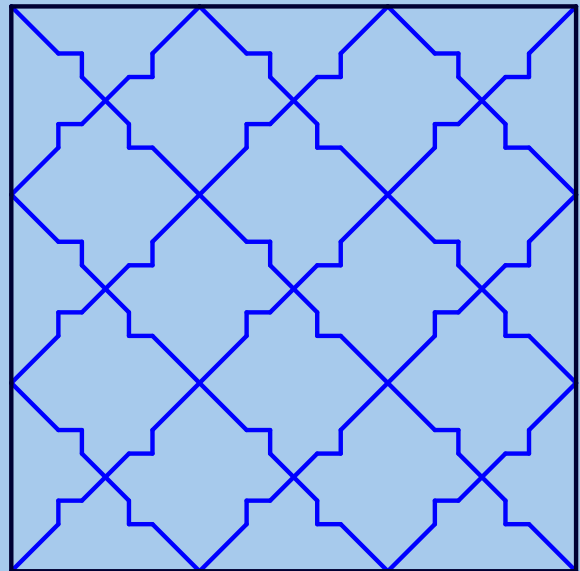
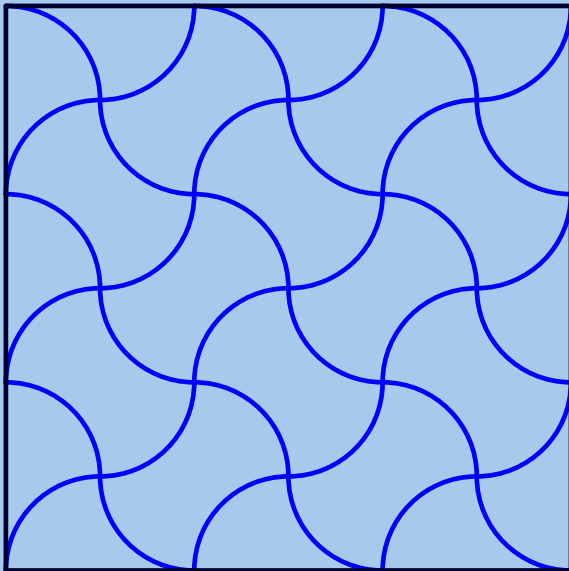


# Learning & Teaching Mathematics

A Journal of  AMESA



# Learning and Teaching Mathematics

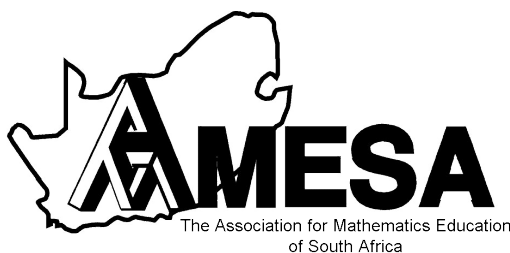
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The main criterion of acceptance is that the article should make a contribution to the improvement of school mathematics teaching and learning. See the inner back cover for more information on the submission of materials and articles for publication.

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## From the Editor

Dear LTM readers

In the first article of LTM 38, Duncan Samson and Moshe Stupel present a simple classroom activity aimed at nurturing creativity and geometrical engagement. In the second article in this issue, Shikha Takker unpacks and explores some of the sources of students' difficulties when working with decimals and suggests mathematical tasks that could be used to address them. The third article, by Orit Broza, demonstrates how students who appear weak or disengaged can, under the right conditions, construct meaningful and lasting mathematical understanding, while in the fourth article Moshe Stupel and Duncan Samson share a simple classroom activity which has the potential to create an *aha!* moment for pupils. Michael de Villiers then explores an interesting geometrical configuration which gives rise to congruent equilateral triangles.

In the sixth article, David Fraivert and Moshe Stupel transform a quadrilateral into a triangle with an equal area, while in the seventh article Christiaan Venter presents an alternative way to explore the concept of a function which introduces students to real-world applications of functions in fields like computer graphics and image processing. In the eighth article Michael de Villiers presents a provocative article exploring different definitions of trapezia. Alejandro Salguero and Ángel Plaza then provide a simple proof without words for the partial sum of a geometric series with common ratio  $1/(r + 1)$ . Issue 38 of LTM concludes with a review of Kate Kitagawa and Timothy Revell's book "The Secret Lives of Numbers: A Global History of Mathematics & Its Unsung Trailblazers".

We hope you enjoy the diverse array of articles in this issue and remind you that we are always eager to receive submissions. Suggestions to authors, as well as a breakdown of the different types of article you could consider, can be found at the end of this journal. If you have an idea but aren't sure how to structure it into an article, you are welcome to email the editor directly – we'd be happy to engage with you about turning your idea into a printed article.

Duncan Samson

# The Beauty of Mathematics

**Duncan Samson<sup>1</sup> & Moshe Stupel<sup>2</sup>**

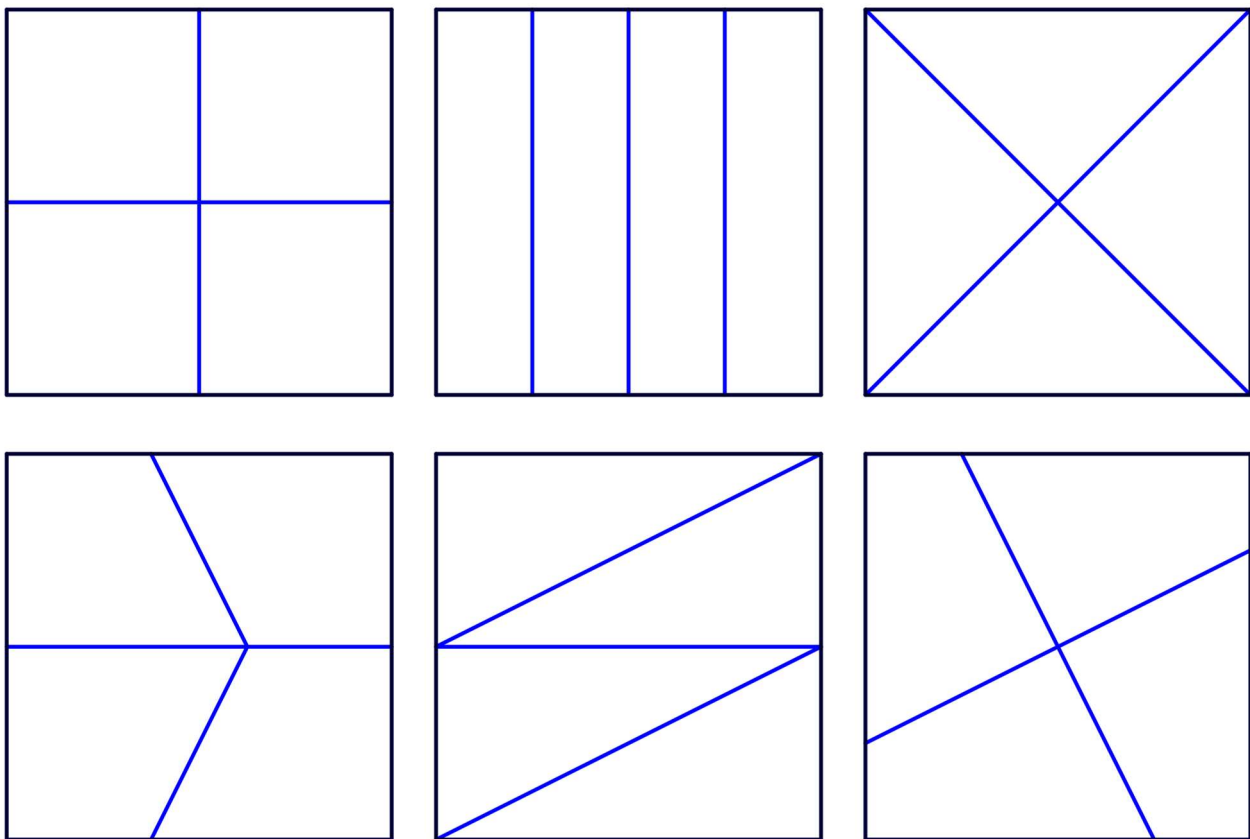
<sup>1</sup>*Victoria Girls' High School, Makhanda (Grahamstown)*

<sup>2</sup>*Gordon Academic College & Givat Washington Academic College, Israel*

*dsamson@vghs.co.za stupel@bezeqint.net*

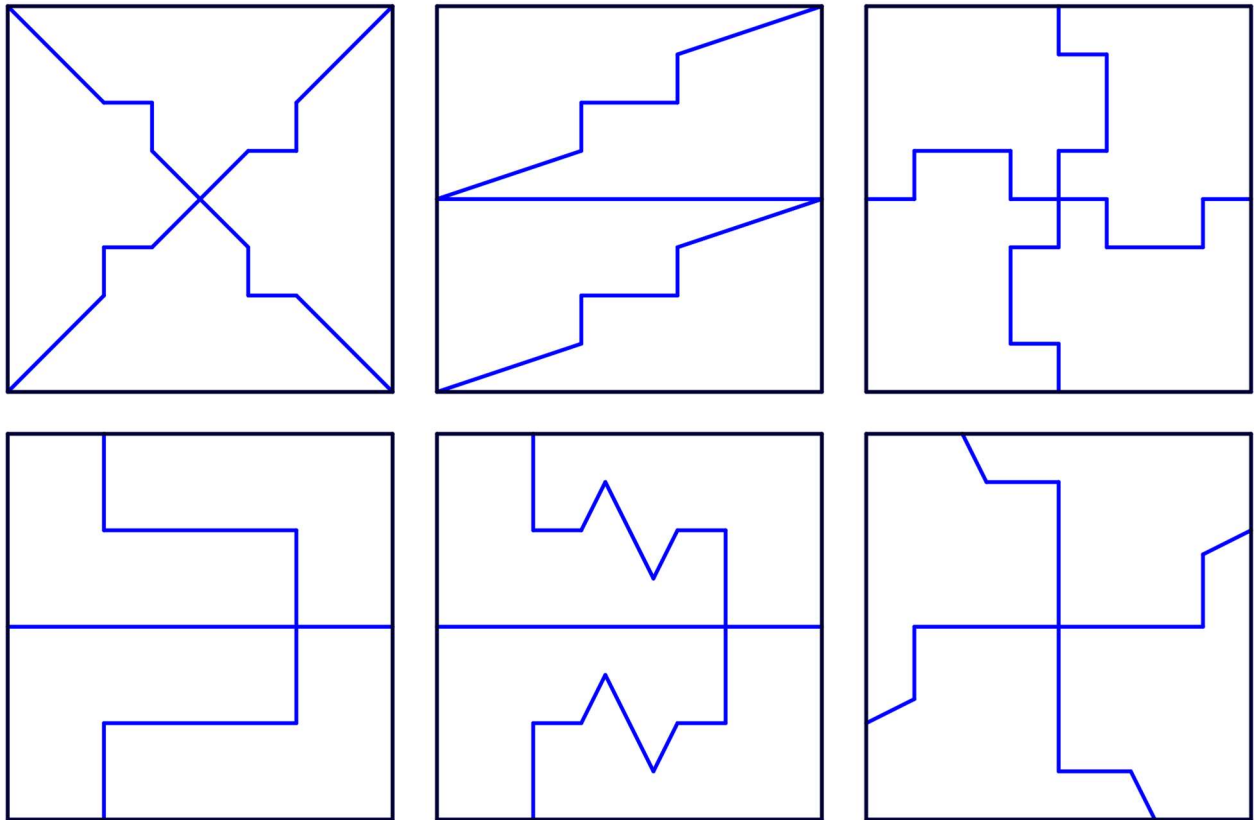
## INTRODUCTION

Here's a simple activity to try out in the classroom. Give pupils a sheet of paper with 16 or so identical squares. The task is for pupils (either individually or in pairs) to subdivide each square differently into four identical parts. The subdivisions could be simple or complex, the only requirement is that the four subdivided parts are identical. Challenge pupils to do this in a variety of ways, and encourage them to be as creative as possible. A few examples of possible subdivisions are illustrated in Figure 1.



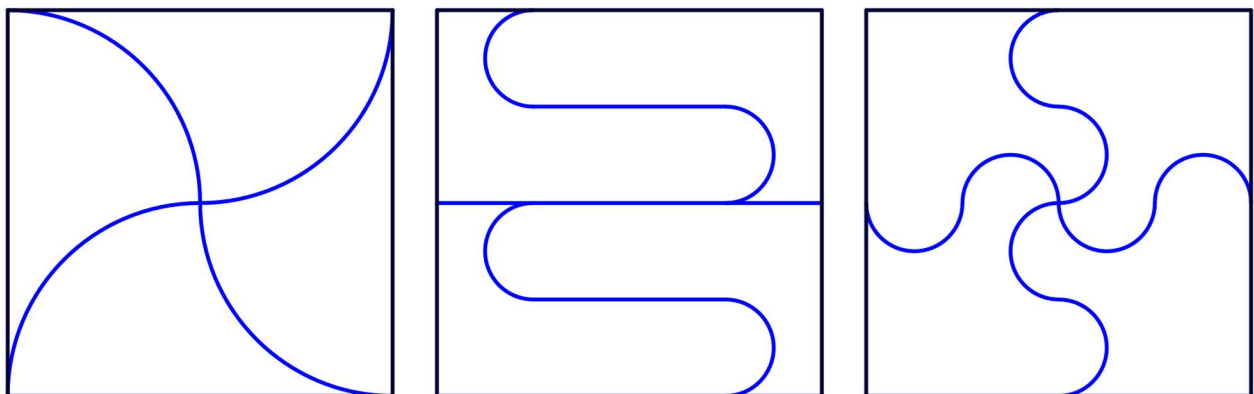
**FIGURE 1:** Subdividing a square into four identical parts.

The examples shown in Figure 1 are all relatively simple subdivisions involving only a few line segments. Figure 2 illustrates some further subdivisions which are somewhat more complex in design.



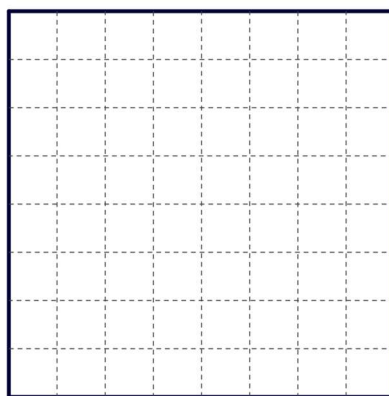
**FIGURE 2:** More complex subdivisions.

Pupils are not restricted to straight line segments in their subdivisions – they can also make use of quarter circles or semicircles, or indeed any arc of a circle, either alone or in conjunction with straight line segments. A few possibilities are illustrated in Figure 3.



**FIGURE 3:** Subdivisions involving quarter circles and semicircles.

In order to help pupils with their subdivision process, it would be useful to give them squares that have an embedded grid structure, as illustrated in Figure 4. This embedded substrate should help not only with accuracy but is likely to support creativity as well.

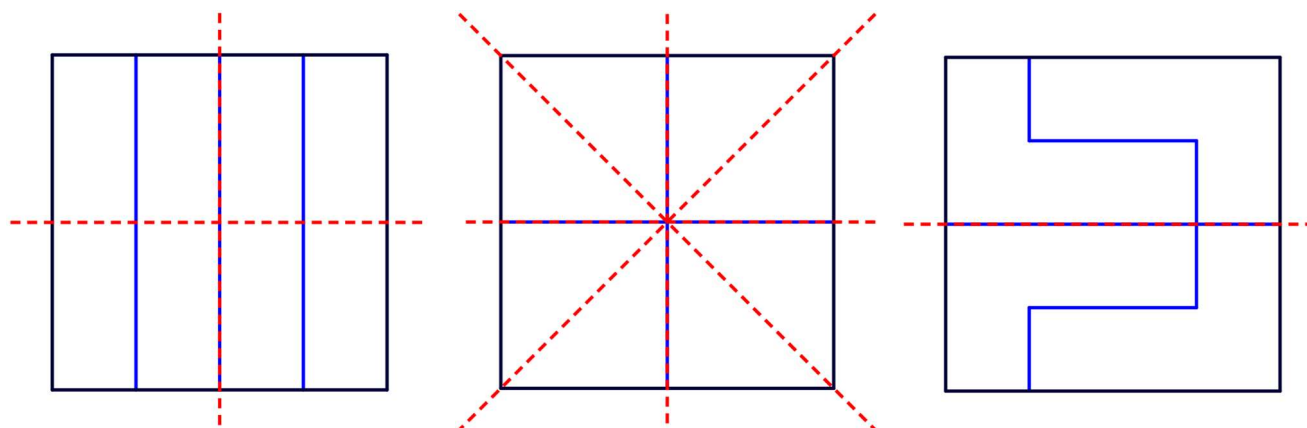


**FIGURE 4:** A square with an embedded grid structure to help accuracy and creativity.

While the activity described is in itself a valuable exercise in creativity, it can readily be extended to explore other aspects of mathematics such as reflectional symmetry, rotational symmetry, tiling patterns and the concept of tessellation, as well as area calculations. We will now touch on each of these in turn.

### REFLECTIONAL SYMMETRY

Once pupils have completed their various subdivisions, they should be challenged to identify those that contain reflectional symmetry, and to draw in the different lines of symmetry. A few examples are illustrated in Figure 5.



**FIGURE 5:** Reflectional symmetry and lines of reflection.

It is worth noting that there are a few subdivisions in those designs previously illustrated (Figures 1 – 3) that, although they don't contain any reflectional symmetry as drawn, could readily do so with some minor adjustments.

### ROTATIONAL SYMMETRY

Pupils should also be challenged to identify those subdivisions that have rotational symmetry and to state the order of rotation as well as draw in the centres of rotation. A few examples are illustrated in Figure 6. In the left-most diagram there is four-fold rotational symmetry around the centre of the square. In the middle diagram, the left and right rectangles each exhibit two-fold rotational symmetry about their respective centres. In the right-most diagram there are three centres of symmetry. The upper and lower rectangles each exhibit two-fold rotational symmetry about their respective centres, but there is also two-fold rotational symmetry about the centre of the square as well.

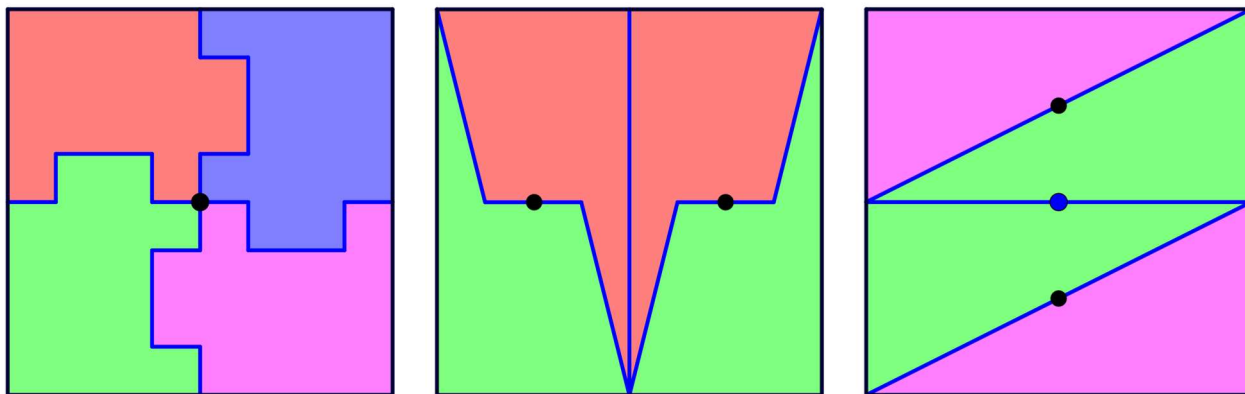


FIGURE 6: Rotational symmetry and centres of rotation.

### TILING PATTERNS & TESSELLATIONS

Pupils should choose a few designs (subdivisions) that they particularly like and make multiple copies of them. Placing these individual square tiles next to one another should create some interesting tiling patterns. A few possibilities are illustrated in Figure 7. Pupils should hopefully discover that some of the subdivisions work more effectively than others. What design characteristics should one consider when subdividing a square intended for a tiling pattern?

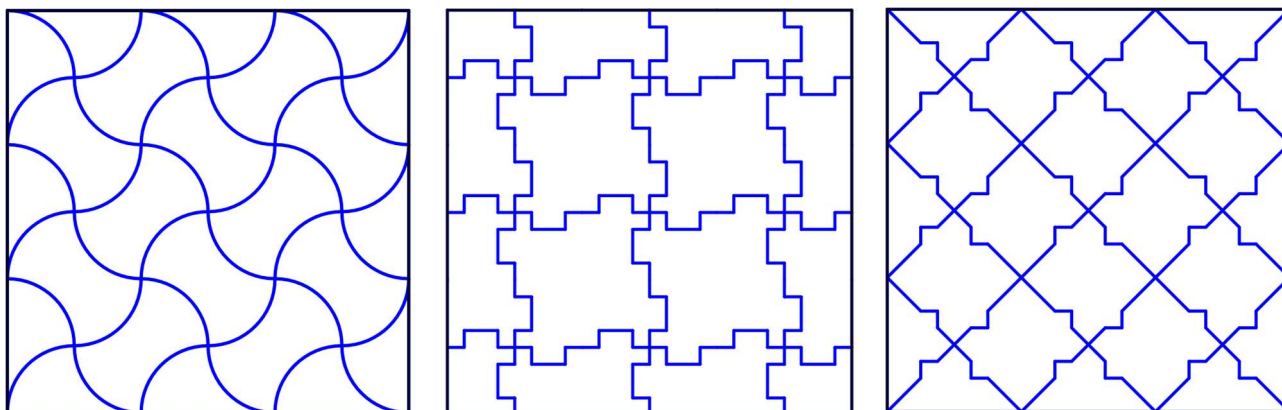
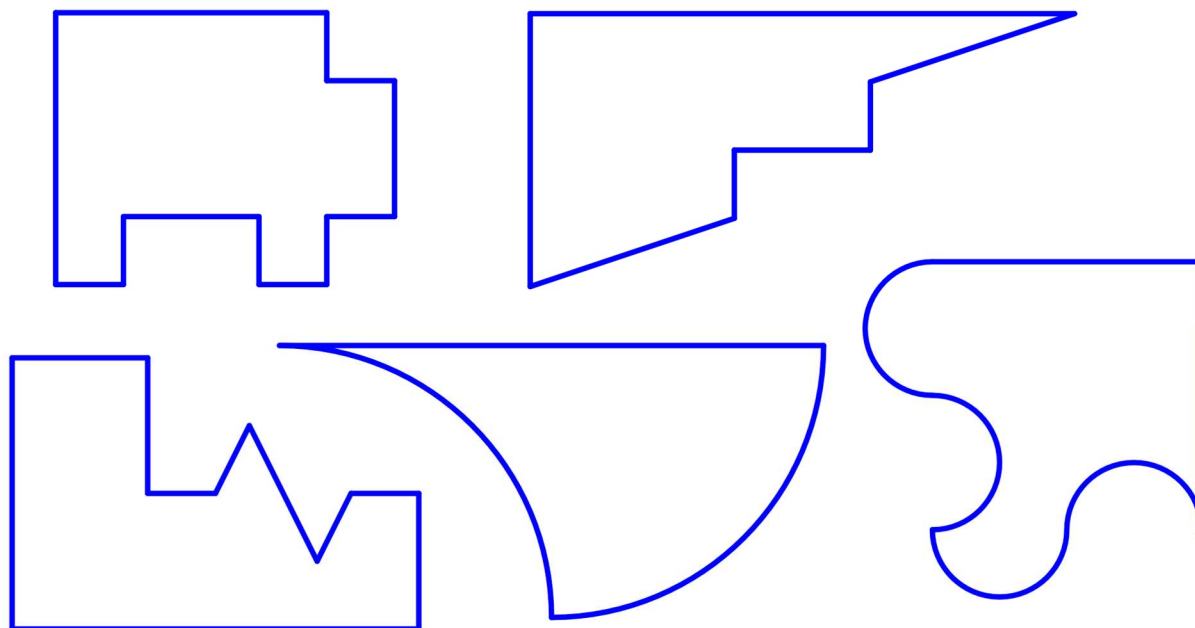


FIGURE 7: Tiling patterns.

From here one could introduce the concept of tessellation, where a plane surface is covered with one or more geometric shapes without any overlaps or gaps. Some of M. C. Escher's wonderful tessellations could be used to illustrate how a simple mathematical concept can lead to intricately beautiful works of art.

### AREA CALCULATIONS

Since we are subdividing the square into four identical parts, it follows that each of these parts will have the same area, and that each part will have an area that is one quarter of the original square. The consequence of this is that each subdivision, from *any* of the various designs, must necessarily have the *same area*. So, for example, each of the shapes illustrated in Figure 8 has the same area, although this may not be immediately obvious.



**FIGURE 8:** Shapes of equal area.

Challenge your pupils to confirm that their various subdivisions do indeed all have the same area. Pupils will need to split each subdivision into standard shapes (squares, rectangles, triangles, parts of circles, etc.) and then sum these together to get the total area of each subdivision. This process could be aided if pupils were originally given squares that have an embedded grid structure, as illustrated in Figure 4. Each square would then have an area of 64 square units, and each subdivision should have an area of 16 square units.

#### **CONCLUDING COMMENTS**

It was the purpose of this article to present a simple classroom activity that has the potential to nurture creativity and geometrical engagement. The initial activity could then be extended to explore other aspects of mathematics such as rotational and reflectional symmetry, tessellations and tiling, mathematics in art, as well as area calculations. Open-ended and differentiated tasks such as these are useful in that different pupils can approach them at a level of complexity with which they are personally comfortable. This helps in allowing each pupil to extract maximum value from the activity.

# Student Difficulties with Decimals: Contexts and Representations

**Shikha Takker**

***Mahindra University, India & University of the Witwatersrand, South Africa***

*shikha.takker@gmail.com*

## INTRODUCTION

Student difficulties with rational numbers are well known. The challenges in interpreting fractions, decimals and percentages, by distinguishing them from whole numbers, is addressed in the middle school. This article focuses specifically on students' difficulties with decimals. Identifying the source of students' difficulties helps us in understanding why students find decimals difficult and in addressing these difficulties conceptually. The source of students' difficulties with decimals often lies in the way decimal representation is used with and without contexts. While contexts are intended to make mathematics meaningful, it is important to be cognizant of what mathematical (or conceptual) compromises are made in the process of contextualisation. Typical contexts used in the teaching of decimals in Grades 6 to 8 involve currency and measurement, and interpretation of decimal representation within these contexts can vary.

This article begins by identifying student difficulties with decimals. This is followed by a discussion of the sources of these difficulties by examining how decimal representation is used in contexts. The article concludes with some tasks that can be used to address students' difficulties with decimals.

## STUDENT DIFFICULTIES AND DECIMAL CONTEXTS

In the South African curriculum, learners are introduced to decimal fractions in Grade 6. Broadly, a basic introduction to fractional place values  $\left(\frac{1}{10}; \frac{1}{100}; \frac{1}{1000}; \dots\right)$  is followed by the comparison and ordering of decimals, conversion between fractions and decimals, and the four arithmetic operations with decimal fractions. Working with and operating on decimal fractions supports a sense of quantity with very small and very large magnitude. Decimal knowledge helps students in understanding measures of length, mass, area and volume, which are expressed using rational numbers. The knowledge is also useful in calculating rates and other quantifiable changes in later grades. The teaching of all these sub-topics involves use of contexts where decimal fractions need to be selected carefully.

The most common contexts used in the teaching of decimals are currency and measurement. Currency contexts involve two kinds of tasks. First, conversions within the units of the national currency, for example rands and cents. Second, conversion between currency of different countries, for example converting South African Rands (ZAR) to United States Dollars (USD) and vice versa. Within measurement, the contexts involve length, weight and volume measures. Some units of measurement, such as the centimetre, kilogram and litre are more popular, while other units of measurement, such as the decimetre, decagram, and centilitre are less typically used.

Students often encounter the contexts in the form of word problems. The contexts are used in word problems so that students can make sense of the quantity by relating it to their experience (Figure 1a). In addition to word problems, contexts are sometimes invoked in numerical problems through the use of units, where the use of units can potentially assist students in 'visualising' the quantities (Figure 1b). Naturally, there are also tasks that are purely numeric or symbolic (Figure 1c).

(a)	Contextual word problem	A container has 0,8 litres of water. You drink a quarter of a litre of water. How much water is left in the container?
(b)	Using context as a reference	0,8 litres – 250 millilitres = _____
(c)	Numeric or symbolic task	What is 0,8 – 0,25?

**FIGURE 1:** Representing a decimal subtraction problem in three different formats.

While the same decimal numbers are used in Questions (a), (b) and (c) in Figure 1, the sense-making is quite different. For example, in Question (a), students first need to rewrite a quarter of a litre from the fraction representation,  $\frac{1}{4}$  of a litre, to a decimal representation such as 0,250. Students are also expected to convert the measures into the same units (litres or millilitres) before performing the operation. This is something that much older students still often find difficult. In addition, students often rely on recall of the relation rather than focusing on the nature of the relationship between the units. This raises three questions for us:

- What are the reasons for students' difficulties with conversions between units?
- How can we, as teachers, make the relationships between units explicit for the students?
- What kind of tasks can we use to help students overcome their difficulties?

The remainder of the article focuses on these three questions.

### SOURCES OF STUDENT DIFFICULTIES

When working with typical units, students often struggle with conversions which include fractional measures. Some examples of students' incorrect conversions are illustrated in Figure 2. One source of difficulty relates to students treating the decimal point as a 'separator' between the units of measurement, where it is literally read as what comes before the decimal comma is the larger unit and what comes after is the smaller unit. By way of example, students who have this incorrect 'separator' notion of a decimal point might write 5 rands and 1 cent as R5,1 or interpret 9,05 *m* as 9 metres and 5 centimetres. Interpretation of the decimal point as a 'separator' masks the relation between the two units of measurement. In other words, students are not thinking about the quantitative relation between rands and cents, or between metres and centimetres, and are treating the conversion merely as a procedure.

2 metres and 8 millimetres → 2,8 <i>m</i>
6 rands and 8 cents → R6,8
2,5 <i>litres</i> → 2 litres and 5 millilitres
9,05 <i>metres</i> → 9 metres and 5 centimetres

**FIGURE 2:** Examples of incorrect responses to conversion tasks.

Another source of students' difficulty is the disconnect between the decimal representation and the relation between measurement units. We notice that student difficulties in conversion between measurement units are linked to their lack of understanding of its links with the decimal notation. We can classify student difficulties into two major categories, based on the reasoning or explanation that accounts for the difficulty. Both these categories are well known to teachers and researchers. However, the interaction between the student difficulties and different representations in decimals is less well known. We will discuss how these difficulties interact with students' understanding of numbers used in decimal contexts.

### 1. When the length of the decimals is not the same

Students are often given tasks where the decimal lengths are not the same – for example, compare 5,72 and 8,1 or subtract 6,05 from 10. The length of a decimal is determined by the place value of the digits. In the examples given, 5,72 and 6,05 have digits at the units, tenths and hundredths positions. In 8,1 there are digits at the units and tenths positions, while in 10 there are digits at the tens and units position, with no tenths or hundredths. Tasks with different decimal lengths require students to first make the decimal lengths the same and then carry out the calculation. For example, it is assumed that students will compare 5,72 and 8,1 by first appending a zero to create a comparable decimal 8,10. Similarly, students will need to represent 10 as 10,00 before subtracting 6,05 from it.

However, many students may assume that 5,72 is greater than 8,1 simply by comparing the number of digits in each decimal (5,72 has three digits which is more than 8,1 which only has two digits). Such incorrect thinking is often referred to as 'longer is larger'. In other words, the longer the length of the decimal number, the greater it is when compared to a number with fewer digits. The source of this difficulty lies in conflating the procedure of comparison of whole numbers with the comparison of decimal fractions. When we compare whole numbers, the number of digits is indicative of a bigger number. Students need to understand why the procedure of comparing whole numbers cannot be extended to the comparison of decimals.

We have found that some student responses also indicate the reverse of this thinking, i.e. 'shorter is larger'. For example, when comparing 6,7 and 4, students might say that 6,7 is smaller than 4 because the fractional place value (tenths, hundredths, etc.) decreases the number. Since 6,7 has tenths and 4 does not, the number with the shorter length is considered larger.

In both these kinds of thinking, 'longer is larger' and 'shorter is larger', students focus on the length of the decimal number and not the quantity that it represents. Students with both these kinds of thinking tend to get some comparisons correct. As teachers we can understand if students are thinking in these ways by carefully selecting examples for decimal comparison. Figure 3 shows a mixed set of examples that can be used to understand whether students' thinking is dominated by the length of decimals during comparison. Students with 'longer is larger' thinking will compare (a) and (b) in Set A correctly but all the other examples incorrectly. Similarly, students with 'shorter is larger' thinking will compare (a) and (b) in Set B correctly but all the other examples incorrectly.

Compare the following decimals by writing in $>$ or $<$ symbols	
Set A	Set B
(a) 4,76 ____ 0,5	(a) 1,7 ____ 0,578
(b) 5,06 ____ 89,34	(b) 9,6 ____ 9,34
(c) 1,67 ____ 2,3	(c) 6,7 ____ 6,81
(d) 0,76 ____ 0,115	(d) 0,5 ____ 1,19
(e) 41,6 ____ 0,705	(e) 0,36 ____ 0,2
(f) 89 ____ 7,12	(f) 14 ____ 19,3
(g) 9,3 ____ 0,250	(g) 9,3 ____ 25,0

FIGURE 3: Comparison tasks to identify student thinking.

## 2. The position of zero in a decimal number

Students are often advised to make the length of decimal numbers the same by appending a zero. For example, to compare 7,5 and 9,05; students are asked to make the lengths of the decimals the same by re-writing 7,5 as 7,50. Importantly, the zero needs to be appended after the last decimal. Students who don't understand this may change 7,5 to 7,05 and then consider them as equivalent decimal fractions. A meaningful way to engage with this misconception would be to compare the place value of the digits of the original number with the number created through insertion of the zero. For example, in creating equivalent decimals for 12,00 the place value of the digits 1 and 2 does not change if the number is re-written as 12 and 12,0, but the place value of the digits 1 and 2 does change when the number is re-written as 120,0 or 102,0.

The consideration of the use of zero is also important in different contexts, specifically currency versus other contexts such as measurement. In the currency context, we use only two units, namely rands and cents, and the relation between them is  $R1 = 100$  cents. As such, R6,05 means 6 rands and 5 cents while R6,50 means 6 rands and 50 cents. The decimal representation R6,5 does not make sense in a currency context, and the decimal representation R6,5 might itself be confusing for students. All currency values should typically be written with digits in the tenths and hundredths positions to avoid this potential confusion.

In measurement contexts we do not have this constraint, and 6,5 m ; 6,50 m and 6,500 m are all meaningful equivalent expressions in the given context. It is important to make these relationships explicit for the students, i.e. (a) a decimal comma is not merely a separator, it represents a specific relationship between positions of digits of a number, and (b) decimal numbers need to make sense in relation to the context within which they are represented.

### TASKS FOR TEACHING DECIMALS

We have seen how the sources of students' misunderstanding can lie in:

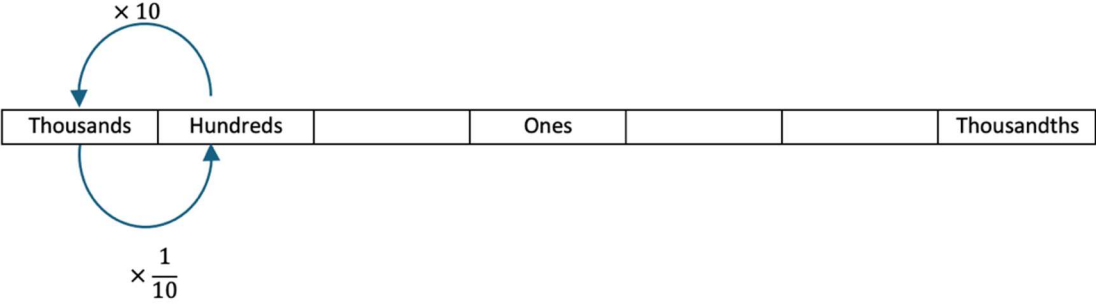
- The lack of conceptual understanding of the meaning of a decimal notation – i.e. the decimal comma is not merely a separator, but represents a particular kind of relation between units in a given context (e.g. currency and measurement).
- Differential meanings of decimal notation with changing contexts – measures represented by decimal numbers need to make sense in relation to the context within which they are represented.

We now discuss three tasks which could help in addressing both these sources of misunderstanding among students. They should also support students in moving beyond the thinking borrowed from whole number comparisons.

#### Task 1: Decimal representation

The purpose of the first task shown in Figure 4 is to make the decimal notation explicit for the students. This task primarily highlights the link between different place values of decimal notation. It can also be extended to discuss links between decimal representation and the units of measurement.

Write all the decimal place values from thousands to thousandths. Then mark the relationship between consecutive place values. The first one has been done for you.



Now use the relations to fill in the blanks.

1 *thousands* = \_\_\_\_ *hundreds*

10 *tens* = \_\_\_\_ *hundreds*

10 *hundredths* = \_\_\_\_ *tenths*

$\frac{1}{10}$  *ones* = \_\_\_\_ *tenths*

$\frac{1}{10}$  *ones* = \_\_\_\_ *hundredths*

FIGURE 4: Decimal representation task.

Task 1 draws students' attention to the relation between place values of a decimal number. The visual representation helps students in three ways. First, it highlights the consistent relation between the consecutive place values using powers of 10. For example, the relation between hundredths and tenths is the same as the relation between hundreds and thousands. Second, it helps students distinguish between hundreds and hundredths (for example) based on their position and relation to other places. Third, it helps students in identifying the relation between non-consecutive place values. An explicit discussion on the relation between place values of a decimal representation will help students in understanding the reasons for conversions. This task can be extended by placing currency and measurement units in a similar table and highlighting the relationships between them. Through the placement of units like the decimal place value, students will be able to see the consistency of relations between different measurement units and a specific kind of relationship between the units of currency.

### Task 2: Is this a decimal point?

Discussing some real-life examples of the decimal comma (or point) might help students in deepening their understanding of the contexts within which a decimal representation is used. A general discussion on where we see separators or points being used around us is a useful starting point. Task 2 gives a list of possible situations for discussion, only some of which represent decimal contexts.

Prices: e.g. R27,50 or 1USD = R18,50
Exercises: e.g. Exercise 1.9 or 2.3
Time on a watch: e.g. 11:50
Time taken to run a race: e.g. 3,25 minutes
Run rate: e.g. 6,22 runs per over
Number of balls bowled: e.g. 14.5 overs
Distances: e.g. 12,5 metres
Dates: e.g. 16.01.2025

**FIGURE 5:** Contexts for discussing separators versus decimal points.

The discussion should highlight that a decimal representation specifically refers to the relationship of powers of 10. Other contexts where this is not the case (e.g. number of balls bowled, time on a watch, dates, etc.) do not represent decimal fractions. Sensitising students to this important distinction should help their conceptual understanding of decimal contexts.

### Task 3: Currency contexts & the position of zero

Task 3 focuses on currency, a popular context for working with decimals, as well as drawing students' attention to the position of zero. In Question 3, each set of decimal fractions purposely makes use of the same non-zero digits so that students can focus on the position of the zero or zeros in each decimal fraction. Students should be encouraged to give reasons for their answers using the place value to explicitly link the decimal representation with the currency context.

**Question 1 (currency context)**

In an advertisement of a mobile operator, Simi saw that the rate for an international call is given as 1 rand and 5 cents per minute. She bought the service for a month. After one month, Simi receives the phone bill. She sees that the phone company has charged her R1,50 for each minute for an international call. Do you think that Simi is being cheated? Give reasons for your answer.

**Question 2 (position of zero)**

Compare the decimal fractions in each of the following sets and identify decimals that are equivalent. Give reasons for your answers.

- |     |      |       |      |       |       |
|-----|------|-------|------|-------|-------|
| (a) | 1,5  | 1,50  | 1,05 | 1,500 | 1,050 |
| (b) | 3,05 | 3,050 | 3,50 | 0,35  | 3,5   |
| (c) | 9,02 | 0,92  | 9,2  | 92,0  | 9,20  |

**Question 3 (currency context & position of zero)**

Re-write each of the following amounts using a decimal comma.

- (a) 5 rands and 2 cents
- (b) 5 rands and 200 cents
- (c) 5 rands and 20 cents

**FIGURE 6:** Position of zero in decimal fractions.

**CONCLUDING THOUGHTS**

This purpose of this article was twofold. First, it was our intention to unpack and explore some of the sources of students' difficulties when working with decimals and to suggest mathematical tasks that could be used to address them. Second, we wanted to highlight the importance of making the relation between decimal representations in typically used contexts explicit. Students' difficulties with decimals are often linked to the way decimal representations are presented to them. We hope that through a careful attention to the contexts within which decimal representations are used, students' conceptual understanding of decimal notation can be supported and nurtured.

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# Even Low Achieving Students Can! Ben's Case Study

**Orit Broza**

*The Academic College Levinsky-Wingate, Tel Aviv, Israel*

*Oritbroza1@l-w.c.il*

## INTRODUCTION

### Rethinking Struggle in Elementary Mathematics

Why do so many students struggle with mathematics as early as elementary school? This question has intrigued educators and researchers across disciplines. Although much research addresses mathematical performance, few studies explore the *learning processes* of underachieving students. Literature often focuses on what these students lack, with little attention paid to how they learn and grow under the right conditions (Roche, Gervasoni & Kalogeropoulos, 2023).

This case study stems from a larger research initiative that examined how low-achieving students construct mathematical understanding in a learning environment tailored to address challenges identified in the literature. The pedagogical approach integrated three broad theoretical pillars: (1) instruction embedded in real-life, meaningful contexts; (2) digital and technological learning tools, with an emphasis on educational games; and (3) constructivist teaching methods, particularly the use of scaffolding and teacher mediation (Broza & Kolikant, 2015; 2021).

The classic literature is rich with studies in which computational strategies emerged intuitively within mathematical contexts outside of school. For example, among young street vendors in Brazil (Nunes, Schliemann, & Carraher, 1993) or among tailors (Lave & Wenger, 1991). However, when these individuals were asked to solve basic problems in a formal test format, they tended to use school-taught procedures, often without success. A common feature of the invented computation methods described in these studies was that they relied on mental procedures, without pencil and paper or calculators, using number decomposition, estimation, and common sense.

On one hand, these studies support the approach that advocates integrating everyday situations into the classroom. On the other hand, they illustrate Brenner's (1998) critique of this type of teaching. According to Brenner, when the connection between everyday computational strategies and the formal school mathematics is not made explicit, it remains unclear to students and is not meaningfully applied. Brenner (1998) investigated the impact of real-life mathematics on school mathematics among preschool and elementary school children by using situations involving money. Through classroom observations and interviews, Brenner found that students used common sense when talking about money. However, she identified a fundamental difference between using money in real life and using it in maths lessons. In the classroom, money was taught as a series of disconnected activities, such as naming coins and translating them into numbers. In this context, money served as a representation of the grouping-by-ten process (e.g., exchanging small coins for larger denominations). In contrast, authentic money situations required making change by breaking down bills and coins into smaller units.

The conclusion drawn from Mary Brenner’s research is that when introducing a real-life situation into the classroom using the money model, it is essential to focus on the meaning of the operations involved in using money – not just to use it as a visual aid for symbolic mathematics. This conclusion is further supported by findings from interviews with students who solved problems using authentic money representations, compared to students who solved problems without such representations (McNeil, Uttal, Jarvin, & Sternberg, 2009). In the group that used money, a higher level of conceptual understanding was observed in solving word problems, and most of their explanations were based on reasoning and sense-making.

The story of Ben – a student who initially struggled deeply with mathematics – offers insight into the value of these approaches. Through interactions with concrete materials, digital tools, and guided peer collaboration, Ben built a personal repertoire of strategies to tackle subtraction of a decimal number from a whole number. This study not only tracks his progress but also asks: What can we reasonably expect from students like Ben? What supports are essential? And how can we sustain progress over time?

## **BACKGROUND**

### **Understanding Ben**

Ben was one of the weakest students in a diverse third-grade classroom. He demonstrated low engagement, especially during maths lessons. Observations revealed a classroom dominated by traditional “ping-pong” discourse, where teacher questions prompted short student answers followed by feedback. This format limited deep thinking and active participation. Ben rarely volunteered answers and often appeared distracted or disengaged.

His teacher reported that Ben had limited number sense, could not clearly explain his mathematical reasoning, and lacked consistent strategies. During a pre-intervention interview, Ben relied on finger counting to solve even basic problems such as  $11 - 8$  or  $31 - 7$ . He struggled particularly with problems requiring regrouping (e.g.,  $50 - 28$ ), though he could solve addition problems with larger numbers, such as  $95 + 10$  or  $395 + 10$ .

### **Phase I: Small Group Learning and Early Observations**

Ben participated in a small remedial group focused on subtraction with decimals. During the first few sessions, he remained reserved, preferring to observe rather than participate actively. However, when tasks involved tangible money, Ben became visibly more engaged. He also responded well to direct prompts from the teacher, and in computer-based partner tasks, he became more vocal, though still hesitant to assert his answers in moments of disagreement.

### **A Turning Point: The Ice Cream Scenario**

In a pivotal session, the teacher posed a contextual task: *“An ice cream costs 7.70 shekels. The customer pays with 20 shekels. How much change will they receive?”* When asked to respond, Ben answered, “Thirteen shekels and thirty agorot,” which was incorrect. Instead of dismissing the answer, the teacher invited Ben to explain his thinking.

He began by subtracting 7 from 20 to get 13, then shifted strategies, suggesting that 0.30 should be added to 0.70 to complete a full shekel. He then stated that 100 agorot equals one shekel. With support from the teacher, he constructed an intermediate representation: 7.100. While not mathematically accurate, this scaffold allowed him to conceptually transition from 7.70 to 8.00. Eventually, he correctly determined the change was 12.30.

1. **Ben:** *You do twenty minus seven and get thirteen.*
2. **Teacher:** *Okay [writes]... First of all, in general, what do you think the correct operation is?*
3. **Ben:** *Add thirty agorot to seventy.*
4. **Teacher:** *Okay... I'll go with you.*
5. **Ben:** *Add thirty agorot to seventy agorot and you get one hundred agorot.*
6. **Teacher:** *Write seventy agorot plus thirty agorot?*
7. **Ben:** *(Nods)*
8. **Teacher:** *Okay. So how do I write it? Tell me exactly.*
9. **Ben:** *Seventy plus thirty.*
10. **Teacher:** *Seventy plus thirty... like that?*
11. **Ben:** *Yes.*
12. **Teacher:** *Yes? Okay.*
13. **Ben:** *Equals one hundred.*
14. **Teacher:** *One hundred what?*
15. **Ben:** *Agorot.*

The teacher's request to write the equation was new for Ben at this stage, as it was the first time she had asked this, in an attempt to move toward formal language. Ben gave an informal explanation using money terms (agorot), which until now was acceptable in the group. The teacher temporarily abandoned writing and decided to follow his lead, perhaps to start from a shared foundation. Ben began by subtracting the whole numbers (utterance 1), ignored the teacher's additional request for formal notation, and used addition to complete the amount to a whole shekel. He likely chose addition because, as noted in an earlier interview, he found subtraction with regrouping harder than addition. In utterance 9, Ben adapted to the teacher's formal writing request, but then she brought him back to contextual language (utterance 14) by asking, "One hundred what?"

In the next step, Ben used a previously learned grouping rule – that one shekel equals 100 agorot – and continued his calculation:

16. **Ben:** *Now we add it to 7.70, and it becomes seven point one hundred agorot. And then when you add that...*
17. **Teacher:** *Wait, wait. What do we do now? I'm writing everything you say. Tell me what to write.*
18. **Ben:** *Then, one hundred agorot is a shekel.*
19. **Teacher:** *Say that again?*
20. **Ben:** *We know that one hundred agorot is one shekel.*
21. **Teacher:** *One hundred agorot is one shekel.*
22. **Ben:** *Then we add it to the seven, and it becomes...*
23. **Teacher:** *You add the shekel to the seven? So make seven shekels...*
24. **Ben:** *Plus one.*
25. **Teacher:** *Plus one shekel. How much is that?*
26. **Ben:** *Eight.*
27. **Teacher:** *Eight shekels.*

This segment demonstrates a process of constructing understanding using the rule that one shekel equals 100 agorot, transitioning from 7 shekels and 100 agorot to 8 shekels. Ben used the rule as an anchor and constructed an intermediate representation (7.100) which served as a scaffold in converting to a new whole. Although 7.100 is not mathematically correct, it made sense to Ben – possibly because the decimal visually separates shekels and agorot (shekels on the left, agorot on the right). This self-created scaffold helped him transition from concrete to formal mathematical language.

This marked the first time Ben proposed an original strategy. His use of informal language and imagery grounded in familiar money concepts served as a bridge to more formal mathematical understanding. Importantly, the teacher validated his reasoning path, allowing him to make and correct errors within a safe, supportive environment.

### **Pedagogical Insight: Language, Form, and Timing**

This episode underscored the teacher’s pedagogical dilemma: how much should she emphasize formal mathematical language versus allowing intuitive, student-generated explanations? Her choice to follow Ben’s reasoning path – while gently guiding him toward formal representations – demonstrated an intentional and flexible teaching approach. Ben, in turn, alternated between resisting and adopting formalization, illustrating the fragile and transitional nature of his mathematical development.

### **Phase II: Regression and Reflection**

Despite his breakthrough, Ben did not consistently apply his new strategy. In a later task ( $50 - 6.40$ ), he attempted to convert 6.40 shekels to 640 agorot but failed to convert 50 shekels, causing confusion. He also reversed the order of subtraction. Recognizing the value in the attempt, the teacher used the mistake as a discussion point. A classmate, Yariv, explained why the subtraction was invalid, and the teacher prompted him to direct the explanation to Ben. This peer mediation helped Ben recognize and articulate his error.

The teacher’s facilitation of this moment illustrates a key practice: treating errors not as failures but as learning opportunities. Her repeated clarification questions and her use of peer explanations served as scaffolds, helping Ben reconstruct the conceptual foundations of the task.

### **Peer Learning and Strategy Adoption**

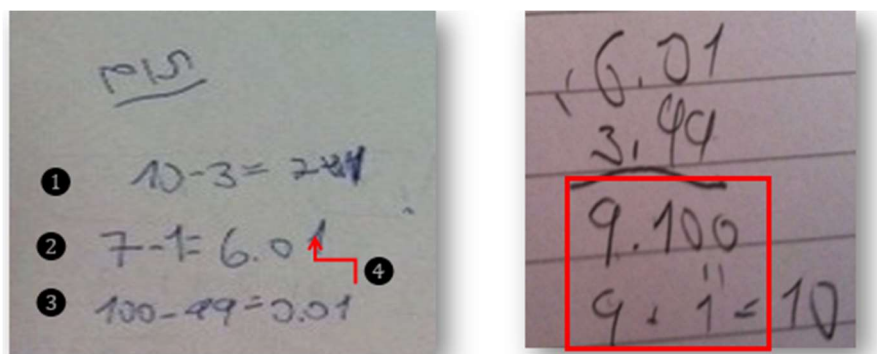
In the ensuing discussion, Shiri suggested a simpler method, subtracting the whole numbers first and then handling the decimals. This approach resonated with Ben, who adopted it in a subsequent task ( $27 - 4.80$ ). He successfully applied the method while maintaining elements of his original scaffolding – such as mentally completing agorot to form a shekel. The teacher acknowledged this transfer and highlighted the learning-from-peers dynamic: “We learn from one another.”

### **Phase III: Written Representation and Self-Monitoring**

Ben continued to struggle with written forms of subtraction, particularly when tasks lacked concrete contexts. In problems like  $50 - 25.20$ , he made procedural errors similar to those seen earlier. His strategies had not yet become robust enough for automatic application across settings.

Recognizing this, the teacher reintroduced subtraction tasks with smaller numbers and more guidance. In one task ( $20 - 4.80$ ), Ben used vertical subtraction. He handled the whole numbers well but mistakenly added rather than subtracted the decimal component. Again, Shiri corrected the mistake through peer explanation.

What stood out in the following lesson was Ben's initiative. In a task involving  $10 - 3.99$ , he not only solved the problem using scaffolding but also documented his full process in writing and independently verified his result. He converted 1 shekel into 100 agorot, subtracted agorot and shekels in turn, and checked his work using addition – a metacognitive behaviour the teacher had long encouraged.



There is a blend of formal and informal writing styles. The focus remains on conveying meaning. His constructed representation functions as an anchor, while the incorrect form of writing is retained.

### DISCUSSION AND BROADER IMPLICATIONS

Ben's case reveals several essential insights:

1. **Conceptual Anchors:** Tangible representations (money) provided an accessible entry point for abstract concepts. Ben repeatedly returned to money as a reliable frame of reference.
2. **Peer Mediation:** Opportunities to hear, evaluate, and adopt peer strategies enriched Ben's understanding and helped validate his thinking.
3. **Flexible Teaching:** The teacher balanced structure with freedom, allowing exploration while gradually guiding toward formal reasoning.
4. **Error as a Resource:** Mistakes were not stigmatized but embraced as stepping stones toward deeper understanding.
5. **Strategy Before Formality:** Ben's case raises questions about the appropriate sequencing of teaching objectives, highlighting that strategic thinking before fluent formal writing may benefit struggling learners.

### CONCLUSION: WHAT CAN WE LEARN FROM BEN?

Ben's mathematical development over a short time illustrates the potential of thoughtfully designed environments to support underachieving students. His journey was not linear but marked by recursive progress, occasional regression, and the steady building of confidence.

While this case reflects local transformation within a specific intervention, it raises important questions for future practice:

- How can we scaffold students toward generalizing strategies across contexts?
- Should instruction prioritize strategy development over formal correctness in early stages?
- How can peer collaboration be systematically integrated into mathematics instruction?

Ultimately, Ben's story demonstrates that even students who initially appear disengaged or weak can, under the right conditions, construct meaningful and lasting mathematical understanding.

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# The Wonders of Hidden Arithmetic

Moshe Stupel<sup>1</sup> & Duncan Samson<sup>2</sup>

<sup>1</sup>*Gordon Academic College & Givat Washington Academic College, Israel*

<sup>2</sup>*Victoria Girls' High School, Makhanda (Grahamstown)*

stupel@bezeqint.net dsamson@vghs.co.za

## INTRODUCTION

One of the great joys of mathematics is experiencing that *aha!* moment when a particular insight or sudden realisation sees the explanation of a problem fall effortlessly into place. In this short article we present an activity that has the potential to create just this effect.

Consider the grid of numbers shown in Figure 1. At first sight one might assume that the image represents a classic magic square, but closer scrutiny soon rules this out. There is nonetheless an interesting mathematical property hidden in the array of numbers.

18	10	12	7	14
22	14	16	11	18
19	11	13	8	15
17	9	11	6	13
21	13	15	10	17

**FIGURE 1:** A square array of numbers with a hidden mathematical property.

## DESCRIPTION OF THE ACTIVITY

Give each of your pupils a copy of the grid and let them individually carry out the following process:

- Choose any number in the grid and circle it. Now cross out all the other numbers in the same row and column as your chosen number.
- Now choose any of the remaining numbers (i.e. that haven't been crossed out) and circle it. Once again cross out all the other numbers in the same row and column as this new chosen number.
- Repeat this process until you have circled five numbers. All the numbers that haven't been circled should hopefully be crossed out at this point.
- Add up the five circled numbers and compare your answer with other members of the class.

If pupils have carried out the process correctly then they should all get the same total, namely 68, irrespective of which numbers they chose. This result should hopefully be rather intriguing and should generate some interesting discussion. Figure 2 shows one possible outcome for the process described.

18	10	12	7	14
22	14	16	11	18
19	11	13	8	15
17	9	11	6	13
21	13	15	10	17

FIGURE 2: One possible outcome giving a sum of 68.

Similar intriguing grids of numbers can be created for any  $n \times n$  square array. For interest, Figure 3 shows a  $6 \times 6$  array which gives a sum of 100 for the six numbers chosen using the same procedure previously described.

21	18	32	20	26	24
10	7	21	9	15	13
13	10	24	12	18	16
19	16	30	18	24	22
14	11	25	13	19	17
8	5	19	7	13	11

FIGURE 3: A  $6 \times 6$  square array with a similar hidden mathematical property.

The reader is encouraged to try the example in Figure 3 to check that they do indeed get a sum of 100 for their six chosen numbers. Some thought should then be given to how the grid, as well as that shown in Figure 1, was constructed to ensure the desired result.

### SCAFFOLDING THE *AHA!* MOMENT

Figuring out why the grid works, and how it was constructed, is not a simple task. However, one can readily scaffold the discovery process by showing pupils the grid illustrated in Figure 4 where additional numbers have been added on the outside of the grid. Ask pupils the following two questions:

- What is the sum of the numbers that have been added on the outside of the grid?
- How do the numbers inside the grid relate to the numbers on the outside of the grid?

The answers to these two questions should hopefully lead to the necessary insight for pupils to make sense of why the sum of the six chosen numbers will always be the same, and why it will always be 100 in this particular case.

	6	3	17	5	11	9	
21	18	32	20	26	24	15	
10	7	21	9	15	13	4	
13	10	24	12	18	16	7	
19	16	30	18	24	22	13	
14	11	25	13	19	17	8	
8	5	19	7	13	11	2	

**FIGURE 4:** An augmented array pointing to its construction.

The sum of the twelve numbers on the outside of the grid is 100, and each number inside the grid is the sum of the two outside numbers in its corresponding row and column. Since the selection process ensures that each row and each column only has one number selected from it, it naturally follows that irrespective of the six chosen numbers, they will always have the same total as the sum of the numbers on the outside of the grid. And there's the *aha!* moment.

### EXPLORING THE IDEA FURTHER

As an additional classroom exercise, pupils can now be encouraged to make up their own square array that has this constant sum property. Pupils first need to decide on the dimensions of their grid. It could be as small as a trivial  $2 \times 2$  grid or any bigger  $n \times n$  grid. Next, pupils need to decide what they want the constant sum to be. Once they have made this decision, they can then place the construction numbers on the outside of the grid. These can be any numbers, provided they add up to the desired constant sum. Pupils can now fill in the missing numbers inside the grid to create their final square array. The construction numbers can then be erased (or a clean grid drawn) so that it can be tested with their classmates.

One can also introduce a bit of algebraic notation to unpack the working of the grid in a more formal way. By way of example, consider the  $3 \times 3$  grid shown in Figure 5.

	<i>a</i>	<i>b</i>	<i>c</i>	
	$a + d$	$b + d$	$c + d$	<i>d</i>
	$a + e$	$b + e$	$c + e$	<i>e</i>
	$a + f$	$b + f$	$c + f$	<i>f</i>

FIGURE 5: Introducing algebraic notation.

Pupils should be able to satisfy themselves that if only one number is chosen from each row and column, then the sum of the three chosen numbers will always be  $a + b + c + d + e + f$ .

One could also introduce the idea of combinatorics and factorial notation. Since the selection process ensures that each row and each column only has one number selected from it, we can proceed as follows. The outside letter  $a$  must be paired with either  $d$ ,  $e$  or  $f$  (i.e. 3 possible pairings). For each of these pairings,  $b$  must now be paired with either of the two remaining letters from  $d$ ,  $e$  and  $f$  (i.e. 2 possible pairings). Finally,  $c$  must be paired with the only remaining letter (i.e. 1 possible pairing). The total number of possible pairings is thus  $3 \times 2 \times 1 = 6$ , or  $3!$ . Pupils can then be challenged to find all six possible scenarios and to cross-check that the sum of the three chosen numbers in each case is  $a + b + c + d + e + f$ .

### CONCLUDING COMMENTS

The purpose of this article was to share a simple scenario which, with suitable guidance, has the potential to create an *aha!* moment for pupils. The scenario can also be extended to explore other aspects of mathematics such as algebraic notation, the concept of generality, and combinations. It is hoped that this activity generates some interesting classroom discussion.

# Attached Regular Pentagons Produce Congruent Equilateral Triangles

**Michael de Villiers**

*RUMEUS, University of Stellenbosch*

*profmd1@mweb.co.za*

## INTRODUCTION

By definition, a problem is not the same as a routine exercise which simply requires the direct application and practice of specific theorems or skills. In the case of a problem, learners should not immediately know how to approach the question, nor what theorems or techniques might apply. Problems chosen by a teacher or textbook should be challenging but not beyond the ability of learners. As David Hilbert said in his famous address to the International Congress of Mathematicians in Paris in 1900: “*A mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts.*” Lastly, a good problem is one that has the potential to surprise learners and elicit their curiosity as to why the result is true (Movshovitz-Hadar, 1988).

The following surprising result was sent to me by Harry Wiggins<sup>1</sup> from the Department of Mathematics & Applied Mathematics at the University of Pretoria. The problem provides a novel yet accessible challenge for learners in Grade 9 to 12, and its beauty and novelty is most strikingly illustrated with dynamic geometry software.

## ATTACHED REGULAR PENTAGONS THEOREM

Given any regular pentagon  $ABCDE$ , attach another regular pentagon  $IFGHD$  to the first pentagon at vertex  $D$  as shown in Figure 1. Next, construct two equilateral triangles  $XAF$  and  $GBY$  as shown. Prove that these two equilateral triangles are congruent.

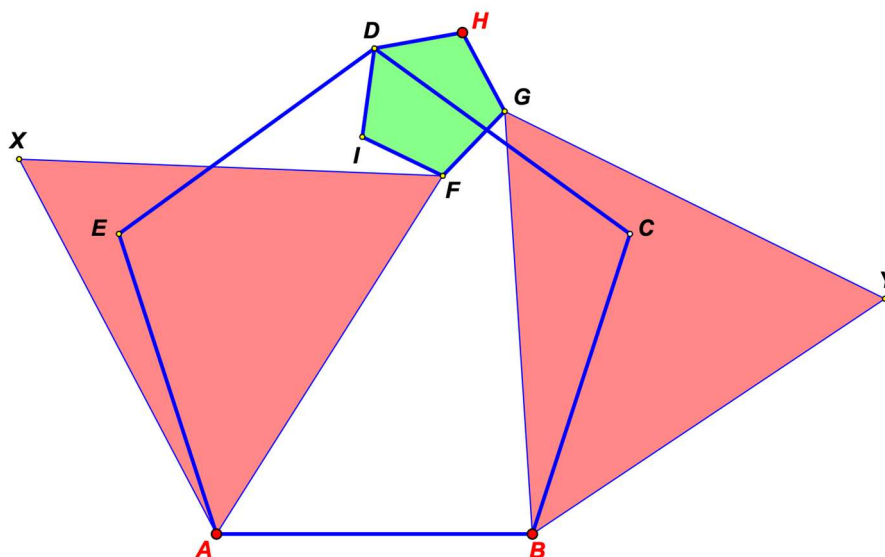


FIGURE 1

<sup>1</sup> Harry is an alumnus of the South African Mathematics Olympiad, finishing in the top ten in the country twice, as well as representing South Africa at the International Mathematical Olympiad (IMO) in 2001 and 2002. In 2011 he also participated in the Survivor Maldives show, and in 2019 won the South African National Scrabble Championships.

Before continuing, readers are encouraged to attempt to solve the problem for themselves. It may assist readers to construct their own dynamic geometry sketch, or to use an online version which is available at: <https://dynamicmathematicslearning.com/pentagon-congruent-equilateral-triangles-result.html>

### THE PROOF

In order to prove the two equilateral triangles congruent, all we need to do is prove that  $AF = BG$  (see Figure 2). This is most easily done by showing that triangles  $DAF$  and  $DBG$  are congruent. Firstly, observe that  $AD = BD$  and  $FD = GD$  since these are two pairs of equal diagonals (which immediately follows from the symmetry of each regular pentagon). Next, note<sup>2</sup> that  $\angle ADF = \angle ADB + \angle BDF$  and  $\angle BDG = \angle FDG + \angle BDF$ . But  $\angle ADB = \angle FDG$  since they are corresponding apex angles in two similar isosceles triangles (of the two regular pentagons). Therefore,  $\angle ADF = \angle BDG$ . Hence, triangles  $DAF$  and  $DBG$  are congruent (SAS) since we have shown that  $AD = BD$ ,  $FD = GD$  and  $\angle ADF = \angle BDG$ . From this we can conclude that  $AF = BG$ .

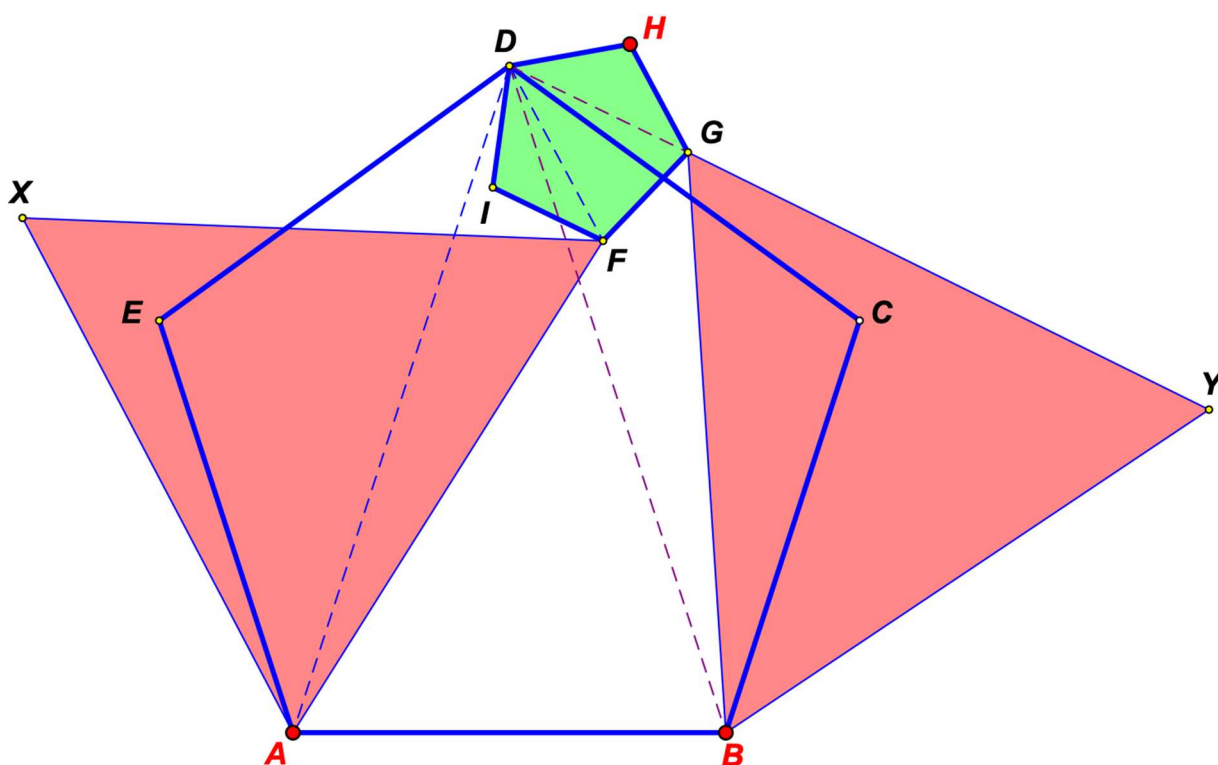


FIGURE 2

### OTHER VARIATIONS AND FURTHER GENERALIZATIONS

Now that we've seen from the proof why the result is true – which illustrates again the *explanatory* function of proof (De Villiers, 1990) – it is quite easy to formulate several other variations or further generalizations of the problem. Firstly, note that the use of equilateral triangles in the original problem was actually a red herring – we could have constructed any two other regular polygons on the respective sides  $AF$  and  $BG$ , such as squares, regular pentagons, regular hexagons, etc.

<sup>2</sup> For complete generality of the proof it is necessary here to assume that all angles are 'directed' angles, since we may need to add or subtract  $\angle BDF$  depending on the configuration. See De Villiers (2020) for further details.

Secondly, we could have chosen other regular polygons to attach at a common vertex. For example, as shown in Figure 3, attaching two equilateral triangles  $ABD$  and  $BEC$  at vertex  $B$  produces, in exactly the same way, two congruent regular polygons – squares in the case shown. Some other variations involving the attachment of regular hexagons and octagons are also available at the URL given earlier.

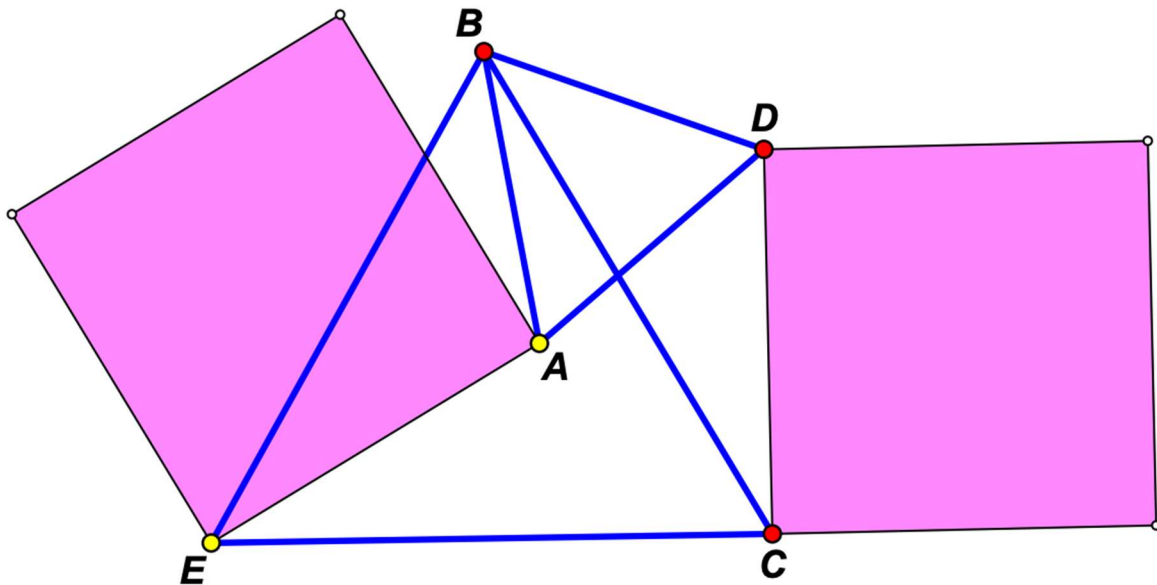


FIGURE 3

Furthermore, we need not necessarily attach regular polygons at *vertices* to get the same type of configuration. As shown in Figure 4, we could attach two squares at a common *midpoint* of two of the sides, and we would produce, in precisely the same way, two congruent regular polygons – equilateral triangles in this particular case. The dotted lines in Figure 4 show the two congruent triangles that are formed, as per Figure 2.

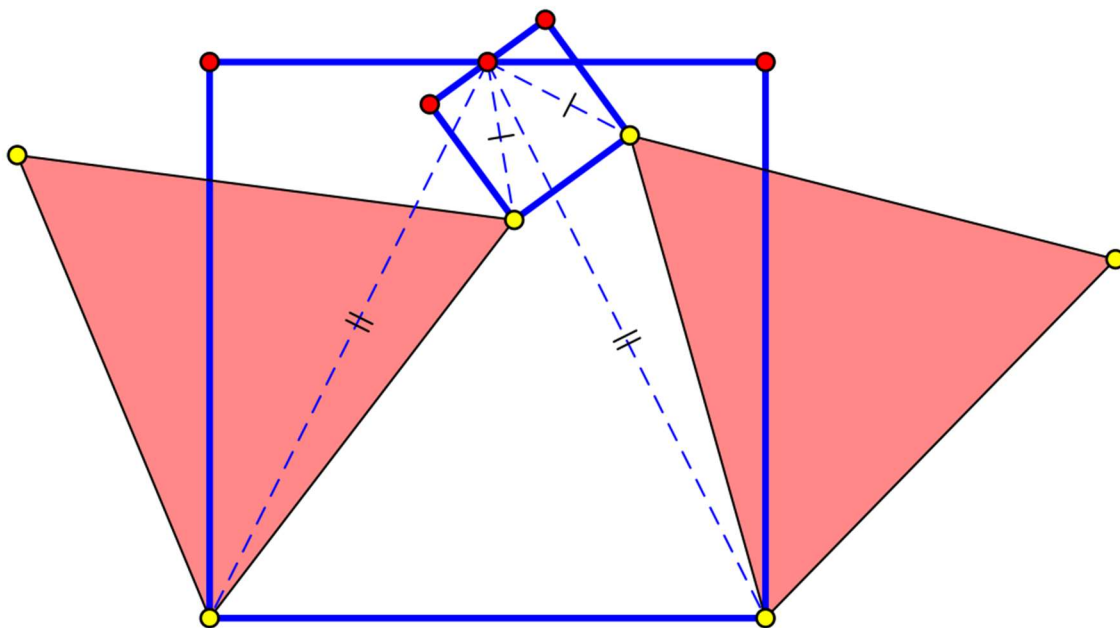


FIGURE 4

## APPLICATION TO A FAMOUS RESULT

The same principle and argument apply to an interesting property of the famous Fermat-Torricelli configuration. By continuously dragging the configuration in Figure 3 (without the squares), we can obtain the first configuration in Figure 5. And by further dragging we can obtain the second configuration shown on the right in Figure 5. This last configuration is equivalent to the Fermat-Torricelli configuration if we construct another equilateral triangle on side  $AC$  of triangle  $ABC$  as shown.

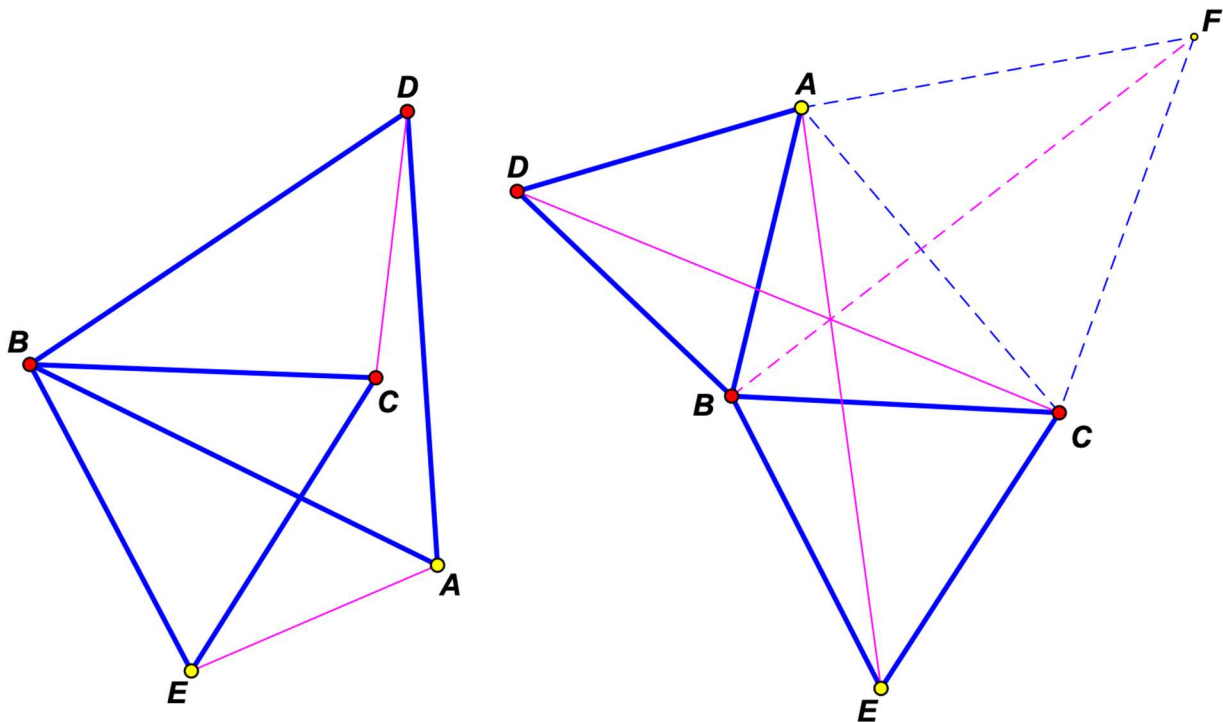


FIGURE 5

Let us now prove that line segments  $DC$ ,  $EA$  and  $FB$  are equal<sup>3</sup>. Exactly as before, we have at each vertex of the base triangle  $ABC$  a pair of equilateral triangles attached at a common vertex. But again we can immediately see, as before, that triangles  $ABE$  and  $DBC$  are congruent (SAS) since  $AB = DB$ ,  $BE = BC$  and  $\angle ABE = \angle DBC$ . Hence,  $AE = DC$ . In the same way we can prove that  $FB$  is equal to either one of the other line segments.

I still fondly remember in 1997 the excitement of two of my final year mathematics education students at the University of Durban-Westville. Working with a dynamic *Sketchpad*<sup>4</sup> sketch, they had noticed by themselves the congruency of triangles  $ABE$  and  $DBC$ , as well as triangles  $FBA$  and  $CDA$ , arriving at the logical conclusion that line segments  $DC$ ,  $EA$  and  $FB$  are equal.

A learning activity and guided proof of the Fermat-Torricelli theorem, with an accompanying *Sketchpad* sketch, is available as a free download in De Villiers (2012).

<sup>3</sup> As illustrated in Figure 5, the three line segments are also concurrent. A learning activity and guided proof of this concurrency, as well as the equality of the line segments, is available in the free download of De Villiers (2012).

<sup>4</sup> The dynamic geometry software, *Sketchpad*, is available as a free download at: <https://dynamicmathematicslearning.com/free-download-sketchpad.html>

**CONCLUDING REMARKS**

Since proving the original result, as well as the many possible variations and generalizations, only requires knowledge of triangle congruency, it is particularly suitable for learners in junior secondary school and higher. Not only does this result provide a novel challenge, it is also rich in numerous variations that a teacher may wish to encourage learners to explore further. Ideally, learners should use dynamic geometry to fully appreciate this beautiful result.

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# Transforming a Quadrilateral into a Triangle with an Equal Area

David Fraivert & Moshe Stupel<sup>1</sup>

<sup>1</sup>*Gordon Academic College & Givat Washington Academic College, Israel*

*davidfraivert@gmail.com stupel@bezeqint.net*

## INTRODUCTION

Consider an arbitrary quadrilateral  $ABCD$  as illustrated in Figure 1. The required task is to construct a triangle  $AED$  with an area equal to the given quadrilateral  $ABCD$ .

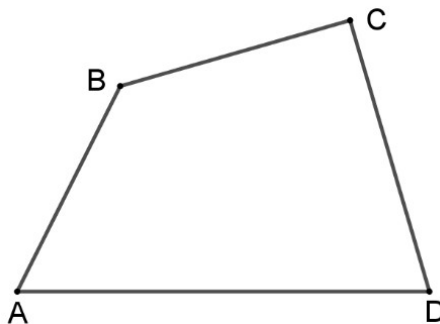


FIGURE 1: Arbitrary quadrilateral  $ABCD$ .

## THE CONSTRUCTION PROCESS

The construction follows the following steps which are illustrated in Figure 2.

1. Extend side  $AB$  of the quadrilateral upwards.
2. Draw in diagonal  $BD$ .
3. Construct  $CE$  parallel to  $DB$  with point  $E$  on the extension of  $AB$ .
4. Draw in line segment  $DE$  and label its point of intersection with  $BC$  as  $G$ .
5. Denote the areas of triangles  $BEG$ ,  $GEC$  and  $GCD$  as  $S_1$ ,  $S_2$  and  $S_3$  respectively.

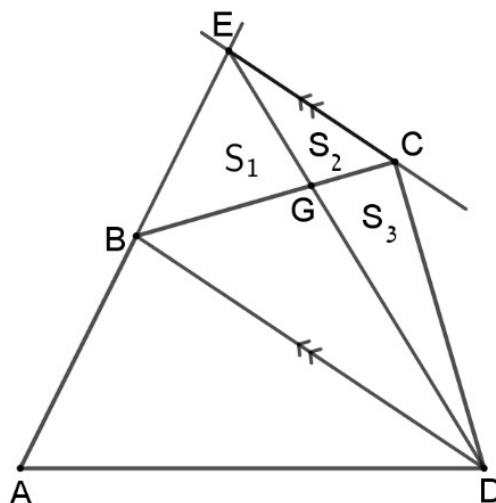


FIGURE 2: The construction process.

### THE PROOF

With reference to Figure 2, since  $EC \parallel BD$  it follows that  $S_1 + S_2 = S_2 + S_3$  (triangles with same base between parallel lines). Thus  $S_1 = S_3$  from which it follows that triangle  $AED$  has the same area as quadrilateral  $ABCD$ .

This result can be explored dynamically by using the following link:

<https://www.geogebra.org/m/wy3afner>

Each of the vertices of the quadrilateral  $ABCD$  can be dragged and the resultant changes in the area of the quadrilateral, as well as the area of the constructed triangle  $AED$ , are shown. The interested reader may also want to explore whether the result holds true for concave as opposed to convex quadrilaterals, or even crossed quadrilaterals.

### GENERALISING THE RESULT

The construction process described in Figure 2 also allows for the transformation of a pentagon into a quadrilateral with equal area, and thence to a triangle with equal area. In general, it is possible to transform an  $n$ -sided polygon into an  $(n - 1)$ -sided polygon with equal area. Through a process of iterative steps it is thus possible to transform any arbitrary polygon into a triangle with the same area. It is left to the interested reader to further explore this beautiful and rather pleasing piece of geometry.

### CONCLUDING COMMENTS

Although the construction described in this article has been known for hundreds of years and appears in numerous texts on geometric construction, it is nonetheless likely to be unfamiliar to many secondary school teachers. It is a simple construction protocol yet has the potential to form the basis of a meaningful classroom-based exploration or guided discovery.

# Pixel Play: Unlocking the Magic of Functions Through Images

**Christiaan Venter**

**Department of Mathematics & Applied Mathematics, University of the Free State**  
 venterc@ufs.ac.za

## INTRODUCTION

An alternative way to explore the concept of functions is by looking at images as mathematical objects. In this approach, a photograph or image is treated as a function where the input is the pixel's position, and the output is its brightness or colour. This can make abstract ideas more tangible but also serve to introduce students to real-world applications of functions in fields like computer graphics and image processing.

This article aims to provide accessible ideas and activities to bring this concept into the classroom. By using greyscale and colour images, students can explore the power of functions through hands-on experimentation. Whether manipulating brightness or unlocking a secret code through contrast enhancement via algebraic or power functions, these activities can make learning functions both engaging and fun. Resources related to this article can be found at [Function Junction](#).

## INTERPRETING IMAGES AS FUNCTIONS

A photograph or an image is a representation of a function if we take as the domain all the positions of the pixels in the image, and as codomain all the possible colours that a pixel might have. This relation between position and colour is a function as each position can only have one colour at a time. To denote a position, we use an ordered pair  $(x, y)$  where both  $x$  and  $y$  are positive integers. To denote a colour, we could use an ordered triple  $(r, g, b)$  where  $r$ ,  $g$  and  $b$  denote the shades of red, green and blue respectively and would be non-negative integers. An alternative colour option would be to use the CMYK colour model – cyan, magenta, yellow and black (key). For greyscale images we are concerned only with intensity, and we obtain different shades of grey by setting  $r = g = b$ . Typically, 256 shades of grey are used, from 0 representing pitch black to 255 representing snow white.

Our colour image,  $C$ , and our greyscale image,  $G$ , can then be denoted as the following functions:

$$C(x, y): Z^+ \times Z^+ \rightarrow Z_{\geq 0} \times Z_{\geq 0} \times Z_{\geq 0}$$

$$G(x, y): Z^+ \times Z^+ \rightarrow Z_{\geq 0},$$

where  $Z^+$  denotes the positive integers and  $Z_{\geq 0} = Z^+ \cup \{0\}$ .

## HEIGHT ABOVE VERSUS SHADE OF GREY

From the study of 1D graphs, a learner will be familiar with the height above/below the horizontal axis as representing the function value. In a greyscale image, it is the shade of grey that represents the function value. In the graph as well as the image, the learner should grasp that *height above* and *shade of grey* are both visual aids making it easier to think about the function and the changes in the function values taking place over different intervals.

The example in Figure 1 shows how *height above* translates to shades of grey. The floor function,  $f(x) = \lfloor x \rfloor$ , is represented on the left as a graph with the function values as heights above the horizontal axis. On the right the same function is represented as an image with each of the 10 different function values having been assigned a specific shade of grey.

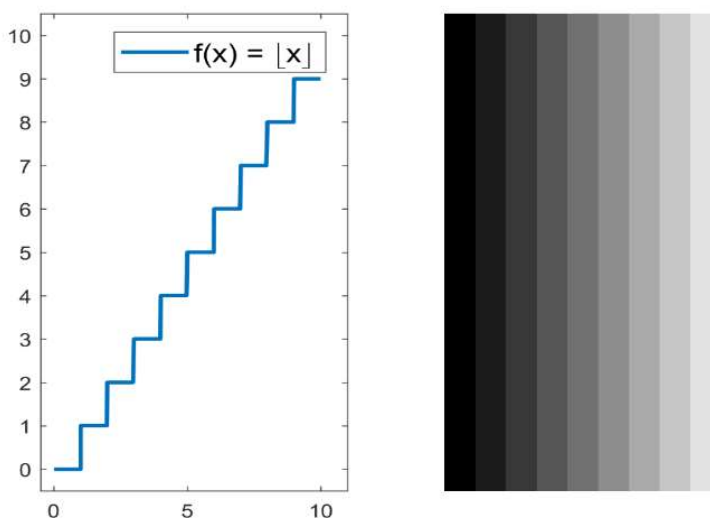


FIGURE 1:  $f(x) = \lfloor x \rfloor$  represented as a graph and an image.

We can easily extend this to a two-dimensional example as illustrated in Figure 2. In Figure 2(a), larger function values are plotted higher above the  $xy$ -plane as usual, but in addition the colour scheme shows that the smallest values are plotted as very dark shades of grey while the largest values as lighter shades of grey. We won't typically call Figure 2(a) an image. If we view this same function from above, i.e. from a viewpoint perpendicular to the  $xy$ -plane as shown in Figure 2(b), we could then regard this as an image. Viewed as an image, we are then dependent on the colours or shades of grey to help us make sense of the function variation and to look for patterns or to generally interpret the essence of the function.

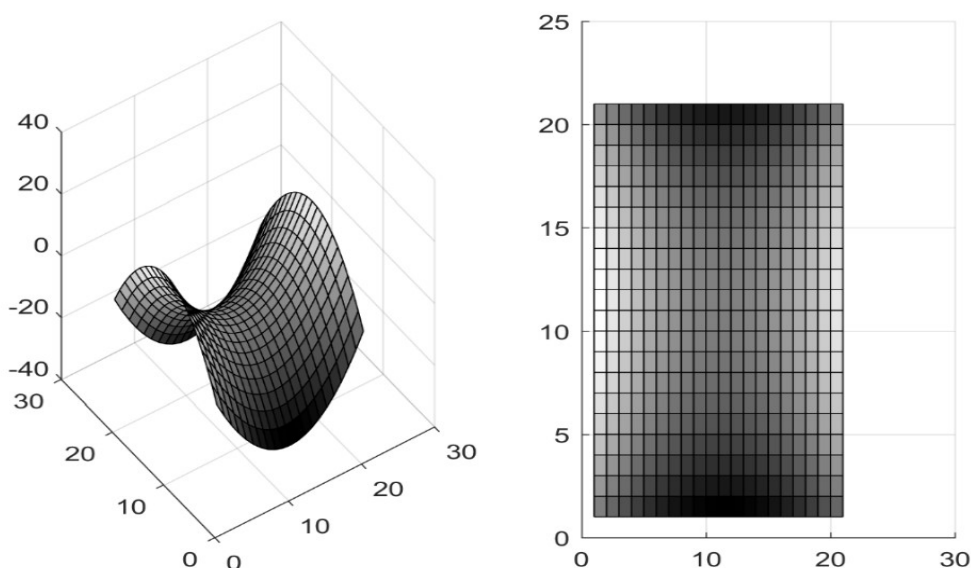


FIGURE 2:  $g(x, y) = x^2 - y^2$  viewed from the side and from above.

### An illustrative example

In Figure 3 we see the same function from the side and from above as in the previous example. Most viewers would easily identify the image of a cat when viewed from above.

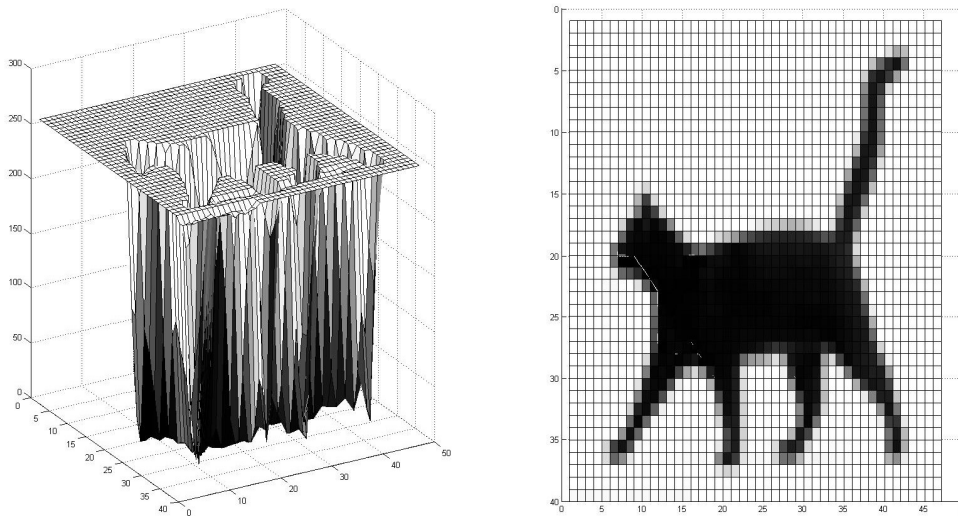


FIGURE 3: Interpreting a function as an image.

We now consider all the usual aspects of functions in light of the image representation:

#### **Domain**

A digital image,  $f(x, y)$ , is a function with both  $x$  and  $y$  being positive integers. Any combination of such an  $x$  and  $y$  will form an ordered pair which will denote the position of a particular pixel in the image. In our example in Figure 3, the image has exactly 40 rows, i.e.  $x \in [1, 40]$  and 47 columns, i.e.  $y \in [1, 47]$ . The domain is thus  $\{(x, y) | 1 \leq x \leq 40, 1 \leq y \leq 47, x \in \mathbb{Z}^+, y \in \mathbb{Z}^+\}$ .

#### **Codomain and Range**

As is typical for greyscale images, the values are chosen as integers in the interval  $[0, 255]$ . The codomain for our function is thus this same interval  $[0, 255]$ . The range will refer to all intensity values and thus all shades of grey *actually* present in the particular image.

#### **The Inverse**

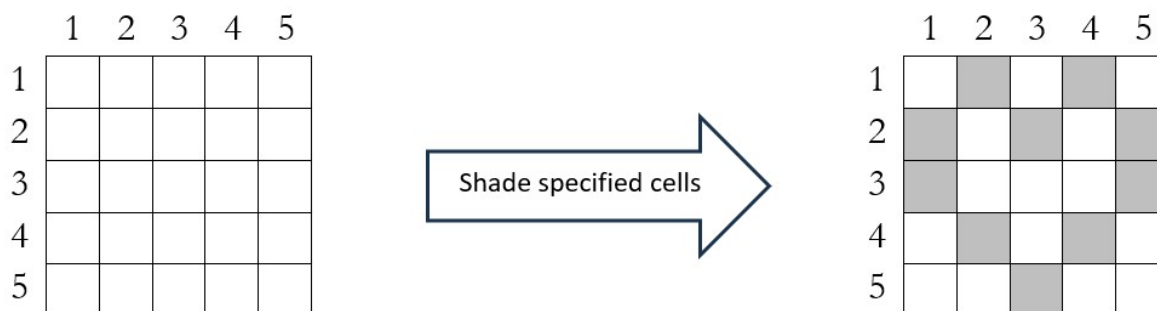
As for any function, the inverse will exist if the function is both injective (one-to-one) and surjective (onto). As photographs typically consist of millions of pixels, yet there are only 256 shades of grey to choose from, or  $256^3 = 16777216$  colours in the RGB colour space, it is indeed very rare for a photograph to be injective. Many pixels will have the same function value; thus, we typically deal with many-to-one functions. As an explorative exercise, learners could be asked to investigate what the equivalence of the so-called 'horizontal line rule' for testing the existence of an inverse would be in the case of images. Also, learners can think about what surjectivity would imply and if it would be likely or rare.

## CLASSROOM ACTIVITIES

### 1. MAKE YOUR OWN IMAGE

Creating small greyscale images can easily be achieved by using paper and a (grey) pencil. Learners can first create an empty table of some convenient size, for example 5 rows and 5 columns. Then they need to shade specified cells (pixels) in the table which should lead to a simple binary image, thus having only two shades of grey/function values. Binary images can be quite useful, for example bar codes and QR codes.

In practising the correct use of terminology, learners should be encouraged, for example, to state what adding new shades of grey would imply regarding the constructed function – thus expanding the codomain and range. Increasing the size of the table would be expanding the domain of the function.



**FIGURE 4:** Empty table to a binary image, representing a two-valued function.

As an alternative to using pencil on paper, Microsoft Excel or a different spreadsheet software could be used to fill in values for specified cells. Then conditional formatting can be used to assign a colour to cells with certain values, making the picture appear. For example, the smiley face in Figure 5 below.

	A	B	C	D	E	F	G	H	I	J	K	L
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	255	255	0	0	0	255	255	0	0	0
6	0	0	255	255	0	0	0	255	255	0	0	0
7	0	0	255	255	0	0	0	255	255	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	255	0	0	0	0	0	0
11	0	0	0	0	0	255	0	0	0	0	0	0
12	0	0	0	0	0	255	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0
16	0	255	0	0	0	0	0	0	0	255	0	0
17	0	255	0	0	0	0	0	0	0	255	0	0
18	0	0	255	0	0	0	0	0	255	0	0	0
19	0	0	0	255	0	0	0	255	0	0	0	0
20	0	0	0	0	255	255	255	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0
22												

**FIGURE 5:** A two-valued function visualised in Excel using conditional formatting.

Examples of questions to ask during this activity:

1. Write down (if a number is available) or describe (in the case of only shading) the function value of the pixel in row 3, column 2.
2. What is the value of  $f(16,2)$ ?
3. Write down or describe the domain, codomain, and range of the function.
4. Is your function one-to-one or many-to-one? Explain.

## 2. CAN YOU IDENTIFY THE INJECTIVE FUNCTIONS?

For each of the five images below (a – e), which represent injective functions? Explain.

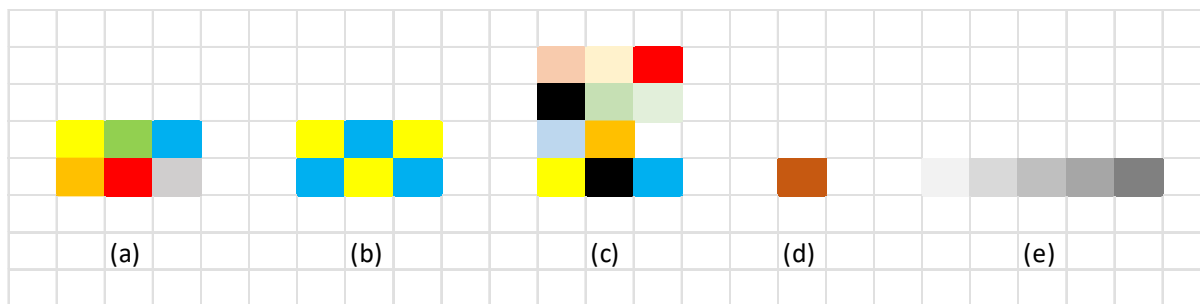


FIGURE 6: Demonstrating injectivity in visualised functions.

## 3. APPLYING FAMILIAR FUNCTIONS TO IMAGES

Let the image function be given by  $I(x, y)$ . Various functions that should be familiar to learners can now be applied to  $I(x, y)$  in order to realise different desired effects. This would constitute function composition. If our transformation function is for example  $f(r) = r^2$ , then we can obtain a transformed image,  $T(x, y) = f(I(x, y))$ . This is an example of a range of methods referred to as point processing (Gonzalez & Woods, 2017).

### (a) Contrast enhancement

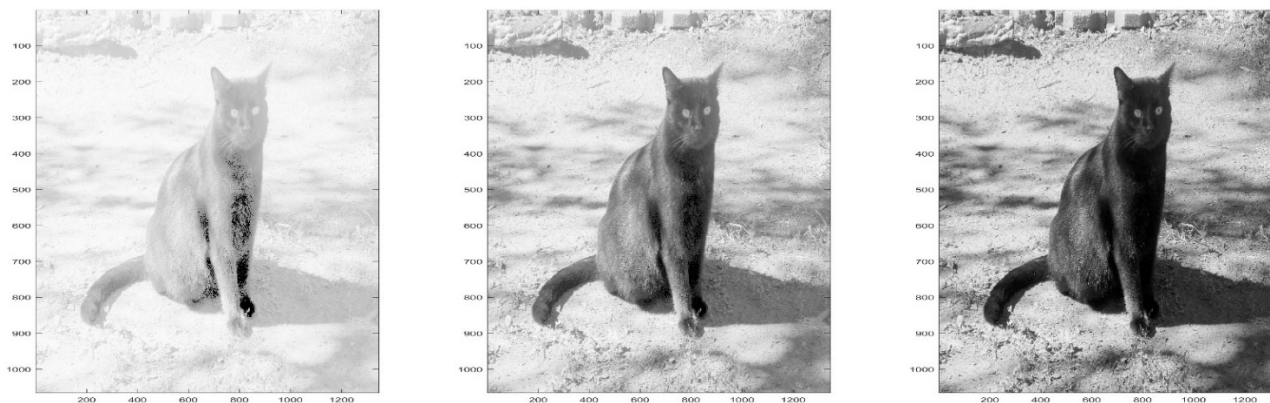


FIGURE 7: (a)  $I(x, y)$ , (b)  $T_1(x, y) = I(x, y)^3$ , (c)  $T_2(x, y) = I(x, y)^6$ .

In Figure 7(a) the detail in the image, especially in the background, is mostly invisible. The image has a washed-out appearance, which is a result of having low contrast. Low contrast means the differences in colour, or shades of grey in this case, of many adjacent pixels are too small for us to be able to notice. As a first attempt to improve the image, we apply the function  $f(r) = r^3$  to obtain the transformed image 7(b). All output values were raised to the power 3, which increases the average difference between intensities in the brighter parts of the image and thus the transformed image has increased contrast. A second attempt is shown in 7(c) which was created by applying  $f(r) = r^6$ . This function has the same effect as in 7(b), but the effect is exaggerated leading to a further increase in contrast. Much more detail is now visible in the background than before.

Why does this work? In Figure 8 we see how a relatively narrow input interval (yellow lines) in the lighter region (closer to 255) is stretched to a wider output interval (green lines). Thus, if this transformation is applied to the image, the contrast in the light regions will be stretched or increased which will assist in making details situated in those regions clearer or even visible for the first time.

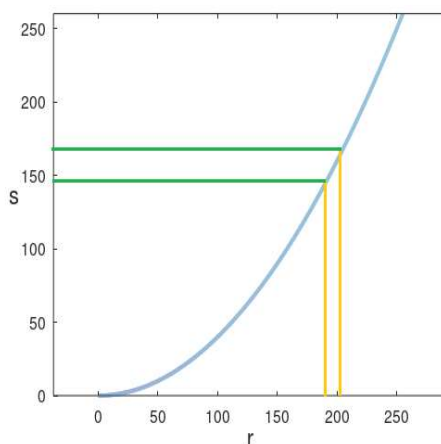


FIGURE 8: Illustration of  $s = cr^3$  with  $c$  a scaling factor to bring the values back to the interval  $[0,255]$ .

### (b) Contrast stretching

In this example (Figure 9) our original photo of the Triumphator calculating machine is quite under-exposed. Insufficient lighting was used when the photograph was taken. To enhance the image, another point-processing technique named *contrast stretching* was used, which typically makes use of a transformation in the form of a piece-wise linear function. In this particular case the function used was:

$$f(r) = \begin{cases} 3.4r, & r \in [0,35] \\ 0.7r + 95.3, & r \in [35,120] \\ -1.3r + 336.4, & r \in (120,255] \end{cases}$$

The original image is quite dark overall, and the detail we would like to see is situated in some of the darkest regions of the image. Therefore, our linear transformation acting on the interval containing the darkest shades of grey needs a slope much larger than one. This way we stretch the dark region's values until the differences become large enough for us to be able to see the detail emerge.

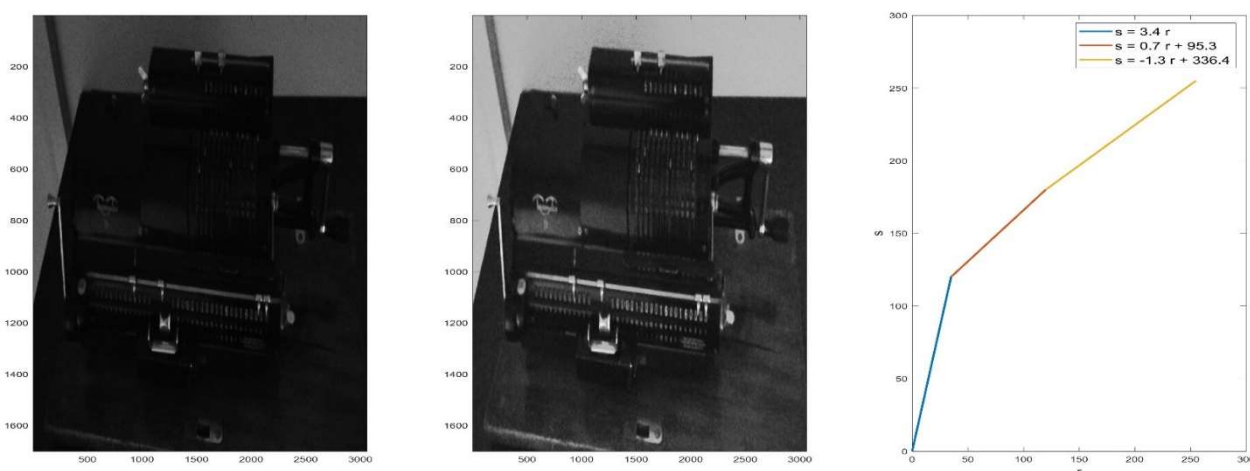


FIGURE 9: (a) Original image, (b) Transformed image, (c) Piece-wise linear transformation.



In Figure 12 we see a  $3 \times 9$  array of values, named  $x$ . Without loss of generality, let us assume our secret code is given by the values E2 to H2, thus [2 8 5 3]. To hide our code, we carry out the encrypting transformation  $y = 2x + 14$ . Someone knowing this transformation can then determine its inverse, apply it to the encrypted array, and then read off the secret code.

	A	B	C	D	E	F	G	H	I	J	K
1	Original values	x	7	1	4	7	2	5	1	3	2
2			5	5	2	8	5	3	4	3	6
3			3	6	7	7	4	3	5	6	1
4											
5	Encrypting	$y = 2x+14$	28	16	22	28	18	24	16	20	18
6			24	24	18	30	24	20	22	20	26
7			20	26	28	28	22	20	24	26	16
8											
9	Decrypting	$x = (y-14)/2$	7	1	4	7	2	5	1	3	2
10			5	5	2	8	5	3	4	3	6
11			3	6	7	7	4	3	5	6	1
12											

**FIGURE 12:** Encrypting and decrypting using a linear function and its inverse.

### CONCLUDING COMMENTS

As humans have such a keen ability to recognize the whole of an image, working with images will allow the early establishment of thinking about functions as objects (Venter, 2020) that can be manipulated and dealt with in all kinds of ways. Using images to explore functions bridges the gap between visual intuition and formal mathematical reasoning. This approach engages students through hands-on activities, deepens their understanding of concepts like domain and transformations, and demonstrates real-world applications of mathematics.

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## 'Apartheid' Definitions of Trapezia Must Fall!

**Michael de Villiers**

**RUMEUS, University of Stellenbosch**

*profmd1@mweb.co.za*

While apartheid in South Africa was finally abolished in 1994, it is sad that we often still seem to find 'apartheid' (exclusive) definitions of trapezia in school textbooks and online websites, and even being loosely used by teachers and learners in the classroom. By way of example, I recently came across the following definition of a trapezium in some South African online materials: "A trapezium is a quadrilateral with one pair of opposite sides parallel." The first problem with this definition is that it is vague and ambiguous. Do the authors mean *exactly* one pair of opposite sides parallel, or do they mean *at least* one pair of opposite sides parallel? The first interpretation of the definition would imply the exclusion of parallelograms as a special case, and it would then be an *exclusive* ('apartheid') definition. The second interpretation, however, is *inclusive* and allows us to consider a parallelogram as a special case of a trapezium.

Further down in the same online materials, however, a classification of quadrilaterals is given (as shown in Figure 1) that seems to suggest that they intended an exclusive definition for the trapezium (as well as the kite), since the classification shows no direct link between a trapezium and a parallelogram (nor between a kite and a rhombus).

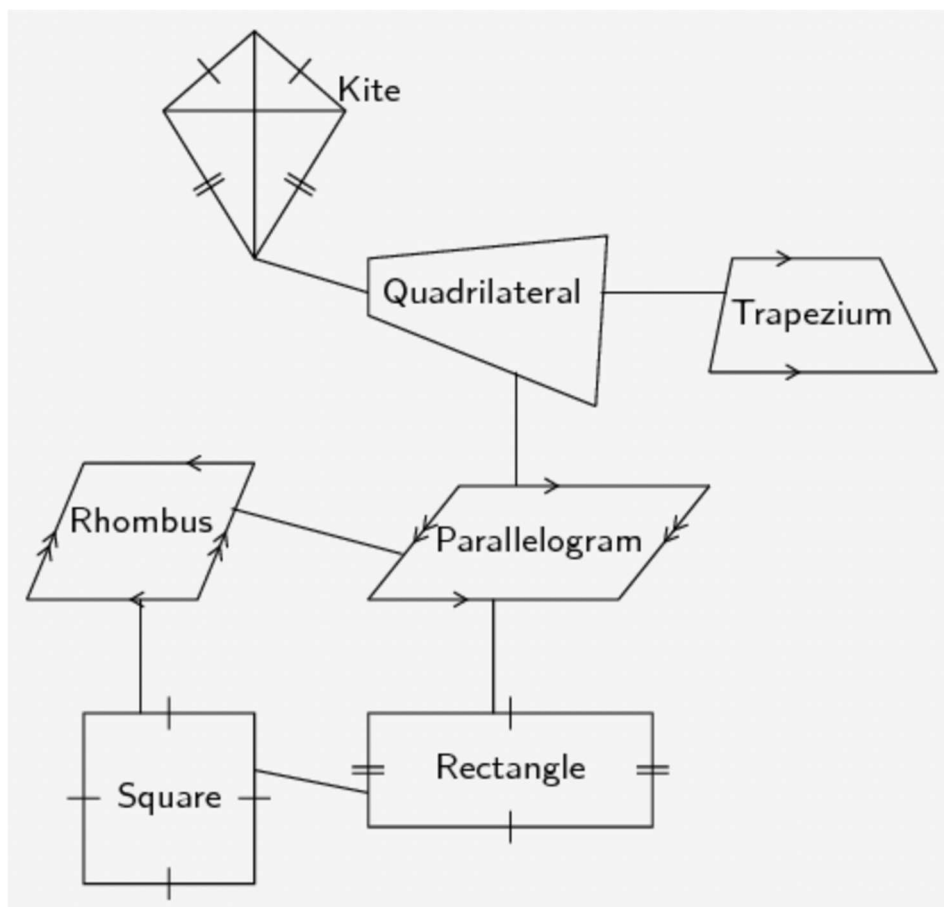


FIGURE 1: A family tree.

Two other South African websites explicitly use exclusive definitions to respectively define a trapezium as follows: “A trapezium is a quadrilateral. It has one pair of parallel sides – the other two sides are not parallel” and “a quadrilateral with only one pair of opposite sides parallel.”

While Euclid in 300 BC himself chose exclusive definitions for the quadrilaterals as can be seen in Joyce (1996) and De Villiers (2011), the mathematical community nowadays generally prefers to use inclusive, hierarchical definitions for the quadrilaterals for the following reasons (compare De Villiers, 1994):

- 1) it leads to more economical definitions of concepts and formulations of theorems
- 2) it simplifies the deductive systematization and derivation of the properties of more special concepts
- 3) it often provides a useful conceptual schema during problem solving
- 4) it sometimes suggests alternative definitions and new propositions
- 5) it provides a useful global perspective

It unfortunately took several centuries (gradually from about the 1600s onwards) for mathematicians to start realizing that it was more beneficial to define quadrilaterals in an inclusive, hierarchical way. Without going into too much detail, one of the major advantages of inclusive, hierarchical definitions (and the corresponding hierarchical classification) is that all the theorems one has proved for a quadrilateral then automatically apply to any of its special cases. For example, having proved that a parallelogram has diagonals that bisect each other, it is not necessary to prove it for a rectangle, rhombus or square. But using an exclusive definition for a parallelogram, one would mathematically be required to prove separately for a parallelogram, a rectangle, a rhombus and a square, in each case, that their diagonals bisect each other – so four proofs are needed instead of just one! Compared to an inclusive hierarchy of quadrilaterals, using exclusive definitions becomes ‘deductively uneconomical’ (De Villiers, 1994).

Similarly, for an inclusive hierarchical definition of a trapezium: 1) if one has proven that triangles  $ABE$  and  $DCE$  for the trapezium shown in Figure 2 have equal areas, or 2) that the centres of squares constructed on its parallel sides are collinear with the intersection of its diagonals as shown, then there is no need to prove these again for a parallelogram, since they would automatically apply. A dynamic sketch illustrating this latter result for a trapezium, as well as three others, is available at:

<https://dynamicmathematicslearning.com/trapezoid.html>

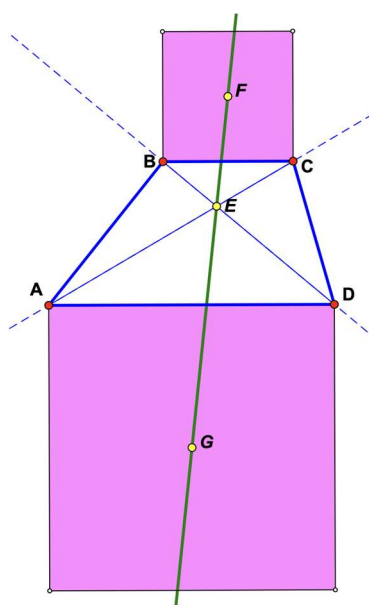


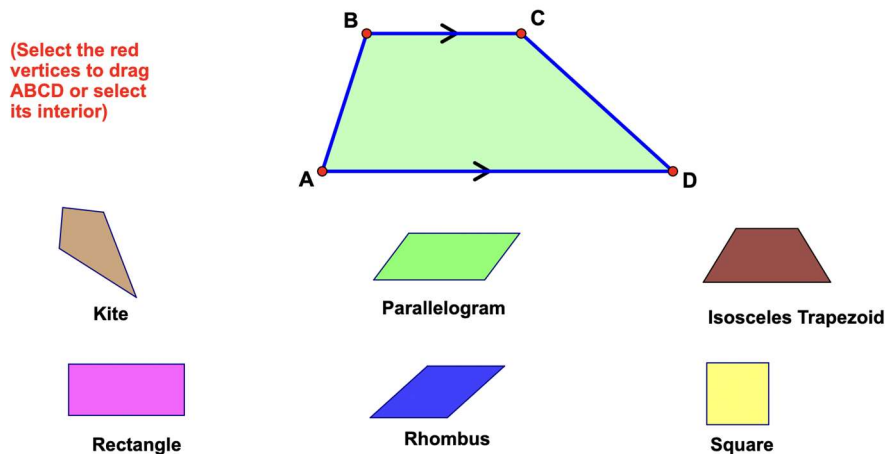
FIGURE 2: Trapezium theorem.

Perhaps the most compelling reason for defining a trapezium in an inclusive, hierarchical way is that if you construct two parallel segments  $AD$  and  $BC$  with dynamic geometry software, and connect the end points to obtain a trapezium  $ABCD$  as shown in Figure 3, then that shape can be dynamically dragged into a parallelogram, rectangle, rhombus or square. It thus does not make sense in a dynamic geometry environment to choose an exclusive definition for a trapezium. A suggested learning activity for young learners involving such dynamic dragging is available online at:

<https://dynamicmathematicslearning.com/trapezium.html>

**A trapezoid (trapezium) is a quadrilateral that looks like  $ABCD$ , but can have different shapes**

1. Which of the figures below  $ABCD$  are ('special') trapezoids?
2. Try dragging the light green trapezoid  $ABCD$  into these shapes.
3. Specifically check by trying to fit trapezoid  $ABCD$  over each of the figures below.



**FIGURE 3:** Suggested dynamically visual learning activity.

Using technology in this way, a primary school teacher can assist learners to move away from a static, fixed or rigid concept image of a trapezium, and instead develop a more robust, dynamically visual concept image that allows special cases. While dynamic activities such as this should not replace traditional geometric manipulatives such as cardboard, paper or plastic representations of various quadrilaterals, such dynamic transformations of one quadrilateral into another are essential for developing a sound conceptual framework before dealing with their formal definitions. With the concrete representations and models for quadrilaterals that are normally used in the primary or elementary school, certain static 'visual prototypes' unfortunately become fixed in the minds of learners, and eventually inhibit or prevent the development of a dynamic mental image of quadrilaterals that can transform into one another, and which would allow them to see inclusive relationships (Fujita, 2012).

In addition to initially providing young learners with similar dynamically visual experiences for the various quadrilaterals (as suggested in Figure 2 for a trapezium), learners will of course need many other carefully planned conceptual experiences. Following Van Hiele (1973), they will next need to extensively explore and learn the properties of each quadrilateral in detail, and finally they will also need to engage in constructing each quadrilateral in various different ways before they are conceptually ready to fully understand and appreciate their formal definitions (Smith, 1940; Govender & De Villiers, 2003).

Furthermore, simply providing learners with inclusive, hierarchical definitions alone will not automatically solve the problem. Unless the prerequisite conceptual framework has already been adequately formed, students may just memorize and regurgitate the given definitions for examination purposes, but their own, personal concept images of the various quadrilaterals are likely to remain static, and will invariably exclude special cases. See for example the short interview with a Grade 9 learner about his own preferred definition for a parallelogram as reported in De Villiers (1994).

While nowadays inclusive, hierarchical definitions are usually preferred whenever possible in regard to quadrilaterals, it is sometimes necessary and useful to classify and define quadrilaterals separately in partitions. For example, it is sensible and meaningful to partition the general concept of a quadrilateral into three disjunct sets, namely, convex, concave and crossed quadrilaterals.

Lastly, it should be mentioned that South Africa is not the only country where ‘apartheid’ definitions, specifically for a trapezium, still persist in some textbooks and online materials. While in Canada and most European countries an inclusive, hierarchical definition for a trapezium is generally preferred, in some countries like the USA and Israel, an exclusive definition for a trapezium is by far still in the majority. For example, Usiskin et al. (2008, p. 27) found in a survey of 80 textbooks in the USA that 76 used exclusive definitions for a trapezium (trapezoid) and only four used inclusive, hierarchical definitions.

Nonetheless, Usiskin et al. (2008, p. 32) write optimistically in their concluding comments of that section as follows: “The preponderance of advantages to the inclusive definition of trapezoid has caused all the articles we could find on the subject, and most college-level geometry books, to favor the inclusive definition. The inclusive definition is also the virtual unanimous choice of geometers and other mathematicians, judging from opinions expressed in the chat-room records of the Math Forum and in essays listed on the internet (Whitely 2002 and Math Forum, Trapezoid definition discussion). If definitions evolve in the 21st century as they have in prior centuries, perhaps by the end of the century the inclusive definition of trapezoid will be the one used by a majority of authors.”

As we are already 25 years into this new century, let’s hope that this will happen around the world in due course – probably not in my lifetime, but perhaps in the lifetime of some of our readers here? Hopefully in some small way this little article will also help contribute towards the achievement of this educational goal, and that eventually ‘apartheid’ definitions of trapezia will fall!

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## Partial Sum of a Geometric Series with Ratio $1/(r + 1)$

**Alejandro L. Salguero & Ángel Plaza**

*University of Las Palmas de Gran Canaria, Spain*

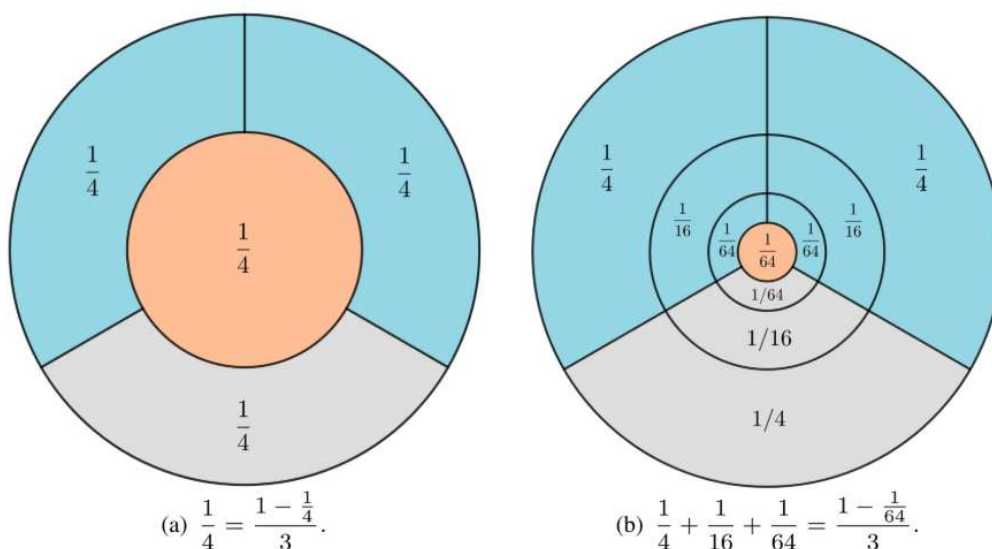
*alejandro.lopez123@alu.ulpgc.es angel.plaza@ulpgc.es*

This article provides a simple ‘proof without words’ for the partial sum of a geometric series with common ratio  $1/(r + 1)$ .

**Theorem:** Let  $r \in \mathbb{N}$ . Then, for  $n \in \mathbb{N}$ :

$$\sum_{k=1}^n \left(\frac{1}{r+1}\right)^k = \frac{1 - \left(\frac{1}{r+1}\right)^n}{r}$$

**Proof:** A proof without words is illustrated below for the case  $r = 3$ . Figures (a) and (b) correspond respectively to  $n = 1$  and  $n = 3$ .



**Corollary:** Let  $r \in \mathbb{N}$ , then:

$$\sum_{k=1}^{\infty} \left(\frac{1}{r+1}\right)^k = \frac{1}{r}$$

**Proof:** Since  $\left(\frac{1}{r+1}\right)^n \rightarrow 0$  as  $n \rightarrow \infty$ , the convergence and the sum of the series follows. For the case where  $r = 3$ , we have:

$$\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{3}$$

A similar proof without words can be found in Tanton (2008).

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## Book Review

### **The Secret Lives of Numbers: A Global History of Mathematics & Its Unsung Trailblazers**

**Kate Kitagawa & Timothy Revell**

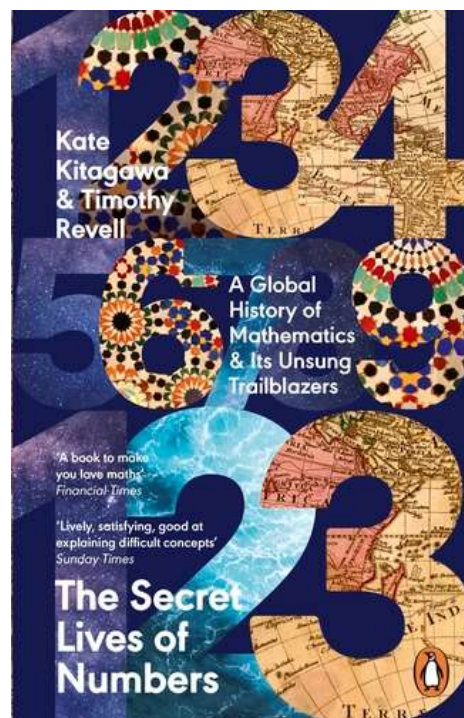
**Penguin Books, 2024**

**ISBN: 9780241994351 (softcover, pp. 320)**

***Reviewed by Gregory Whittaker***

Kate Kitagawa's journey through mathematics is marked by a unique blend of storytelling, global curiosity, and an inclusive perspective on history. Her book, *The Secret Lives of Numbers*, co-authored with Timothy Revell, exemplifies her approach to mathematics as an ever-evolving, interconnected story that transcends borders, gender, and time. This work explores mathematics from early tally marks to the complexity of quantum physics, celebrating unsung figures whose contributions have been overlooked due to historical biases. The book's focus on global perspectives challenges traditional Eurocentric narratives, showcasing mathematics as a collaborative "international team sport".

Kate is regarded as one of the world's leading experts on the history of mathematics. She is currently the Director of the Space Education Office at the Japan Aerospace Exploration Agency and was kind enough to answer a few questions that I posed to her about her research and her book.



**Greg:** You have conducted research in many countries such as the United States, United Kingdom and Germany. Please tell us about some of the research you have conducted in South Africa and when that took place.

**Kate:** I went to South Africa for the first time in 2018. I visited Johannesburg with mathematicians who gave summer lectures at the University of the Witwatersrand. It was a short visit, but my former boss at the Ministry of Foreign Affairs was then Ambassador of Japan to South Africa. I learned that the Embassy of Japan was helping the then Centre for Japanese Studies at the University of Pretoria (which has now merged into another centre) and celebrating the centenary of the Japanese Consulate in Cape Town (established in 1918). I went back to South Africa that year to take part in the celebration and gave lectures in Johannesburg, Pretoria, and Cape Town.

In Pretoria, I met a mathematician who invited me to do more research there and I entered the university's Visiting Professor Programme in 2019. The host faculty, Professor Eder Kikianty and I wrote a paper "A History of Mathematics in South Africa: Modern Milestones" published in *The Mathematical Intelligencer*.

Greg: What was your motivation behind writing *The Secret Lives of Numbers*?

Kate: The motivation behind the book is to broaden the historical narrative of mathematics by uncovering and celebrating the diverse contributions of cultures and individuals often overlooked in mainstream accounts. The book seeks to challenge the Eurocentric lens that frequently dominates mathematical history, emphasizing that mathematical innovation is neither linear nor exclusive to Western, white male figures. Instead, contributions from cultures in China, India, the Middle East, and elsewhere are highlighted, tracing important discoveries like calculus to earlier work in 14th-century India.

I want readers to see mathematics not only as a set of abstract principles but as a deeply human endeavour, influenced by social, political, and cultural contexts across continents and eras. Ultimately, my goal is to initiate a conversation and inspire further explorations into this inclusive and interconnected history of mathematics.

Greg: I would like to focus on Chapter 15 of your book entitled “Number-crunching” as that has close ties to South Africa. As described by G. H. Hardy, the Skewes Number, found by South African mathematician Stanley Skewes in the 1930s, was “the largest number which has ever served any definite purpose in mathematics”.

Background for readers:

- Let  $\pi(x)$  denote the number of primes less than  $x$
- Let  $\text{Li}(x)$  denote the logarithmic integral:  $\int_2^x \frac{dx}{\log x}$

The prime number theorem states that  $\lim_{x \rightarrow \infty} \frac{\pi(x)}{\text{Li}(x)} = 1$ . Numerical evidence tends to suggest that  $\pi(x) - \text{Li}(x) < 0$  as shown in Table 1:

Table 1: Comparison of  $\pi(x)$  and  $\text{Li}(x)$

$x$	Number of primes less than $x = \pi(x)$	$\text{Li}(x)$
1,000	168	178
10,000	1,229	1,246
100,000	9,592	9,630
1,000,000	78,498	78,628
10,000,000	664,579	664,918
100,000,000	5,761,455	5,762,209
1,000,000,000	50,847,534	50,849,235
10,000,000,000	455,052,511	455,055,614

Littlewood proved however that the number  $P(x) = \pi(x) - \text{Li}(x)$  actually changes sign infinitely often as  $x$  increases to infinity. Skewes found that  $\pi(x) - \text{Li}(x)$  changes sign for the first time at the number:

$$10^{10^{10^{34}}}$$

Could you provide some background as to who Stanley Skewes was, his training and how he happened upon this number?

Kate: Skewes graduated with a degree in civil engineering from the University of Cape Town and later pursued mathematics at King's College, Cambridge. His transition from engineering to mathematics was challenging, as the depth of mathematical knowledge required at Cambridge forced him to essentially restart his studies from undergraduate level. He eventually earned a PhD under the supervision of John Edensor Littlewood, a renowned mathematician. After his studies in Cambridge, Skewes returned to South Africa and took a teaching position at the University of Cape Town. He was known as an inspiring educator.

Skewes' Number emerged from his work on prime numbers and the distribution of primes, particularly in relation to the Riemann Hypothesis. He made use of Littlewood's theorem, which involved complex analyses of the behaviour of the prime-counting function and its relationship to the logarithmic integral function. To reach his results, Skewes assumed the truth of the Riemann Hypothesis, though some found this reliance controversial. His results showed that there existed a number (Skewes' Number) where  $\pi(x)$  would exceed  $\text{Li}(x)$ , contrary to the previous understanding based on smaller computations. The exact number was extraordinarily large, making it a standout example in mathematical history.

Skewes' work gained attention for its theoretical implications and the use of innovative methods to address longstanding questions in mathematics, even if some aspects relied on unproven hypotheses.

Greg: Skewes went on to inspire many South African students, including Dona Strauss. Could you give an overview of her career as she will probably be little known to many people?

Kate: Dona Strauss was one of Stanley Skewes' notable students at the University of Cape Town and went on to have an illustrious career in mathematics, making significant contributions to her field. Strauss entered the University of Cape Town at just 15 years old. By 18, she had completed a Bachelor of Science degree, and by 19, she had earned a Master of Science degree. Strauss credited Stanley Skewes with preparing her well for advanced studies, including her transition to Cambridge University. She was particularly influenced by Skewes' teaching and his focus on challenging mathematical ideas.

After completing her MSc, she joined the PhD program at Cambridge, an environment where women mathematicians were still relatively rare. In her PhD thesis, Strauss pioneered "pointless topology", a novel approach that eliminated reliance on specific points, offering a broader conceptual framework in topology.

Strauss had a prolific career, co-authoring over 200 papers and books with many collaborators across pure and applied mathematics. Her work extended into applications in computer science, biology, and theoretical physics. Her contributions were widely recognized. In 2009, Cambridge University hosted a conference in her honour on her 75th birthday, celebrating her profound impact on mathematics.

Greg: Alan Turing is synonymous with the crucial role he played in code breaking during World War 2. You uncover an extremely interesting story between Skewes and Turing in your book. Please could you share that with us?

Kate: The story between Alan Turing and Stanley Skewes is a fascinating tale of mathematical camaraderie and shared interest in number theory, especially concerning the Riemann Hypothesis. Both mathematicians were deeply interested in the Riemann Hypothesis and its implications for prime number theory. Their discussions often centred around the computational challenges and theoretical aspects of the hypothesis.

Turing and Skewes co-authored an unpublished manuscript, which reportedly included both their names. Skewes later suggested that this work could be published posthumously under Turing's name as a tribute.

The relationship between Turing and Skewes remains a tantalizing “what if” in mathematical history. Their collaboration hinted at greater discoveries that might have been realized had they worked more closely together.

Greg: Finally, those in the mathematical sciences will find a lot of motivation and inspiration from reading your book. Do you have any advice for young students in the mathematical sciences?

Kate: As we conclude this book with hope for the future of mathematics, I believe that the next generation will bring fresh perspectives and write new chapters in the history of this ever-evolving field. Our book aims to inspire readers to see the infinite potential for mathematical innovation. Good luck! Dream big!



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# Suggestions to writers

## What is this journal for?

*Learning and Teaching Mathematics* is a journal of the Association for Mathematics Education of South Africa (AMESA). This journal aims to provide a medium for stimulating and challenging ideas, offering innovation and practice in all aspects of mathematics teaching and learning. It seeks to inform, enlighten, stimulate, correct, entertain and encourage. Its emphasis is on addressing the challenges that arise in the learning and teaching of mathematics at all levels of education. It presents articles that describe or discuss mathematics teaching and learning from the perspective of a practitioner.

## What type of submissions are we calling for?

The types of articles considered for publication in *Learning and Teaching Mathematics* are:

- *Ideas for teaching and learning*: articles in this section report on classroom activities and good ideas for teaching various mathematics topics. This includes worksheets, activities, investigations etc.
- *Letters to the editors*: discussion pieces that raise important issues on the teaching and learning of mathematics and current curriculum innovations. Views and news on current initiatives.
- *Kids say and do the darndest things*: personal anecdotes of something mathematical that has happened in a classroom.
- *Window on a Child's Mind*: description of a classroom event that you want the Journal to respond to.
- *A day in the life of ...* includes stories about a head of department, a maths teacher, an NGO worker etc.; it could also be an account of a visit to another mathematics classroom... another school... another country...
- *Reviews*: reviews of maths books, school mathematics textbooks, videos and movies, resources including apparatus and technology etc.
- *Webviews*: reviews of mathematics education related websites.
- *Help wanted* is a question and answer column: teachers can send their questions on teaching specific topics or aspects to this column for fellow colleagues in the AMESA community to respond to.

## What are the technical requirements for the submission of articles?

Articles should not exceed 3 000 words and must be written in English. Articles as short as 300 words are also accepted and of course many of our categories such as “Question and Answers”, “Kids say and do the darndest things”, “Letters to the editors” and so forth can be even shorter. Articles should include the title, author's name, institution and full postal address, email and contact telephone numbers of the author.

Send your articles by e-mail (in a Word compatible format) to [LTM@amesa.org.za](mailto:LTM@amesa.org.za).

