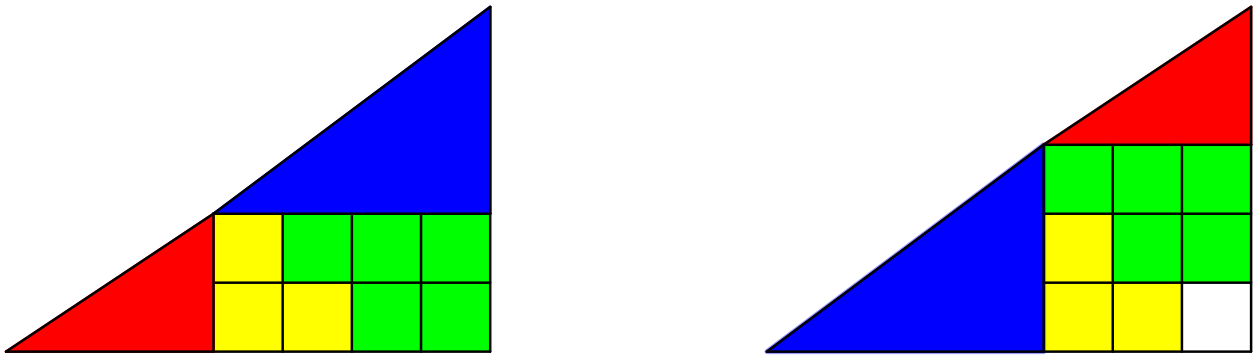

FIGURE 7: A new "extra square" illusion

Figure 8 shows a further puzzle created from the factor pairings of 2 & 4 and 1 & 9.

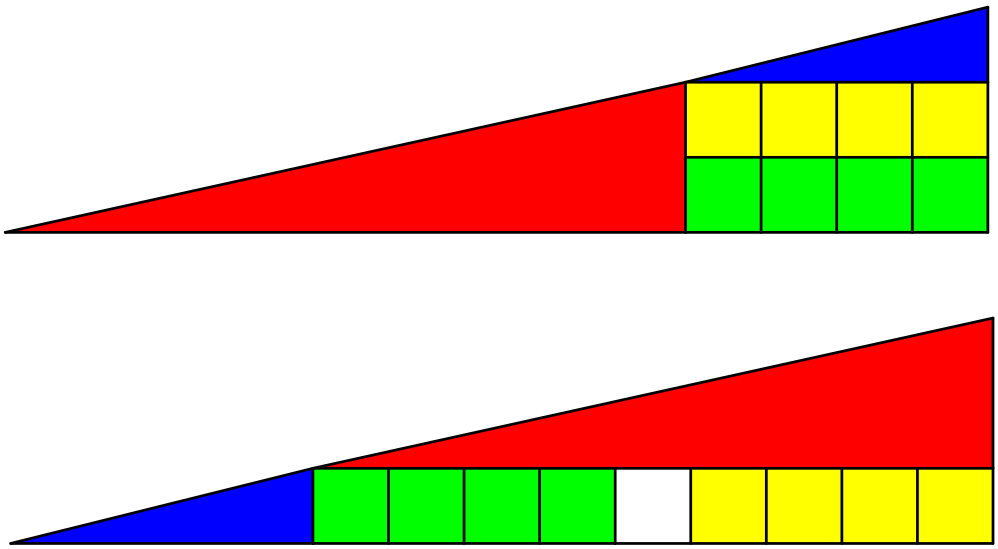

FIGURE 8: A further "extra square" illusion

CONCLUDING COMMENTS

These kinds of visual illusions are not only fun to explore in the classroom, but they also sensitise pupils to the dangers of making assumptions. Just because three points "look" collinear, one shouldn't assume that they necessarily are! The process of unpacking why the puzzle/illusion works is an interesting exercise in itself, and extending the idea to create other similar puzzles has the potential to lead to some interesting mathematical discussions. Are some factor pairs more visually effective than others? In how many different ways can one successfully partition the first rectangle? Other interesting observations may emerge. Notice, for example, that if we choose consecutive integers 20 and 21 then the factor pairs 4 & 5 and 3 & 7 result in some pleasing symmetry (Figure 9). Can this result be generalised?

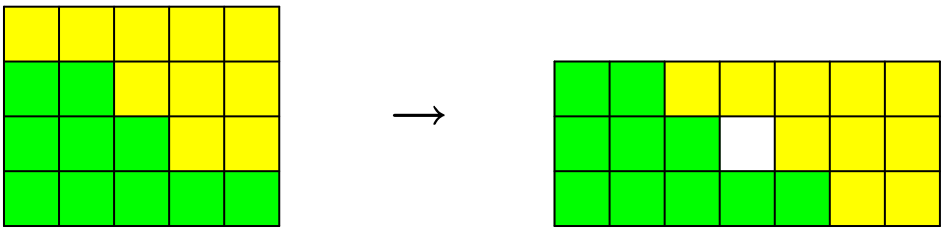

FIGURE 9: A pleasing symmetry

Activities to Encourage Numerical Reasoning and Flexibility

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INTRODUCTION

The ability to perform numerical calculations with flexibility relies to a large degree on having a sound numerical basis. Such a basis would include not only the ability to perform basic operations such as addition, subtraction, division and multiplication, but also an understanding of the properties of numbers (related, for example, to their factors), approaches to partitioning numbers, and general divisibility rules. In this short article we present two activities which encourage precisely this kind of engagement.

ACTIVITY 1

The first activity is a variation on the classic 'Four 4s'. In this variation the objective is to reach the number 100 with the use of five identical digits (from 1 to 9) along with any combination of mathematical operations/symbols. These would include addition, subtraction, multiplication, division, use of brackets, concatenation (e.g. putting two 3s together to make 33), exponentiation, square rooting, and the use of the decimal point. Depending on the age of the pupils one could also include use of factorials and logarithms.

In most cases there are several different ways one could arrive at the target of 100. A few examples are shown below to illustrate the potential creativity that this activity could give rise to.

1	$(11-1)^{1+1}=100$	$111-11=100$
2	$[(22-2)\div 2]^2=100$	$[2\times(2+2)+2]^2=100$
3	$33\times 3+(3\div 3)=100$	
4	$\left(44+4+\sqrt{4}\right)\times\sqrt{4}=100$	$(4!+4\div 4)\times\sqrt{4\times 4}=100$
5	$(5+5)^{\frac{5+5}{5}}=100$	$\left(5-\frac{5}{5}\right)\times 5\times 5=100$
6	$\frac{6!}{6}-\frac{6!}{6\times 6}=100$	
7	$7\times(7+7)+\log_{\sqrt{7}}7=100$	
8	$88+8+\sqrt{8+8}=100$	$\left(8+\sqrt{\sqrt{8+8}}\right)^{\sqrt{\sqrt{8+8}}}=100$
9	$99+\frac{\sqrt{9\times 9}}{9}=100$	$99+\left(\frac{9}{9}\right)^9=100$

ACTIVITY 2

In this activity, students are presented with twelve envelopes numbered from 1 to 12. They are also given twelve sheets of card numbered from 110 to 121. The object of the activity is to place one card into each envelope so that the number on the card is exactly divisible by the number on the envelope.

One possible starting point is to list the proper divisors (up to and including 12) of each of the card numbers (110 – 121).

Card Number	Proper divisors
110	1, 2, 5, 10, 11
111	1, 3
112	1, 2, 4, 7, 8
113	1
114	1, 2, 3, 6
115	1, 5
116	1, 2, 4
117	1, 3, 9
118	1, 2
119	1, 7
120	1, 2, 3, 4, 5, 6, 8, 10, 12
121	1, 11

Since 113 is prime, that particular card will have to go into the envelope No. 1. We can also see that the card numbered 115, which only has proper divisors of 1 and 5, must go into envelope No. 5, and the card numbered 118, with proper divisors of 1 and 2, will have to go into envelope No. 2. Following a similar process of considering the proper divisors of each number, students should hopefully be able to arrive at the following pairings:

Envelope	1	2	3	4	5	6	7	8	9	10	11	12
Card	113	118	111	116	115	114	119	112	117	110	121	120

CONCLUDING COMMENTS

The two activities presented in this article are aimed at promoting flexibility and creativity of numerical computation while at the same time sensitising students to the inherent properties of numbers as well as nurturing logical reasoning. There are many other activities of a similar nature that one can readily access in mathematical puzzle books and puzzle-based websites, and these are wonderful tasks to explore in the context of the mathematics classroom.

Repeated Terms in a Quadratic Sequence

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INTRODUCTION

Arithmetic sequences and geometric sequences, if we ignore trivial cases of $d = 0$ in the former and $r = 1$ in the latter, never contain repeated terms. However, quadratic sequences, i.e. sequences with a general term given by $T_n = an^2 + bn + c$ ($a \neq 0$), may contain repeated terms. This article considers the circumstances under which quadratic sequences contain (i) no repeated terms, (ii) a single consecutive pair of repeated terms, and (iii) many repeated terms.

THE GENERAL TERM OF THE FIRST DIFFERENCES

A quadratic sequence has the property that the differences between consecutive terms (known as the first differences) themselves form a linear sequence. The nature of the general term for this sequence of first differences plays a pivotal role in the number and positioning of repeated pairs. For a quadratic sequence of the form $T_n = an^2 + bn + c$ ($a \neq 0$), the terms of the first differences form an arithmetic sequence with a general term $T_m = T_1 + (m-1)d$, where T_1 is the 1st term, m the position of the m^{th} term, and d the common difference between any two consecutive terms, i.e. $T_{m+1} - T_m$. Since the common second difference of a quadratic sequence is $2a$, it follows that $d = 2a$. Further, the first term of the first differences of a quadratic sequence is $3a + b$ and as such $T_1 = 3a + b$. We can thus write the general term of the first differences as $T_m = 3a + b + (m-1) \times 2a$, which simplifies to:

$$T_m = 2am + (a + b)$$

REPEATED TERMS IN A QUADRATIC SEQUENCE

For a quadratic sequence to have a repeated term, it is a necessary requirement that a term in the sequence of first differences must be zero. From the above result this means that:

$$2am + a + b = 0 \quad \rightarrow \quad m = -\frac{b}{2a} - \frac{1}{2}$$

From this it should be clear that for a quadratic sequence to have at least one repeated term:

- The sequence of its first differences must contain a zero term
- $b \neq 0$ (since m is a positive integer)
- a and b must be of *opposite sign*
- $\frac{b}{a}$ must be an *odd integer*, with $\frac{b}{a} \leq -3$

The last bullet point is due to the fact that $m \geq 1$ ($m \in \mathbb{N}$) and thus $-\frac{b}{2a} - \frac{1}{2} \geq 1$ from which $\frac{b}{a} \leq -3$.

We may therefore choose any values for a and b satisfying the above conditions in order to produce a quadratic sequence with at least one pair of repeated terms.

Note that the value of c is irrelevant. By way of example, based on the above constraints, we may choose: $T_n = -2n^2 + 14n + c \rightarrow m = 3$, and thus T_3 and T_4 of the quadratic sequence will be the same. We have:

$$12 + c \;\; ; \;\; 20 + c \;\; ; \;\; 24 + c \;\; ; \;\; 24 + c \quad \rightarrow \quad T_3 = T_4$$

NUMBER AND STRUCTURE OF REPEATED TERMS

The value of m leading to a zero term in the first differences will dictate the number and positioning of a pair (or pairs) of repeated terms in the quadratic sequence. In the above example, $m = 3$, and there will be three pairs of repeated terms. If we arbitrarily set $c = 10$ then the quadratic sequence is:

$$\begin{array}{cccccccccc} 22 & & 30 & & 34 & & 34 & & 30 & & 22 \\ & 8 & & 4 & & 0 & & -4 & & -8 & \\ & & -4 & & -4 & & -4 & & -4 & & \end{array}$$

The third term of the first differences is zero, leading to three pairs of repeated terms: $22\,;30\,;34$.

For the case of $m = 1$ leading to the zero term in the first differences, the repeated terms will be at the start of the quadratic sequence, with $T_1 = T_2$. In this case we have:

$$T_m = 2am + (a + b) \quad \rightarrow \quad 0 = 2a + a + b \quad \rightarrow \quad \frac{b}{a} = -3 \quad or \quad b = -3a$$

By way of example, from $T_n = 3n^2 - 9n - 7$ we have:

$$\begin{array}{ccccccc} -13 & & -13 & & -7 & & 5 \\ & 0 & & 6 & & 12 & \\ & & 6 & & 6 & & \end{array} \qquad T_1 = T_2 \qquad T_1 = 0$$

Clearly if the first two terms are repeated, there will be no further repeated terms in the quadratic sequence, and in every case $T_3 = c$. In general, if $T_m = 0$ there will be m pairs of repeated terms in the quadratic sequence and these will be positioned as follows:

$$T_{n-k} = T_{n+1+k} \;\; , \;\; k \geq 0 \;\; , \;\; n - k \geq 1 \;\; , \;\; n = m$$

By way of example, $T_n = 2n^2 - 26n + 24$ will lead to $T_m = 4m - 24$ from which the zero term will occur at $m = 6$. From the above, $T_{6-k} = T_{7+k}$ and the following six terms will be repeated:

$$T_6 = T_7 \; ; \; T_5 = T_8 \; ; \; T_4 = T_9 \; ; \; T_3 = T_{10} \; ; \; T_2 = T_{11} \; ; \; T_1 = T_{12}$$

A FURTHER CONSIDERATION

The number of repeated terms may also be established from the following equation:

$$an^2 + bn + c \;\; = \;\; a(n + r)^2 + b(n + r) + c \;\; , \;\; r \geq 1$$

This simplifies to $r = -\frac{b}{a} - 2n$, from which $-\frac{b}{a} - 2n \geq 1$ or $n \leq \; -\frac{b}{2a} - \frac{1}{2}$.

In the above example, $a = 2$ and $b = -26$, thus $n \leq 6$ and since $n \geq 1$ we have $1 \leq n \leq 6$. There are thus six repeated terms, $T_1 \rightarrow T_6$, with matching terms at $T_{12} \rightarrow T_7$ respectively.

THE SPECIAL CASE OF TWO CONSECUTIVE ZEROS

If $T_1 = T_2 = 0$ we have $a + b + c = 0$ and $3a + b = 0$, from which $b = -3a$ and $c = 2a$. We may therefore choose any value for a in the following general term equation and the quadratic sequence will have a pair of leading zeros:

$$T_n = an^2 - 3an + 2a$$

The derivation of the general formula for a pair of consecutive zeros lying *anywhere* in the sequence, and based only on the value of a, is more challenging. If these consecutive zeros lie at $n = t$ and $n = t + 1$, it is left to the reader to verify or develop such formula to be:

$$T_n = an^2 - (2t + 1)an + (t^2 + t)a$$

Setting $t = 1$ collapses this into the previous formula for a pair of leading zeros. Further, note that:

$$2t + 1 = t + (t + 1) \quad \text{and} \quad t^2 + t = t(t + 1)$$

The first "coefficient" of a is the *sum* of the numbers of the two consecutive terms under consideration, and the second "coefficient" of a is the *product* of these two term numbers. By way of example, if we require the 8th and 9th terms to be consecutive zeros, the general term would be:

$$T_n = an^2 - 17an + 72a \quad \text{(for any choice of } a \neq 0\text{)}$$

CONCLUDING COMMENTS

Knowledge of the nature, number, and position of repeated pairs of terms in a quadratic sequence, if any, certainly facilitates better understanding of the structure of these sequences. It further adds to useful classroom investigation and discussion, together with the setting of appropriate questions in assessments.

A Surprise Equilateral Triangle

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INTRODUCTION

Recently, while working on another problem, I came across the following surprising equilateral triangle result:

> Given any hexagon $ADBFCE$ with $\angle D = \angle E = \angle F = 120°$ as shown in Figure 1, then the respective circumcentres D', F' and E' of the three circumcircles of triangles ADB, BFC and CEA form an equilateral triangle.

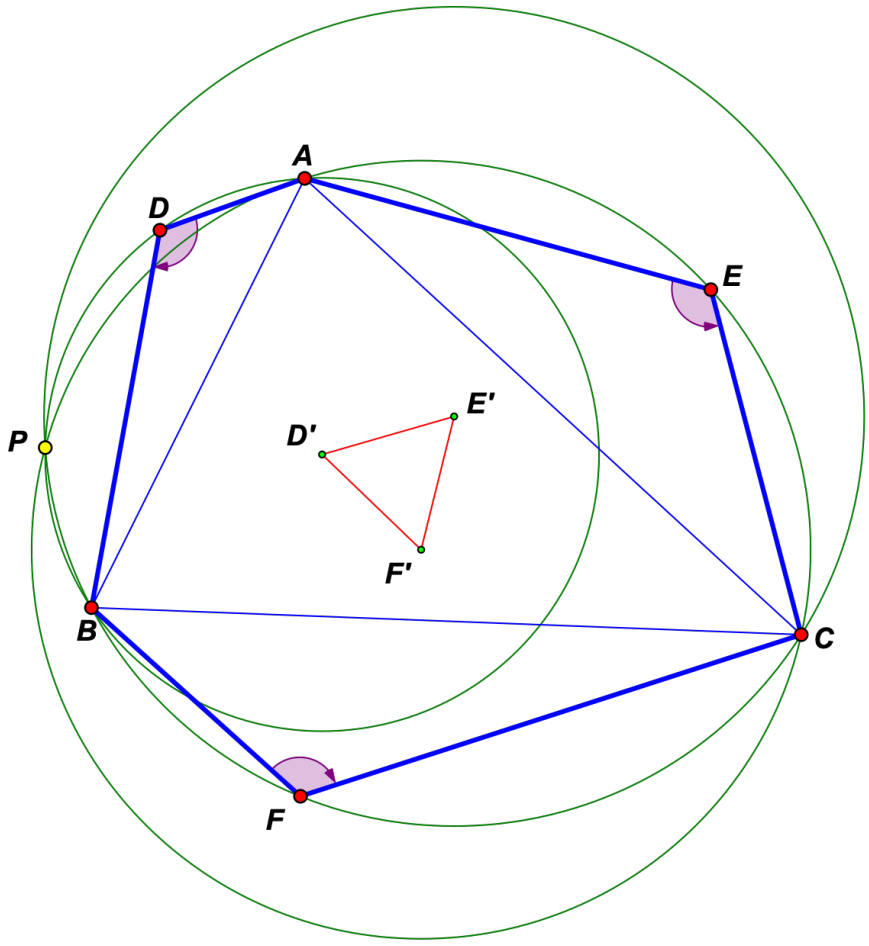

FIGURE 1: The formed equilateral triangle

Note that the result can also be formulated in the following equivalent form:

> Given any triangle ABC with triangles ADB, BFC and CEA constructed on its sides so that $\angle D = \angle E = \angle F = 120°$, then the respective circumcentres D', F' and E' of the three circumcircles of triangles ADB, BFC and CEA form an equilateral triangle.

The reader may wish to check the result using dynamic geometry for themselves, but a dynamic geometry sketch is available online for use at: https://dynamicmathematicslearning.com/circle-concurrencies.html (navigate to the correct sketch by clicking on the 'Link to Special Napoleon Variation' button).

While dragging and moving a figure around in dynamic geometry is a very useful and convincing way to check the validity of results, it does not explain *why* the result is true. Before continuing the reader is encouraged first to try and prove the result themselves.

The astute reader may already have recognized that the result is directly related to Napoleon's theorem discussed in my paper in *Learning and Teaching Mathematics*, No. 33 (De Villiers, 2022). In that paper a generalisation of Napoleon's theorem to some more general triangles is given, but for the purpose of proving the above result we only need the original form of Napoleon's theorem, namely: "If equilateral triangles are constructed on the sides of any ΔABC, then the centroids of the equilateral triangles form an equilateral triangle".

Note that, as shown in Figure 2, the equilateral triangles can be constructed to the exterior as well as to the interior of the base triangle. Obviously, the respective centroids of these equilateral triangles coincide with their respective circumcentres. Many proofs for this famous theorem are available in books as well as on the internet. A free classroom worksheet with related downloadable Sketchpad sketches[1] is also available in De Villiers (2012). The proof of the case when the equilateral triangles are drawn (or dragged in a dynamic sketch) towards the interior is a little more difficult to visualise but is exactly the same.

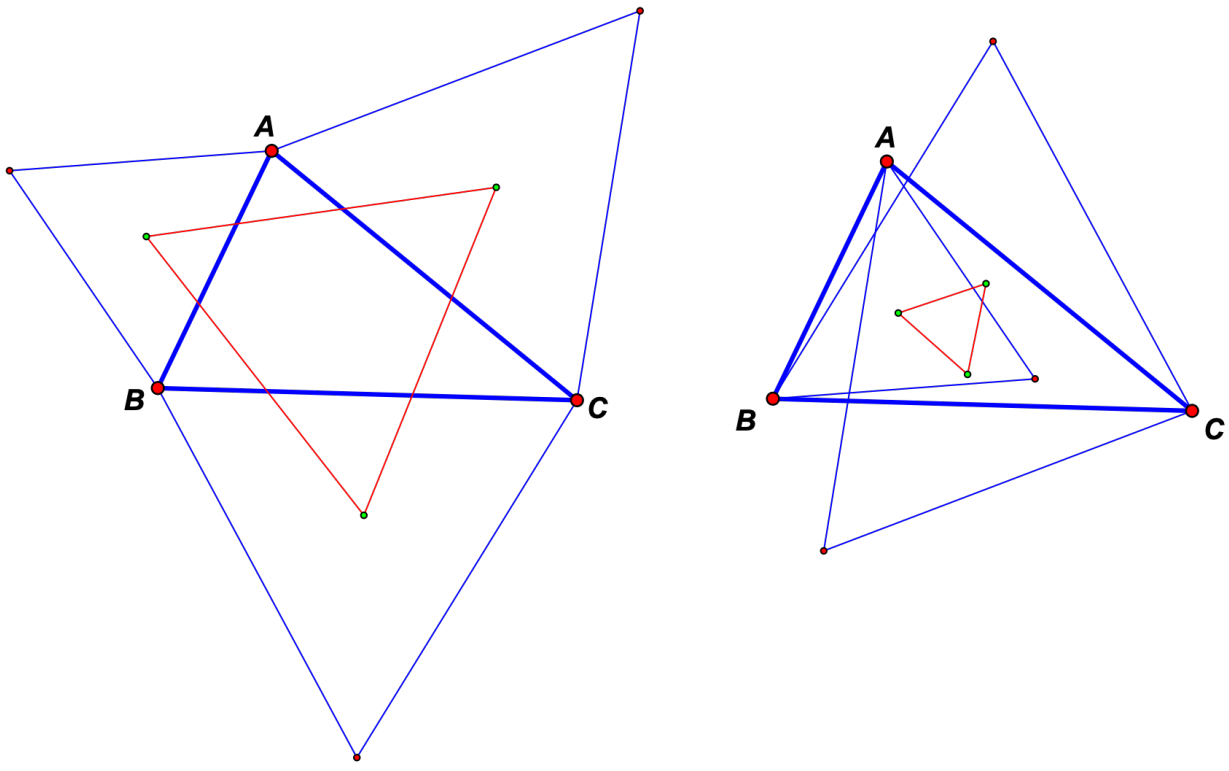

FIGURE 2: Napoleon's Theorem

[1] *Sketchpad* is now free to download at: http://dynamicmathematicslearning.com/free-download-sketchpad.html

PROOF OF THE SURPRISE RESULT

We are now ready to prove the surprise result at the start. Consider again Figure 1 and draw an equilateral triangle inwardly on side AB, as shown in Figure 3, and construct its circumcircle. Since $\angle D = 120°$ and $\angle AXB = 60°$, it follows that X lies on the circumcircle of ΔADB. Therefore, its circumcentre D' coincides with that of the circumcentre of equilateral ΔAXB. If we repeat the same construction on the other sides, it should now be clear that we simply obtain the same Napoleon equilateral triangle as in the second case in Figure 2. This completes the proof.

In essence the surprise result turns out to be merely a simple variation of the standard theorem of Napoleon with the equilateral triangles constructed inwardly. Obviously if we construct triangles ADB, BFC and CEA inwardly, we will merely obtain the outward Napoleon equilateral triangle.

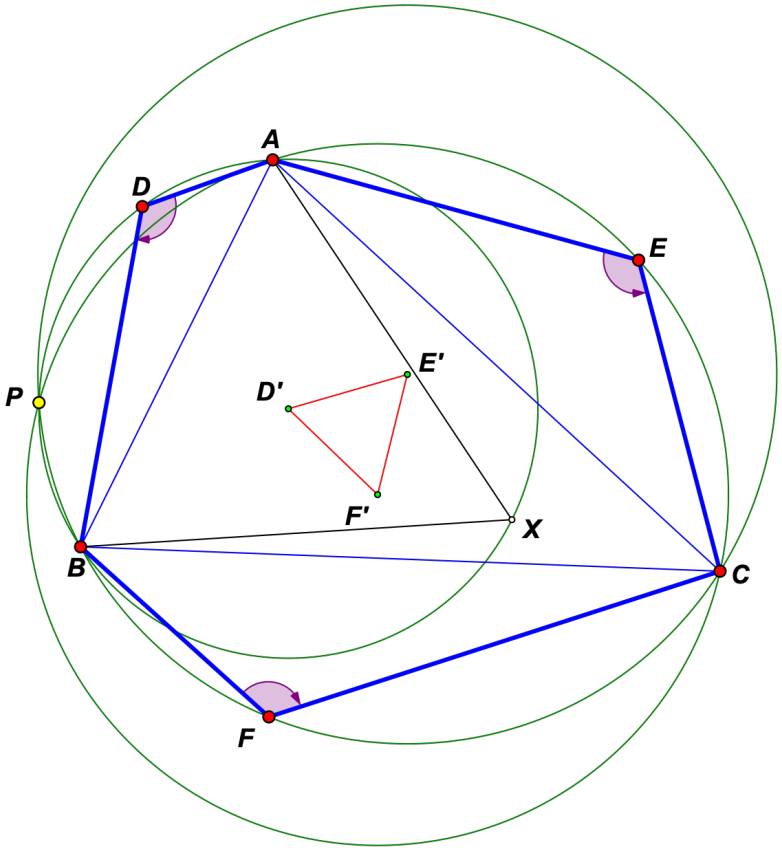

FIGURE 3: Proof of the surprise equilateral triangle result

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The Three Squares Problem

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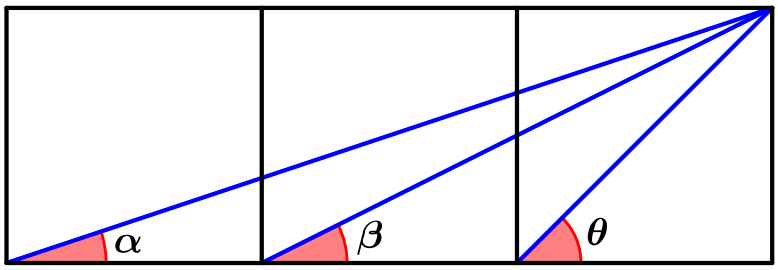
INTRODUCTION

If three identical squares are placed side by side and three diagonals are drawn as illustrated in Figure 1, then the sum of the three angles α, β and θ is $90°$. While this is an immensely pleasing result, it is not immediately obvious why the result holds true.



$$\alpha + \beta + \theta = 90°$$

FIGURE 1: The *Three Squares Problem*

In *Learning and Teaching Mathematics* No. 23, Yiu-Kwong Man presents a visual proof of the Pythagorean theorem which rather neatly links to a setup which can be used to prove the Three Squares Problem. Inspired by that article we decided to explore the Three Squares Problem in more detail. Drawing on our own ideas, as well as suggestions in online discussion threads, in this article we present eight different proofs of this intriguing result.

PROOF 1

The first proof makes use of basic geometry. A second row of identical squares is placed above the first row and three diagonals are drawn to create angles α and β (Figure 2).

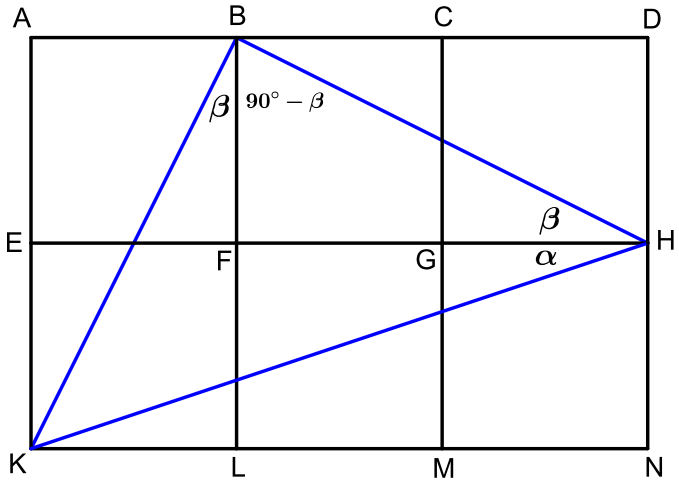


FIGURE 2: Proof using basic geometry

Since angles α and β are formed by the diagonals of a 3-by-1 and 2-by-1 rectangle respectively, we have $K\hat{H}E = \alpha$ and $K\hat{B}L = B\hat{H}E = \beta$. From the angle sum of triangle BFH we have $F\hat{B}H = 90° - \beta$ from which it follows that $K\hat{B}H = 90°$. Then, since triangle KBH is isosceles, we have $B\hat{K}H = B\hat{H}K = 45°$. But $B\hat{H}K = \alpha + \beta$. And since it is clear that $\theta = 45°$ given that it is the diagonal of a square (Figure 1) we obtain the result $\alpha + \beta + \theta = 90°$.

PROOF 2

The second proof is a subtle variation of the first proof. Using the regular grid structure created by the squares, we can superimpose the image on the Cartesian plane (Figure 3). Using the gradients of line segments OA and AB we see that $m_{OA} = 2$ and $m_{AB} = -\frac{1}{2}$ and since these are negative reciprocals of one another it follows that $OA \perp AB$. Furthermore, since $OA = AB = \sqrt{5}$ we have $A\hat{O}B = 45°$, thus $\alpha + \beta = 45°$ and the result follows as before.

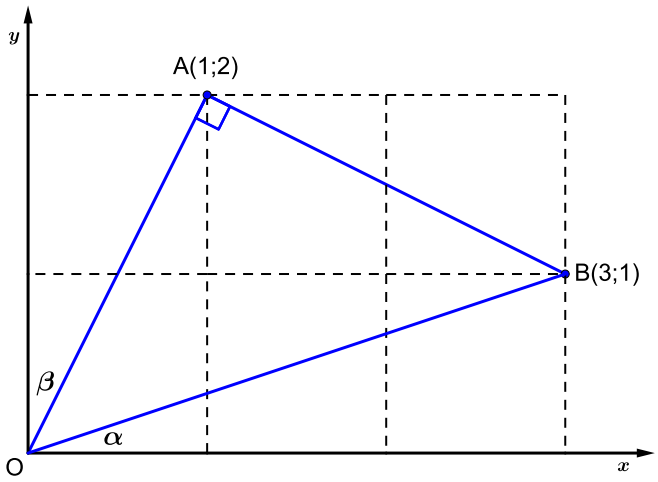


FIGURE 3: Superimposing the image on the Cartesian plane

PROOF 3

Let us now return to the original configuration (Figure 4). In triangle ADC we have $\tan\alpha = \frac{1}{3}$. Similarly, in triangle BDC we have $\tan\beta = \frac{1}{2}$. We can now use the tangent compound angle formula:

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha . \tan\beta} = \frac{\frac{1}{3} + \frac{1}{2}}{1 - \frac{1}{3} \times \frac{1}{2}} = \frac{5/6}{5/6} = 1$$

We thus have $\alpha + \beta = \tan^{-1}(1) = 45°$ from which the result follows.

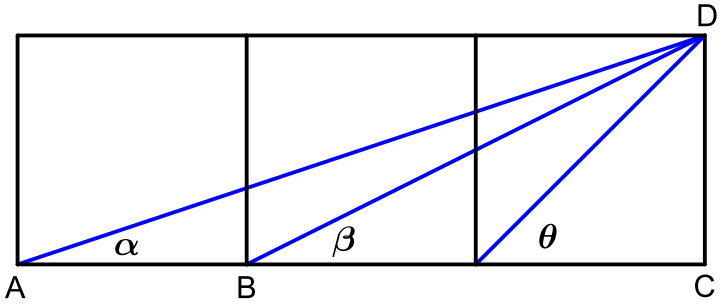


FIGURE 4: Using the compound angle formula for $\tan(\alpha + \beta)$

PROOF 4

The fourth proof makes use of similarity. Figure 5 shows the original configuration with an extra row of squares on top and two extra diagonals drawn in.

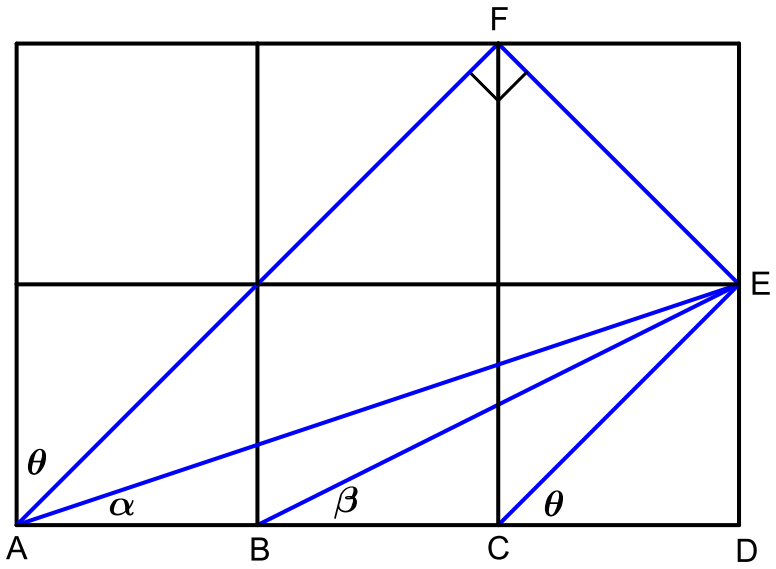


FIGURE 5: Using similarity

Note that triangles AFE and BDE are similar since they are both right-angled with their perpendicular sides in the ratio 1:2. This means that $E\hat{B}D = E\hat{A}F = \beta$, which rather neatly proves that $\alpha + \beta + \theta = 90°$.

PROOF 5

The next proof makes use of the cosine rule. If we let the side length of each square be one unit, then with reference to Figure 6 we have $AE = \sqrt{1^2 + 3^2} = \sqrt{10}$, $BE = \sqrt{1^2 + 2^2} = \sqrt{5}$ and $CE = \sqrt{1^2 + 1^2} = \sqrt{2}$.

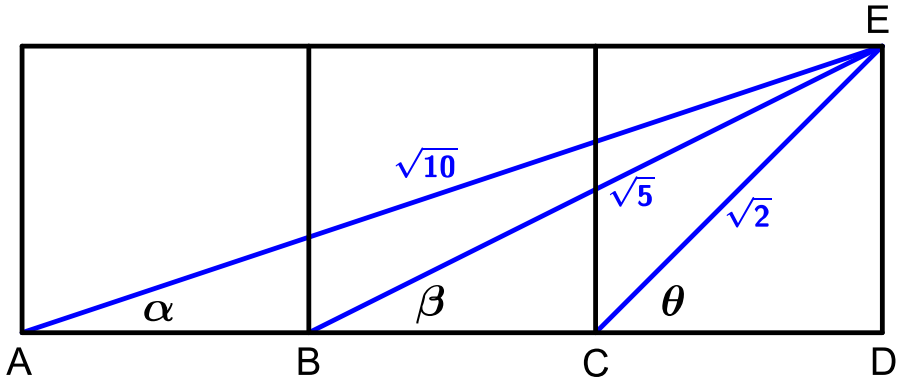


FIGURE 6: Using the cosine rule

Using the cosine rule in triangle BEC:

$$1^2 = \left(\sqrt{5}\right)^2 + \left(\sqrt{2}\right)^2 - 2\left(\sqrt{5}\right)\left(\sqrt{2}\right)\cos B\hat{E}C$$

$$1 = 5 + 2 - 2\sqrt{10}\cos B\hat{E}C$$

$$\therefore \quad \cos B\hat{E}C = \frac{3}{\sqrt{10}}$$

Using the cosine rule in triangle EAB:

$$\left(\sqrt{5}\right)^2 = 1^2 + \left(\sqrt{10}\right)^2 - 2(1)\left(\sqrt{10}\right)\cos\alpha$$

$$5 = 1 + 10 - 2\sqrt{10}\cos\alpha$$

$$\therefore \quad \cos\alpha = \frac{3}{\sqrt{10}}$$

Since $\cos B\hat{E}C = \cos\alpha$ and both $B\hat{E}C$ and α are acute, it follows that $B\hat{E}C = \alpha$. Using the exterior angle of triangle ECB we thus have $\alpha + \beta = \theta$, and since $\theta = 45°$ it follows that $\alpha + \beta + \theta = 90°$.

PROOF 6

This proof also makes use of the cosine rule, but in a slightly more elegant way. Figure 7 shows a grid of nine unit squares with E the midpoint of BC. Using Pythagoras we have $AF = \sqrt{10}$, $AE = \frac{3\sqrt{5}}{2}$ and $EF = \frac{5}{2}$.

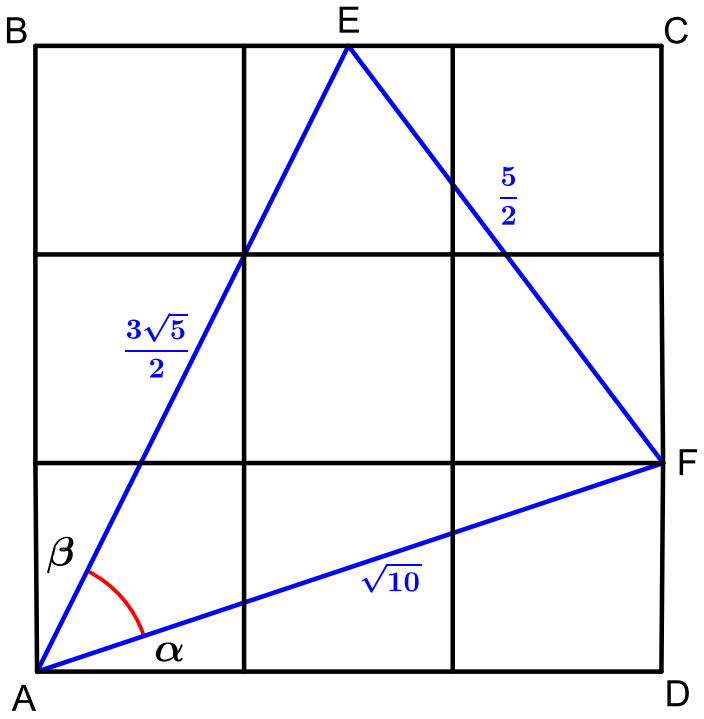


FIGURE 7: Using the cosine rule in a grid of nine unit squares

Using the cosine rule in triangle AEF:

$$\left(\frac{5}{2}\right)^2 = \left(\frac{3\sqrt{5}}{2}\right)^2 + \left(\sqrt{10}\right)^2 - 2\left(\frac{3\sqrt{5}}{2}\right)\left(\sqrt{10}\right)\cos E\hat{A}F$$

$$25 = 45 + 40 - 60\sqrt{2}\cos E\hat{A}F$$

$$\therefore \quad \cos E\hat{A}F = \frac{1}{\sqrt{2}}$$

Since $E\hat{A}F$ is acute, it follows that $E\hat{A}F = 45°$, from which it neatly follows that $\alpha + \beta = 45°$, and hence that $\alpha + \beta + \theta = 90°$.

PROOF 7

The second proof involved overlaying an image on the Cartesian plane and using a coordinate geometry approach. In this next proof we treat the endpoints of the line segments creating the three angles as complex numbers in an Argand diagram (Figure 8).

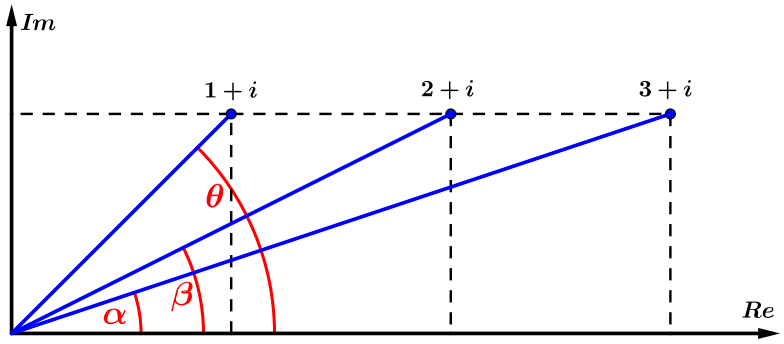


FIGURE 8: Using complex numbers in an Argand diagram

When multiplying complex numbers the result is a spiral dilation. The modulus (magnitude) of the product is simply the product of the moduli of the complex numbers being multiplied, while the argument (i.e. the angle to the real axis) is simply the sum of the arguments of the complex numbers being multiplied. Multiplying the three complex numbers $3 + i$, $2 + i$ and $1 + i$ gives:

$$\begin{aligned}(3+i)(2+i)(1+i) &= (3+i)(2+2i+i+i^2)\\ &= (3+i)(1+3i)\\ &= 3+9i+i+3i^2\\ &= 10i\end{aligned}$$

Since the complex number $10i$ lies on the imaginary axis (since its real component is zero) it has an argument of $90°$. In other words the sum of the arguments of the complex numbers $3 + i$, $2 + i$ and $1 + i$ is $90°$, i.e. $\alpha + \beta + \theta = 90°$.

PROOF 8

The final proof makes use of the converse of an interesting circle geometry theorem[2]. Given a circle and a tangent segment PT along with a secant from the same external point P, as illustrated in Figure 9, then the square of the tangent segment is equal to the product of the secant segment PA and its external portion PB.

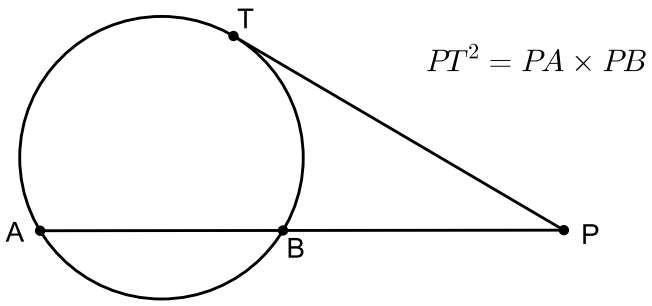


FIGURE 9: An interesting circle geometry theorem

[2] http://dynamicmathematicslearning.com/tangent-squared-equals-product-chords.html

We can now return to our original configuration and add in the circumcircle of triangle ABD as illustrated in Figure 10.

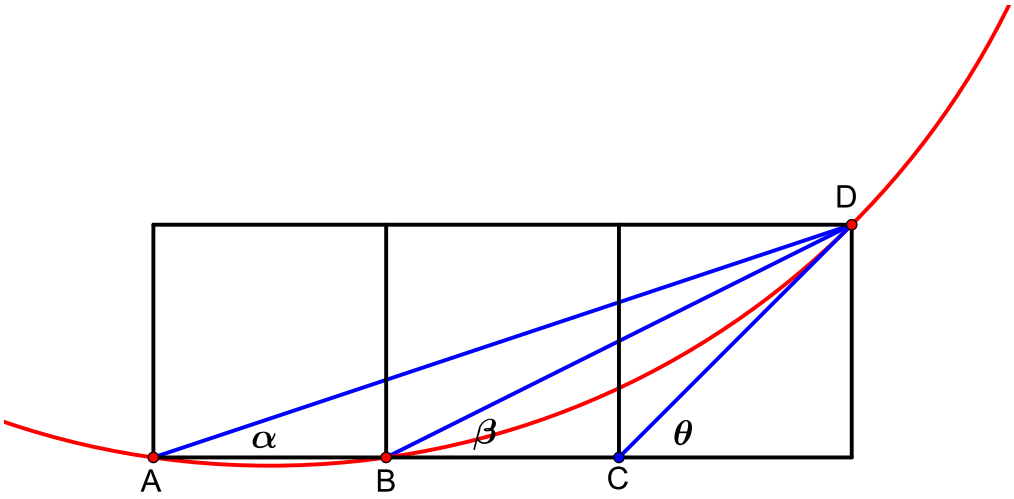


FIGURE 10: The original configuration with the circumcircle of triangle ABD

Treating the side length of each square as one unit, then $AC = 2$, $BC = 1$ and $CD = \sqrt{2}$. We thus have $CD^2 = AC \times BC$, and from the converse of the theorem illustrated in Figure 9 it follows that CD is a tangent to the circle passing through A, B and D. From this it follows that $B\hat{D}C = \alpha$ (angles in the same segment). Using the exterior angle of triangle $B\hat{C}D$ we have $\alpha + \beta = \theta$, from which $\alpha + \beta + \theta = 90°$ as before.

CONCLUDING COMMENTS

In this article we have presented eight different proofs of the *Three Squares Problem*. There are no doubt other ways of proving this interesting result, but the sheer variety of approaches illustrated in this article is both surprising and pleasing, and is certainly the hallmark of a good starting premise. The methods and techniques illustrated include use of basic geometry, the Pythagorean theorem, coordinate geometry, the tangent compound angle formula, the cosine rule, similarity, complex numbers and circle geometry. Almost all of these approaches are well within the grasp of most high school pupils, and the original problem lends itself beautifully to guided classroom investigation and exploration. Each proof also has the potential to open up broader discussions which could add to the richness of pupils' mathematical experience.

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The Sine Rule Disguised

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INTRODUCTION

Mathematics education is more than simply chasing after good marks in the matric examination. Of course, good marks are important, but good mathematics education should also develop what Skemp (1974) has called 'relational understanding'. Most examinations still focus largely on testing 'instrumental proficiency' – i.e. the ability to apply and carry out learnt procedures and algorithms correctly. In contrast, relational understanding is about developing meaningful connections between concepts, as well as developing the ability to think a bit more creatively, to be able to solve novel problems, and to begin posing one's own problems. Of course, relational understanding is much harder to establish and assess, but that doesn't mean we shouldn't constantly strive towards this as a goal.

A particularly useful problem-posing strategy that one can readily nurture in one's students is the so-called "what if" strategy (Brown & Walter, 1990). For example, after encountering a particular result about triangles in class, it would be natural to consider what would happen if the triangle was a quadrilateral, or a pentagon. While not all such questions necessarily lead to meaningful further exploration, some may, so it is important to continue cultivating such a mindset in students, as discovering something for oneself as a learner is immensely rewarding and empowering.

WHAT IF?

Here is a simple example to consider. Take any triangle. What would happen if we now construct arbitrary triangles on each side of the original triangle as illustrated in Figure 1? Are there any features that remain invariant, i.e. unchanged? Readers may wish to explore such a context by replicating the situation depicted in Figure 1 with their own dynamic geometry software. Alternatively, an online dynamic sketch is available for readers to explore interactively at:

http://dynamicmathematicslearning.com/invariant-product-triangles-on-sides-plus-Anghel.html

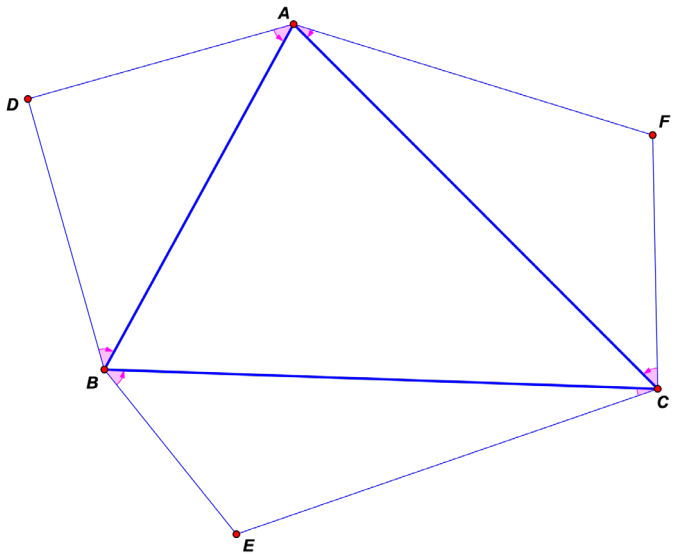

FIGURE 1: Arbitrary triangles on the sides of a central triangle

At first glance this may seem like a silly, rather meaningless scenario to consider. After all, what could possibly remain invariant (unchanged) in such a situation? However, the configuration has more than one property that remains invariant. The most obvious one is if we consider the (convex or concave) hexagon $ADBECF$ which has an angle sum of 360° (which is easy to see, and prove, from the diagram). Perhaps not so easy to 'see', however, is the following hidden invariant result involving the product of the ratios of the sides and the sine ratios of the angles of the respective outer triangles:

$$\frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA} \times \frac{\sin D\hat{A}B}{\sin D\hat{B}A} \times \frac{\sin E\hat{B}C}{\sin E\hat{C}B} \times \frac{\sin F\hat{C}A}{\sin F\hat{A}C} = 1$$

Note that the result remains true even if the hexagon $ADBECF$ becomes concave or crossed, and readers are encouraged to confirm this for themselves by dragging points D, E and F in their own sketches or using the interactive sketch at the link given earlier.

PROOF

Why is this somewhat surprising relationship true, and how can we prove it? The result may remind one of the sine rule, and it is indeed simply the sine rule disguised in a somewhat novel situation. Consider, for example ΔADB. Applying the sine rule to it we obtain:

$$\frac{AD}{\sin D\hat{B}A} = \frac{DB}{\sin D\hat{A}B} \Rightarrow \frac{AD}{\sin D\hat{B}A} \times \frac{\sin D\hat{A}B}{DB} = 1 \Rightarrow \frac{AD}{DB} \times \frac{\sin D\hat{A}B}{\sin D\hat{B}A} = 1$$

Since the respective products of the other ratios in triangles BEC and CFA are similarly equal to 1, the result follows.

GENERALISATION

It is now also easy to see that the result will similarly generalise to any polygon with triangles constructed on the sides, and is yet another illustration of the so-called 'discovery' function of proof.

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Suggestions to writers

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- *Help wanted* is a question and answer column: teachers can send their questions on teaching specific topics or aspects to this column for fellow colleagues in the AMESA community to respond to.

What are the technical requirements for the submission of articles?

Articles should not exceed 3 000 words and must be written in English. Articles as short as 300 words are also accepted and of course many of our categories such as "Question and Answers", "Kids say and do the darndest things", "Letters to the editors" and so forth can be even shorter. Articles should include the title, author's name, institution and full postal address, email and contact telephone numbers of the author.

Send your articles by e-mail (in a Word compatible format) to LTM@amesa.org.za.