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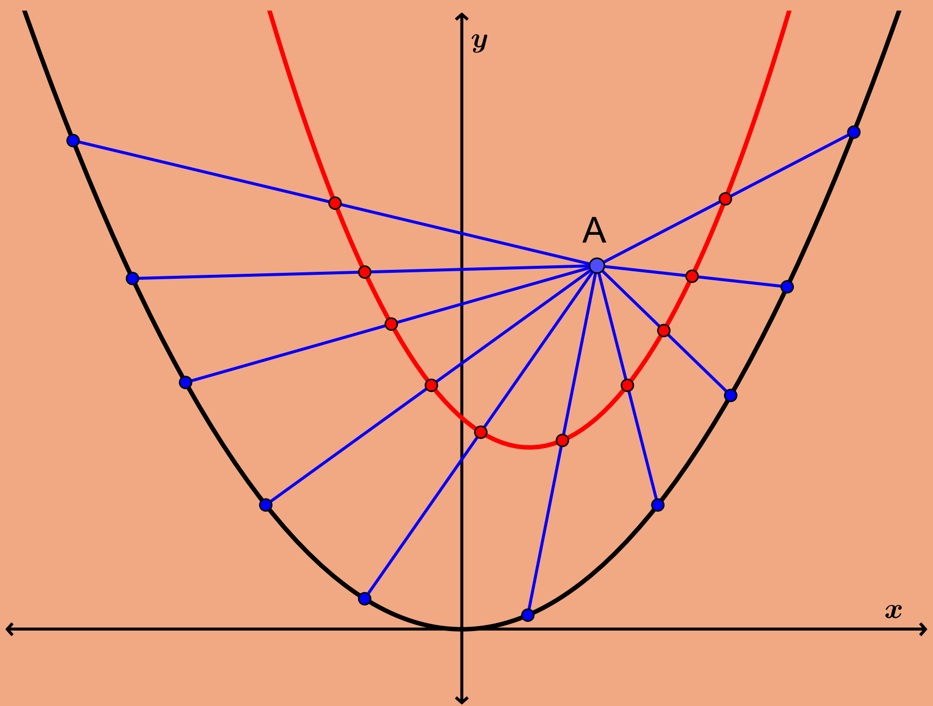

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Learning and Teaching Mathematics

Learning and Teaching Mathematics is a journal of the Association for Mathematics Education of South Africa (AMESA). This journal is aimed at mathematics teachers at primary and secondary school level and it provides a medium for stimulating and challenging ideas, offering innovation and practice in all aspects of mathematics teaching and learning in school. Learning and Teaching Mathematics aims to inform, enlighten, stimulate, challenge, entertain and encourage mathematics educators. Its emphasis is on addressing the challenges that arise in the mathematics classroom. It presents articles that describe or discuss mathematics teaching and learning through the eyes of practising teachers and learners. While this journal 'listens' to research and considers it in the activities, lesson ideas, and teaching strategies that it publishes, it is not a research publication.

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Articles submitted will be reviewed by the editors and members of the Editorial Board. The Board will ensure that the papers make a contribution to our understanding of mathematics learning and teaching, that the mathematics presented is correct, and that the language and layout used is user friendly. Support will be provided by the editors to contributors in relation to meeting the above requirements.

The main criterion of acceptance is that the article should make a contribution to the improvement of school mathematics teaching and learning. See the inner back cover for more information on the submission of materials and articles for publication.

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From the Editor

Dear LTM readers

In the first article of LTM 39, Kgaugelo Lefala, Siphuxolo Sjula and Lynn Bowie share their journey of using fingers as a way of helping Senior Phase learners make sense of negative numbers and the addition of integers. In the second article in this issue, Brad Uy and James Metz explore the use of coloured tokens as a tangible way of demonstrating that $a - b = a + (-b)$. The third article, by Duncan Samson, illustrates the efficacy of using algebraic representations of multi-digit numbers in making sense of interesting numerical results, while in the fourth article Letuku Moses Makobe presents a multiple solution task which he explored with his Grade 8 learners. Moshe Stupel and David Katsir then propose an interesting geometric activity to try out in the classroom.

In the sixth article, Duncan Samson and Moshe Stupel explore an interesting locus result involving midpoints, while in the seventh article Michael de Villiers presents a curious trigonometric identity which he discovered while applying the sine rule to the cosine rule. Alan Christison then highlights the need for care when solving trigonometrical equations of the form $\sin x + \cos x = m$. In the final article of issue 39, Michael de Villiers provides a personal example of a discovery made by one of his students in a classroom discussion, and highlights the importance of learner exploration and discovery.

We hope you enjoy the diverse array of articles in this issue and remind you that we are always eager to receive submissions. Suggestions to authors, as well as a breakdown of the different types of article you could consider, can be found at the end of this journal. If you have an idea but aren't sure how to structure it into an article, you are welcome to email the editor directly – we'd be happy to engage with you about turning your idea into a printed article.

Duncan Samson

Negative Fingers

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INTRODUCTION

We have been working with Senior Phase (SP) learners for a number of years and have found the teaching of negative numbers particularly challenging. Getting learners to master this topic with understanding is difficult. While counting numbers have a clear meaning, integers include values below zero which do not really have a tangible counterpart. Thus, by their nature, negative numbers are abstract entities. This makes it challenging to provide learners with meaningful representations to support their sense-making.

We have seen that many teachers resort to teaching learners how to add and subtract negative numbers by instructing them to follow rules similar to the following:

If the signs are the same then add the numbers and take the sign of the numbers

If the signs are different then subtract the numbers and take the sign of the bigger number

Although these rules "work", we worry that teaching learners via these "rules without reasons" leads to a view of mathematics as a series of arbitrary, disconnected rules and does not allow learners to build a powerful and coherent understanding of mathematical ideas.

We have learnt in our work that starting immediately with a discussion of a concept in the abstract is often not accessible to all SP learners. Instead, we make use of the CIA approach – Concrete, Imagined, Abstract – an adaptation of Bruner's (1966) "concrete, pictorial, abstract" approach advocated by many primary school educators. In the CIA approach, where possible, learners start by working with a concrete representation of a concept. We then ask them to picture that representation in their mind and get them to engage with the concrete representation using mental imagery rather than the physical manipulative itself. This imagined phase then provides the bridge to the abstract. One of the challenges with this approach is that schools often don't have a sufficient budget for concrete manipulatives. Furthermore, the administrative hassle of storing, handing out and collecting in the manipulatives may result in them simply not being used. Our aim was thus to find an affordable, easy-to-use representation to help make addition and subtraction of negative numbers meaningful to our learners.

The technique we came up with uses fingers to help make sense of addition of integers. Fingers pointing up represent positive integers while fingers pointing down represent negative integers. This provides learners with a visual representation which they can use to make sense of working with integers. What follows is the story of our journey using this representation. We describe how we use it for different calculations and then discuss some of the mathematical issues and limitations of the representation.

Developing the idea: From game to representation

A game that we have found useful for practising single digit calculations with both our primary and high school learners is *1, 2, 3, Show*! Learners play this game in pairs. Both learners start with their hands behind their back. After saying "*1, 2, 3 show*!" they then simultaneously reveal their hands, each learner with their choice of fingers pointing up. Then, depending on the skill one wants the learners to practise, the pair can either be asked to compare, add or multiply the number of fingers they are showing. The first learner to correctly call out the answer is the winner. The cartoon below, adapted from the *Bala Wande* workbooks, illustrates the game as played by primary school learners doing simple addition. Here they are only using one hand in order that the numbers remain small.

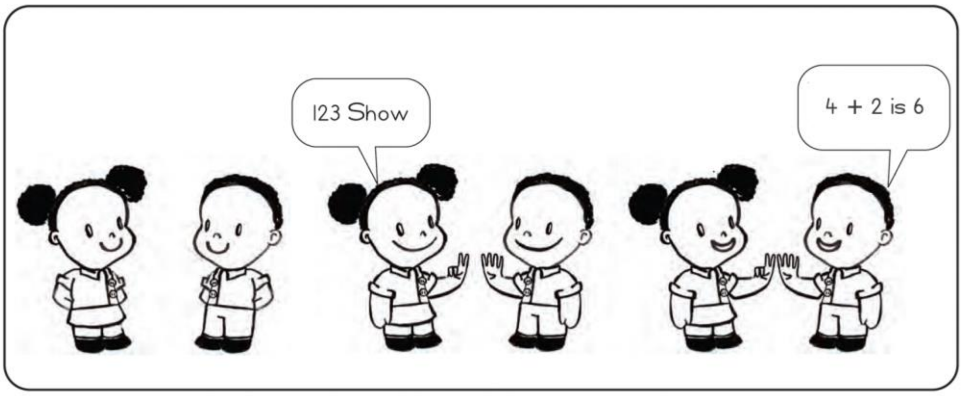


Figure 1: Playing the game *1, 2, 3, Show*!

One of the authors adapted this game to include negative numbers so that pupils could practise the addition of integers. In this version of the game each learner could make the choice to hold their fingers either up (representing positive integers) or down (representing negative integers).

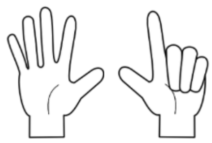
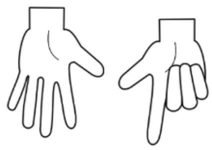
Learner A	Learner B	
		-5 + 7 = 2
		-3 + (-7) = -10

Figure 2: Adapting the game to include negative numbers.

Although this was initially intended simply as a game to provide learners with a fun way to practise integer addition, we soon saw that "fingers up, fingers down" could be a useful and meaningful representation to aid understanding. We expand on this in the following section.

USING NEGATIVE FINGERS TO MODEL ADDITION

We noticed that getting learners to physically match up fingers that added to zero, what we will refer to as "zero pairs", helped them to calculate the correct answer. This process of matching up zero pairs is illustrated in Figure 3 and Figure 4.

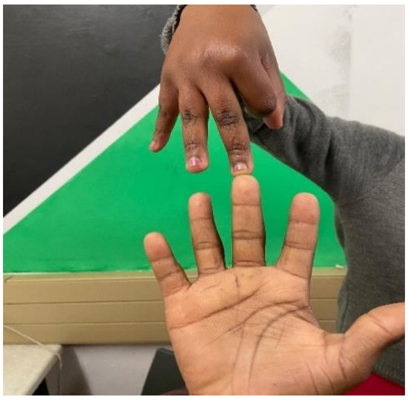
$-3+5$	
	
-3 is represented by three fingers pointing down while 5 is represented by five fingers pointing up.	Learners align their fingers to create zero pairs. We have three zero pairs: -3 + 3 = 0. We then see that what remains is two fingers pointing up, which means the answer is 2.

FIGURE 3: Making use of zero pairs to calculate $-3+5$.

$5+3$	$-5+3$	$-3+(-5)$
		
5 is represented by five fingers pointing up while 3 is represented by three fingers pointing up. There are no zero pairs. We see in total we have eight fingers pointing up so our answer is 8.	-5 is represented by five fingers pointing down while 3 is represented by three fingers pointing up. Learners align their fingers to create zero pairs. There are three zero pairs: -3 + 3 = 0. Two fingers remain pointing down, so our answer is -2.	-5 is represented by five fingers pointing down while -3 is represented by three fingers pointing down. There are no zero pairs. In total we have eight fingers pointing down so our answer is -8.

FIGURE 4: Further examples of fingers up and fingers down.

This technique gives a powerful concrete visualization of integer addition, giving meaning to what learners are doing instead of just providing rules to follow. It emphasizes the idea of, for example, 3 and -3 being additive inverses which when added together gives 0.

After sufficient practice with these concrete representations, we can move on to larger numbers, e.g. -30 + 50, asking the learners to use mental imagery rather than physical fingers. They would need to picture -30 as 30 fingers pointing down and 50 as 50 fingers pointing up. With their eyes closed they then need to visualize the 30 imaginary down fingers sliding to meet 30 of the 50 imaginary up fingers, leaving them with 20 free imaginary up fingers, giving a final answer of 20.

As learners work with these concrete and imagined representations we want to help connect this to the abstract calculation. For instance, we want them to see that in calculating -30 + 50 we end up looking for the difference between the up and down fingers and, because there were more up fingers, the answer will be positive.

THE COMPLEXITIES OF EXTENDING THE MODEL TO SUBTRACTION

Although we have found this representation very useful in dealing with integer addition, as with all attempts to find concrete representations of integers, there are limitations. The main issue occurs with subtraction of integers. Consider for example 5 – 3. Would we want learners to do this by combining five up fingers with three down fingers? 5 – 3 is actually 5 subtract 3, but by using the finger representation we would be treating it as 5 + (-3). Although we know that these are equivalent, the question is whether we are happy glossing over the distinction of "–" as an operation and "–" as the sign of the number. In Grade 1, learners would have seen 5 – 3 as five fingers up and then taking three of them away. So we are now giving a different meaning to the expression. Do we need to discuss that with the learners? We would have a similar issue with -3 – 5.

If we are happy to gloss over the distinction between "–" as an operation as opposed to the sign of the number, we could calculate 5 – 3 using five up fingers and three down fingers. Similarly, -3 – 5 could be calculated as three down fingers and a further five down fingers. However, things get more complicated when we consider 5 – (-3) or -5 – (-3). In this case teachers usually go straight into teaching learners to multiply signs, but without a thorough explanation.

We can expand our finger work so that we can deal meaningfully with things like 3 – 5 or 5 – (-3), but to do so necessitates making the model more complex. To do this we use an understanding of subtraction as "take away", so we remind learners that 9 – 4 means from nine objects you take away four. Then, importantly, each calculation needs to start with a zero hand. This involves one learner having five fingers up matched against their partner having five fingers down, together representing zero. Learners then use their other hands to represent the starting number.

The next few examples illustrate this modified use of the "fingers up, fingers down" representation.

3 – 5		
		
Learners use one hand each to make their zero hand.	To make 3 we need three "floating" fingers pointing up. One learner does this using their other hand.	We then need to take away 5, so we must remove five up fingers. We begin by removing the three floating up fingers. We then need to remove a further two up fingers which is done by removing two up fingers from the zero hand. This will leave us with two floating down fingers giving us our answer of -2.

FIGURE 5: Adapting the model to calculate $3 - 5$.

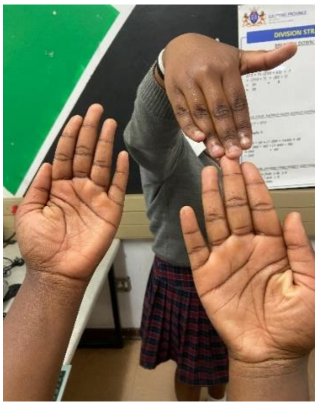
5 – (-3)		
		
Learners begin by using one hand each to create their zero hand.	To make 5 we need five floating fingers pointing up. One learner does this using their other hand.	We now need to take away -3 so we must remove three down fingers. These will have to be taken from the zero hand. This means that in the zero hand we now have three floating up fingers. In total we thus have eight floating up fingers, giving the answer of 8.

FIGURE 6: Adapting the model to calculate $5 - (-3)$.

<table>
<tr><th colspan="3">-5 – (-3)</th></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td>Learners begin by using one hand each to create their zero hand.</td><td>To make -5 we need five floating fingers pointing down. One learner does this using their other hand.</td><td>We need to take away -3 so we must remove three down fingers. These can be taken from the five down floating fingers. We are then left with only two floating fingers pointing down, giving an answer of -2.</td></tr>
</table>

FIGURE 7: Adapting the model to calculate $-5-(-3)$.

As you can see, dealing meaningfully with subtraction using this model makes it a lot more complex. It is also harder to match up hands to do the physical movements quickly and easily.

CONCLUDING COMMENTS

Reflecting on our experiences, we have found it useful to expose learners to a variety of representations – for example using number lines or analogies like credit and debt. We have found this engagement to be important and helpful in developing learners' understanding of the addition and subtraction of integers. Particularly in the initial stages, we have found that the tactile, concrete nature of the "negative fingers" model is often more accessible to learners, and as such provides a useful image that learners can use to bridge the gap between the concrete and the abstract.

The negative fingers representation can be extended to subtraction of negative numbers by the introduction of a "zero hand". This is similar to the use of zero pairs used in the two-sided or two-coloured counter representation (see for example https://www.youtube.com/watch?v=mxqlOa8y0is). Our experience suggests the power of the negative finger representation lies in it being an easy and quick concrete representation, and a simple image to recall to support the abstract calculation. However, we do not necessarily recommend using it for subtraction as the physical and mental complexity of introducing the zero hand is a bit cumbersome. We suggest that after providing learners with a slow and careful justification of why, for example, subtracting a negative is the same as adding a positive, it is possible to think of subtraction calculations as additions (e.g. -3 – (-5) = - 3 + 5) and those additions can then quickly and easily be supported by the negative fingers representation.

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Pause to See Why $a - b = a + (-b)$

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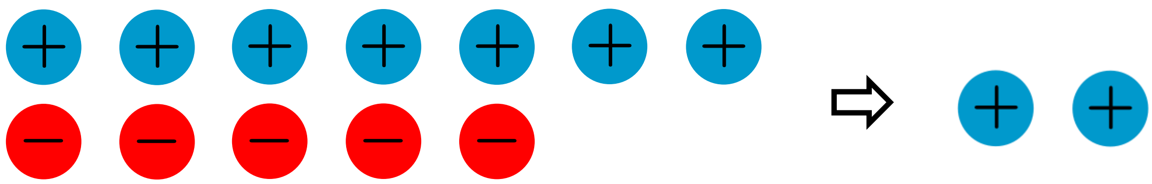
INTRODUCTION

The use of coloured tokens is a well-established hands-on way of engaging with addition and subtraction of integers. It is also a useful way of seeing that $a - b$ is equivalent to $a + (-b)$. Because the use of tokens is inherently visual, learners will be able to 'see' the equivalence of $a - b$ and $a + (-b)$. Learners who are already confident with the use of tokens for the addition of integers will find this a particularly beneficial approach.

As a reminder of the process of using tokens for addition, let us consider $7 + (-5)$. To perform this calculation, learners begin by putting down 7 positive tokens. They then place 5 negative tokens below them. Pairing up negative and positive tokens (which cancel each other out to give zero) learners are left with 2 positive tokens. Thus $7 + (-5) = 2$.

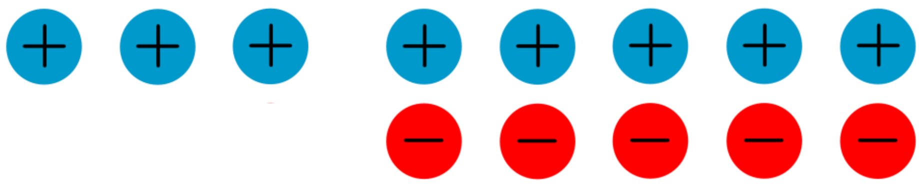


Let us now consider a few examples of subtraction.

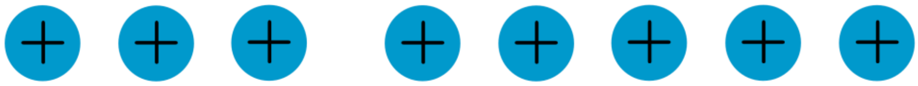
EXAMPLE 1: $3 - (-5)$

We begin by putting down 3 positive tokens.

We now need to remove (subtract) 5 negatives from the 3 positives. However, since we don't have any negatives to subtract, we add 5 zeros (in the form of negative and positive pairs of tokens). In effect we are now representing 3 as:



We can now subtract 5 negatives to give a final answer of 8.

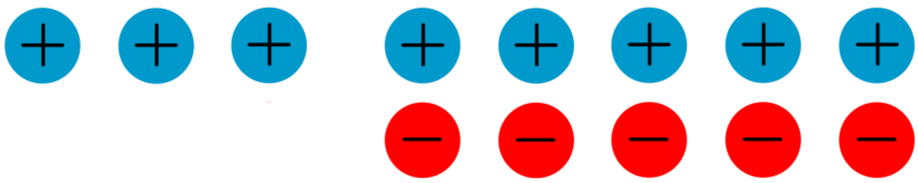


We thus see that $3 - (-5)$ is equivalent to $3 + 5$. The reader is now invited to try $-3 - 5$.

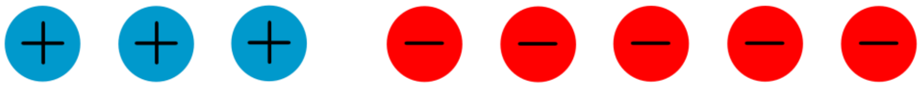
Example 2: $3 - 5$

We again begin by putting down 3 positive tokens.

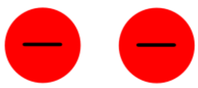
We now need to remove (subtract) 5 positives from the 3 positives. Although we lack only 2 positives, let us add 5 zeros (in the form of negative and positive pairs of tokens) as before, once again representing 3 as:



After subtracting 5 positives we are left with a visual representation of $3 + (-5)$.



After making 3 zeros by pairing positive and negative tokens, we are left with the final answer of -2.

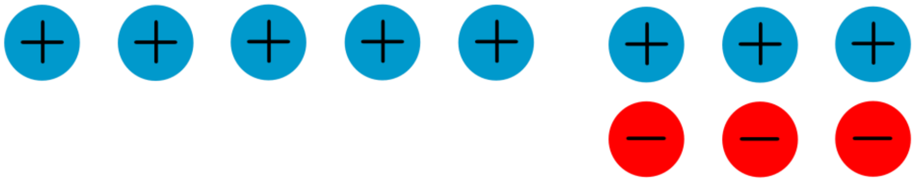


This elegantly shows that $3 - 5 = 3 + (-5) = -2$. The reader is now invited to try $-3 - (-5)$.

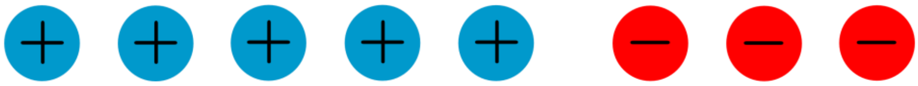
Example 3: $5 - 3$

In the previous example we used a strategy that involved adding zeros (pairs of positive and negative tokens) so that we were able to remove (subtract) what we needed to entirely from the introduced set representing zero. Sticking with this strategy we now consider $5 - 3$. We begin by putting down 5 positive tokens.

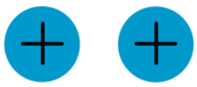
Even though we can readily subtract 3 positives from these 5 positives, we stick with our strategy of adding 3 zeros (pairs of positive and negative tokens) as follows:



We can now remove the 3 positives from those that form part of the added zeros. This leaves us with:



At this juncture we can pause to reflect that this arrangement represents $5 + (-3)$. After pairing up positives and negatives to make zeros, we are left with the final answer of 2.



This rather elegantly shows the equivalence of $5 - 3$ and $5 + (-3)$. Learners can of course easily verify this simply by removing 3 positive tokens from the original 5.

The interested reader is now invited to try $-5 - (-3)$.

CONCLUDING COMMENTS

By using a strategy of 'adding sufficient zeros', subtraction problems can be transformed into equivalent addition problems, with which learners may perhaps be more familiar. The cost is that we sometimes add more zeros than we really need, but the benefit is that we are able to use the process of addition for every problem. It also allows us to really 'see' that $a - b = a + (-b)$.

From Flard Cards to Algebra

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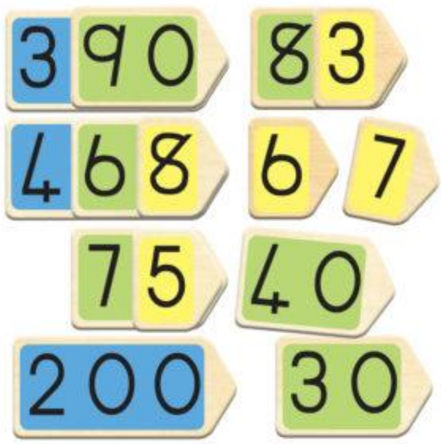
INTRODUCTION

A strong understanding of place value is a crucial aspect of number sense. Flard cards are a wonderful way to teach place value and develop number sense. Pupils can use Flard cards to visualise place value by breaking numbers down into their constituent parts (units, tens, hundreds, thousands etc.) or by building them up from their constituent parts. Flard cards encourage active learning through hands-on manipulation, and this hands-on approach nurtures a deep understanding of how numbers are structured and how different numbers relate to one another. The strong visual representation strengthens pupils' understanding that the position of a digit determines its value by connecting abstract number concepts with tangible representations.



AN ALGEBRAIC ANALOGUE

Flard cards afford a strong visual and tangible representation of place value. By way of example, a number such as 6253 would be represented by first placing down a 6000 card, and then placing a 200 card on top of it, followed by a 50 card and finally a 3 card. In essence, 6253 is built up as follows:

$$6000 + 200 + 50 + 3 \quad \rightarrow \quad 6253$$

An algebraic analogy of this process is to represent a general number of the form $ABCD$ as:

$$1000A + 100B + 10C + D \quad \rightarrow \quad ABCD$$

This algebraic representation of a multi-digit number is a useful way of exploring and making sense of interesting numerical results. As a simple example, take any 2-digit number, reverse the digits, and then add the two numbers. The resulting number will always be divisible by 11. By way of example:

$$26 + 62 = 88 \qquad 41 + 14 = 55 \qquad 79 + 97 = 176 \qquad 85 + 58 = 143$$

While the first two sums (88 and 55) are self-evidently divisible by 11, a quick check shows that so too are 176 and 143. We can readily make sense of this result by representing the two 2-digit numbers algebraically:

$$\begin{aligned}(10A + B) + (10B + A) &= 11A + 11B \\ &= 11(A + B)\end{aligned}$$

From this we can clearly see that the sum is divisible by 11.

In similar vein, any number of the form $ABBA$ will also always be divisible by 11, as illustrated below:

$$\begin{aligned}1000A + 100B + 10B + A &= 1001A + 110B \\ &= 11(91A + 10B)\end{aligned}$$

Numbers with repeated digits

Let us now consider numbers of the form $ABCABC$, for example 528528. Somewhat surprisingly, all numbers of this form are divisible by 7, 11 and 13. Even more surprisingly, dividing the 6-digit number by 7, and then dividing the result by 11, and then finally dividing this new quotient by 13, the final result will always be the 3-digit number ABC. If one hasn't seen this result before it can be quite striking!

However, we can readily make sense of the result by representing the original 6-digit number algebraically:

$$\begin{aligned} &100000A + 10000B + 1000C + 100A + 10B + C \\ &= 100100A + 10010B + 1001C \\ &= 1001(100A + 10B + C) \end{aligned}$$

From this we can see that all numbers of the form $ABCABC$ are divisible by 1001. And since 1001 rather pleasingly prime factorises to $7 \times 11 \times 13$, the result follows.

As a slight variation, let us now consider numbers of the form $ABABAB$, for example 787878. Somewhat surprisingly, all numbers of this form are divisible by 3, 7, 13 and 37. Even more surprisingly, dividing the 6-digit number systematically by 3, 7, 13 and 37 results in the 2-digit answer AB. As previously:

$$\begin{aligned} &100000A + 10000B + 1000A + 100B + 10A + B \\ &= 101010A + 10101B \\ &= 10101(10A + B) \end{aligned}$$

From this we can see that all numbers of the form $ABABAB$ are divisible by 10101, and since 10101 prime factorises to $3 \times 7 \times 13 \times 37$, the result follows.

It is left to the reader to explore numbers of the form $ABCDABCD$.

The answer is always 37

Here's a curious little trick. Ask a friend to think of a three-digit number where all three digits are the same, for example 444. Now get them to add the three digits together (e.g. $4 + 4 + 4 = 12$) and then to divide the original number by this sum ($444 \div 12 = 37$). No matter what three-digit number they start with, the answer will always be 37.

We can make sense of this result by representing the original three-digit number algebraically as follows:

$$100A + 10A + A = 111A$$

$$\frac{111A}{3A} = \frac{111}{3} = 37$$

Since $111 \div 3 = 37$, the answer will always be 37 no matter what the original 3-digit number was. The above illustration should also explain why a similar trick wouldn't work with a 4-digit number with repeated digits (since 1111 isn't divisible by 4). Nor would it work for 5-digit or 6-digit numbers with repeated digits. The next interesting scenario happens with 9-digit numbers where dividing the original number by the sum of the digits will always give an answer of 12345679. By way of example:

$$333333333 \div 27 = 12345679$$

MULTIPLICATION BY 11

There's a useful little technique for quickly multiplying any 2-digit number by 11 in your head. Take the 2-digit number and separate the two digits in your mind (effectively moving the tens digit of the original number into the hundreds position). Now add the two digits together and place the result between the separated digits. If the number placed between the two digits is greater than 9, then keep the units digit in the middle and add the 1 (i.e. the tens digit of the middle number) onto the number on the left (i.e. the hundreds digit of the 3-digit answer). The result is 11 times the original 2-digit number.

By way of example, let us consider multiplying 53 by 11:

- Separate the digits: 5_3
- Add the two digits and put the result between the separated digits: 583
- And that's the answer: $53 \times 11 = 583$

Let us now consider multiplying a larger number like 76 by 11:

- Separate the digits: 7_6
- Add the two digits and put the result between the separated digits: $7\mathbf{13}6$
- Since the sum of the two digits is greater than 9, keep the units digit (3) in the middle, and add the 1 (i.e. the tens digit of the middle number) to the number on the left (i.e. the hundreds digit of the final answer): 836.
- And that's the answer: $76 \times 11 = 836$

We can make sense of this little mental trick as follows. Imagine you want to multiply the 2-digit number AB by 11. The steps represented algebraically would look like this:

$$10A + B$$

$$100A \quad + \quad B$$

$$100A \ + \ 10(A + B) \ + \ B$$

$$= 110A + 11B$$

$$= 11(10A + B)$$

The result is 11 times the original 2-digit number.

THE CLASSIC 1089 RESULT

A well-known party trick is the following. Take any 3-digit number containing three different digits. Now reverse the order of the digits to create a new 3-digit number. Next, subtract the smaller number from the larger number. Take this answer and once again reverse the digits, but this time *add* the two numbers together. Irrespective of what 3-digit number you begin with, the final sum will always be 1089.

By way of example, let us arbitrarily choose 278 as our initial 3-digit number:

- Original number: 278
- Reversed digits: 872
- Subtract smaller number from larger number: $872 - 278 = 594$
- Reverse the digits: 495
- Add the last two numbers: $594 + 495 = 1089$

This self-working "number trick" never ceases to surprise, particularly since the final answer of 1089 seems somewhat arbitrary at first glance. However, we can once again make use of our algebraic representation of multi-digit numbers to unpack the process and make sense of what's happening behind the calculations.

Let the original 3-digit number be ABC. Without loss of generality let us assume that $A > C$. The original number can thus be represented as $100A + 10B + C$. When we reverse the order of the digits we create the number $100C + 10B + A$. Since $A > C$, our original 3-digit number is the larger of the two. The algebra becomes a little tricky now as we need to do some carrying. In order to make sense of the carrying, the process is summarised in table form.

Reversing the original 3-digit number and subtracting:

	Hundreds	Tens	Units
Original	A	B	C
Reversed	C	B	A
Subtracting	$(A-1)-C$	$\big((B-1)+10\big)-B$ $=9$	$(C+10)-A$

Reversing the above result and adding:

	Hundreds	Tens	Units
Original	$(A-1)-C$	9	$(C+10)-A$
Reversed	$(C+10)-A$	9	$(A-1)-C$
Adding	$9+1$ $=\mathbf{10}$	$\mathbf{8}$	$\mathbf{9}$

As a slightly simplified explanation we could proceed as follows. Once again represent the original 3-digit number as $100A + 10B + C$, with $A > C$. Reversing and subtracting yields:

$$(100A + 10B + C) - (100C + 10B + A)$$
$$= 99A - 99C$$
$$= 99(A - C)$$

Thus, at the end of the first part of the process we will always end up with a multiple of 99. The 3-digit multiples of 99 are:

$$198 \ ; \ 297 \ ; \ 396 \ ; \ 495 \ ; \ 594 \ ; \ 693 \ ; \ 792 \ ; \ 891$$

Notice that in all cases the middle digit is 9 and the outer two digits add up to 9. This means that when we reverse and add we will always get 900 from the hundreds digits, 9 from the units digits, and two lots of 90 from the tens digits: $900 + 2 \times 90 + 9 = 1089$.

CONCLUDING COMMENTS

It was the purpose of this article to illustrate the efficacy of using algebraic representations of multi-digit numbers in exploring and making sense of interesting numerical results. This kind of representation can be seen as an algebraic extension of Flard cards, and can be used to nurture a more nuanced appreciation of place value.

A Multiple Solution Task

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INTRODUCTION

Inspired by previous multiple solution tasks presented in *Learning and Teaching Mathematics*, I decided to give my Grade 8 learners a geometry question that could be solved in a number of different ways. My motivation for doing this was to provide learners with an opportunity to engage with a question that could be approached using different strategies, thereby encouraging creativity. I also hoped that the question would give rise to classroom discussion as different approaches were shared and explored.

The question I gave my Grade 8 class is shown in Figure 1. The diagram shows four straight line segments with $AB \parallel DE$. Given that $A\hat{B}C = 140°$ and $C\hat{D}E = 100°$, learners were asked to determine the size of $B\hat{C}D$, indicated on the diagram as x.

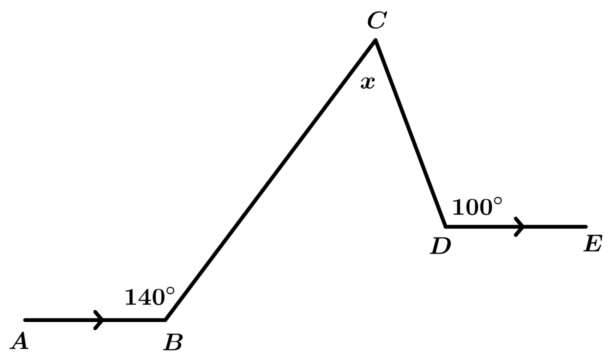

FIGURE 1: The question presented to the Grade 8 class.

I was pleasantly surprised by the number of different approaches learners explored as they engaged with the diagram. Although some approaches needed scaffolding and occasional prompting, what follows is a synthesis of the different solution strategies that came to light.

STRATEGY 1

This strategy involved drawing a line FG passing through C parallel to AB and DE (Figure 2).

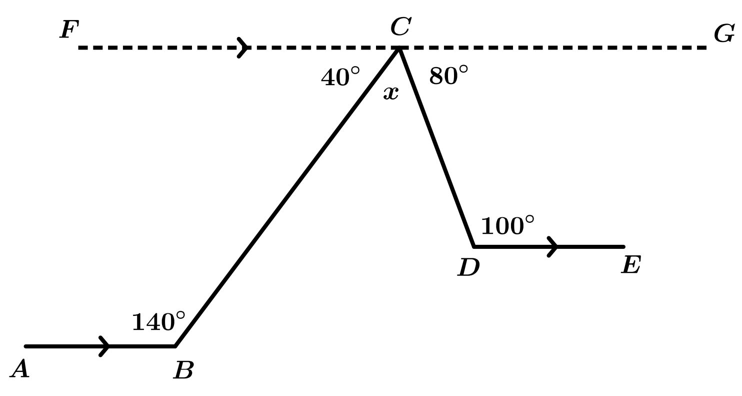

FIGURE 2: Drawing a line through C parallel to AB and DE.

Learners made use of the fact that co-interior angles on parallel lines are supplementary to establish that $F\hat{C}B = 40°$ and $D\hat{C}G = 80°$. Then, using the property that angles on a straight line are supplementary, it follows that $x = 60°$.

STRATEGY 2

In this strategy, line DE was extended to the left (Figure 3).

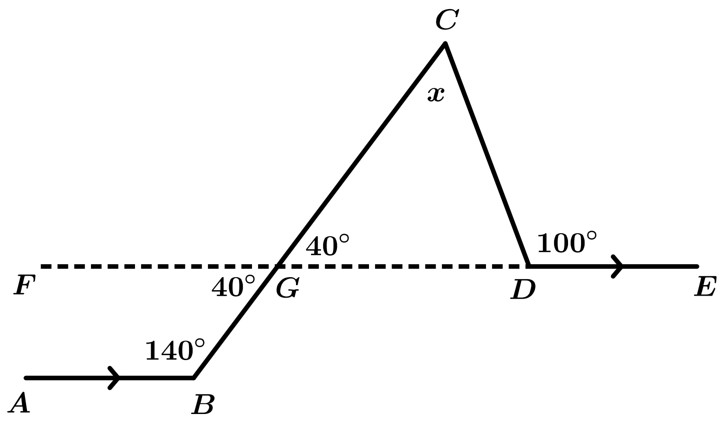

FIGURE 3: Extending line DE to the left.

This construction creates supplementary co-interior angles $A\hat{B}G$ and $B\hat{G}F$, as well as triangle CDG. Learners were able to establish that $B\hat{G}F = 40°$ (co-interior angles on parallel lines) and thus $C\hat{G}D = 40°$ (vertically opposite angles). Then, using the fact that the exterior angle of a triangle equals the sum of the two opposite interior angles, we have $x + 40° = 100°$, from which $x = 60°$.

STRATEGY 3

This strategy is similar to the previous one, but instead of extending line DE to the left, line AB is extended to the right (Figure 4).

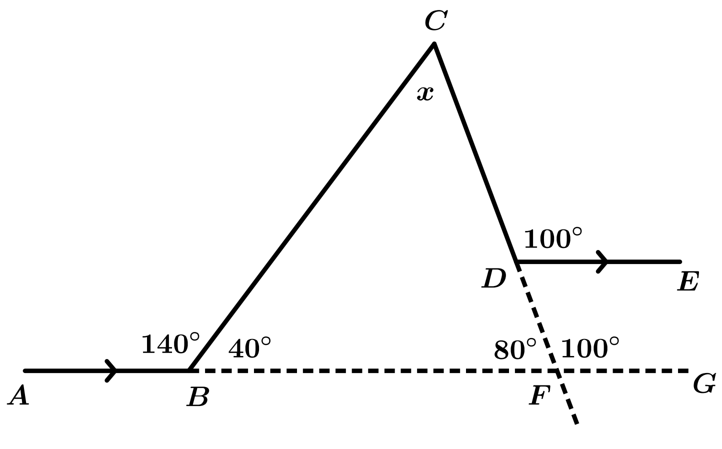

FIGURE 4: Extending line AB to the right.

This construction creates equal corresponding angles $C\hat{D}E$ and $D\hat{F}G$, as well as triangle CBF. Using corresponding angles on parallel lines we have $D\hat{F}G = 100°$. Then, using supplementary angles on straight lines, we have $C\hat{B}F = 40°$ and $C\hat{F}B = 80°$. It then follows that $x + 40° + 80° = 180°$ (interior angles of triangle CBF) from which $x = 60°$.

STRATEGY 4

This strategy is a subtle variation of the first strategy, but it gives rise to some different thinking. In this approach a line is drawn from C to the right, parallel to DE (Figure 5).

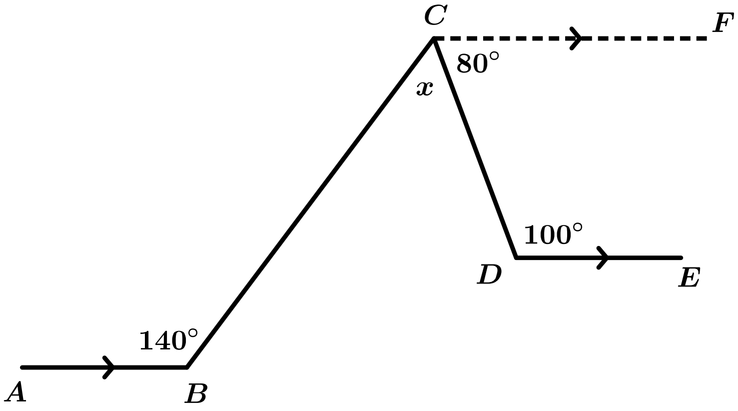

FIGURE 5: Drawing a line from C to the right, parallel to DE.

The construction of the line CF creates supplementary co-interior angles $C\hat{D}E$ and $F\hat{C}D$, from which we have $F\hat{C}D = 80°$. We can now make use of the equal alternate angles $A\hat{B}C$ and $B\hat{C}F$ to establish that $140° = x + 80°$, from which $x = 60°$.

STRATEGY 5

This strategy involved creating two right angles by extending line AB to the right, and dropping a perpendicular from D onto this extended line (Figure 6).

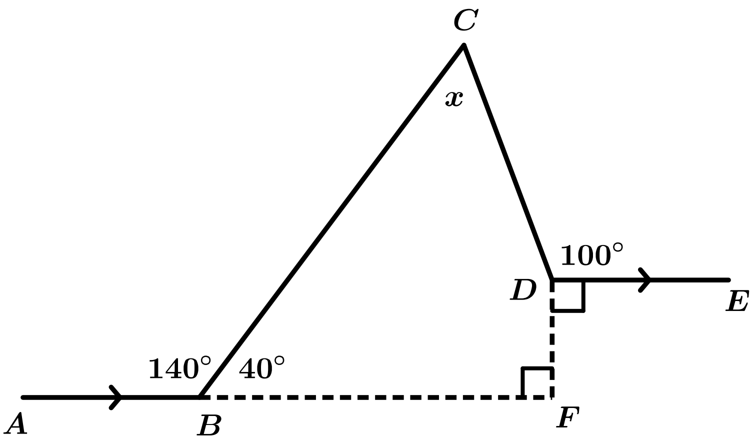

FIGURE 6: Creating two right angles.

Using supplementary angles on a straight line, we first establish that $C\hat{B}F = 40°$. Then, using angles round a point, we have $C\hat{D}F = 360° - 100° - 90° = 170°$. Finally, focusing on the interior angles of quadrilateral $CBFD$ we have $x + 40° + 90° + 170° = 360°$, from which $x = 60°$.

STRATEGY 6

As a final strategy, consider creating two right-angled triangles as illustrated in Figure 7.

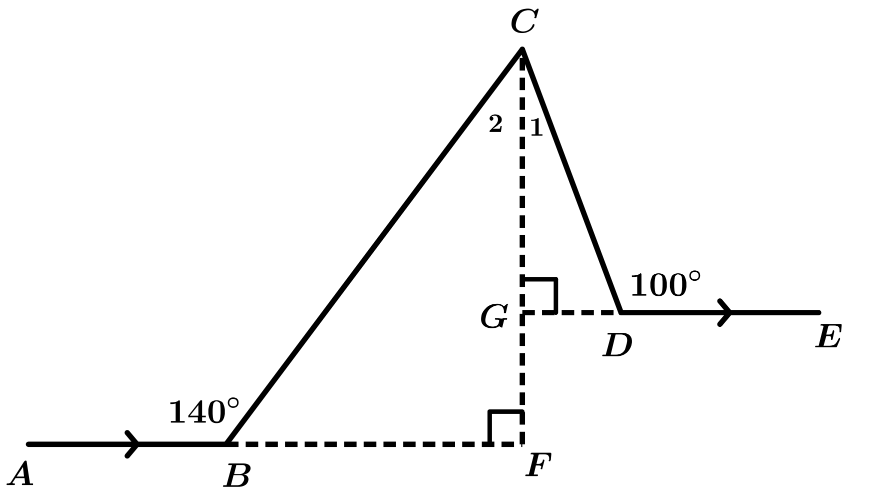

FIGURE 7: Creating two right-angled triangles.

Although the construction is perhaps a little more complex, the solution rather elegantly uses only a single geometric theorem, namely that the exterior angle of a triangle equals the sum of the opposite interior angles. In ΔBCF we have $\hat{C}_2 + 90° = 140°$, from which $\hat{C}_2 = 50°$. Similarly, in ΔCGD we have $\hat{C}_1 + 90° = 100°$, from which $\hat{C}_1 = 10°$. From this it follows that $x = \hat{C}_1 + \hat{C}_2 = 10° + 50° = 60°$.

CONCLUDING COMMENTS

Learners enjoyed this activity, and the number of different approaches and strategies used highlights the value that can be extracted from a single suitable question. Almost all the geometry theorems that the learners knew were made use of:

- Co-interior, corresponding and alternate angles on parallel lines
- Supplementary angles on a straight line
- Interior angles of triangles and quadrilaterals
- Angles around a point
- Exterior angle of a triangle

Not only did this activity allow learners to be creative in their engagement with the diagram, but it also opened their eyes to multiple solution strategies, and the fact that there is not only "one correct way" to get to an answer. Learners were also able to take ownership of their particular approach, and sharing this with the rest of the class gave added meaning to their mathematical engagement.

Triangular Tours

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Here's an interesting activity to try out in the classroom. Get each learner to draw an arbitrary triangle ABC. On side AB, chose a random point D that is not the midpoint of AB. From point D draw a line parallel to side BC until you intersect the triangle on side AC. Label this point E. From point E draw a line parallel to side AB until you meet the triangle again at point F. From F draw a line parallel to AC until you meet the triangle at point G. Now continue this process until you create a closed path. If you do this process correctly, the sixth line that you draw should take you back to point D, the starting point, as illustrated in Figure 1.

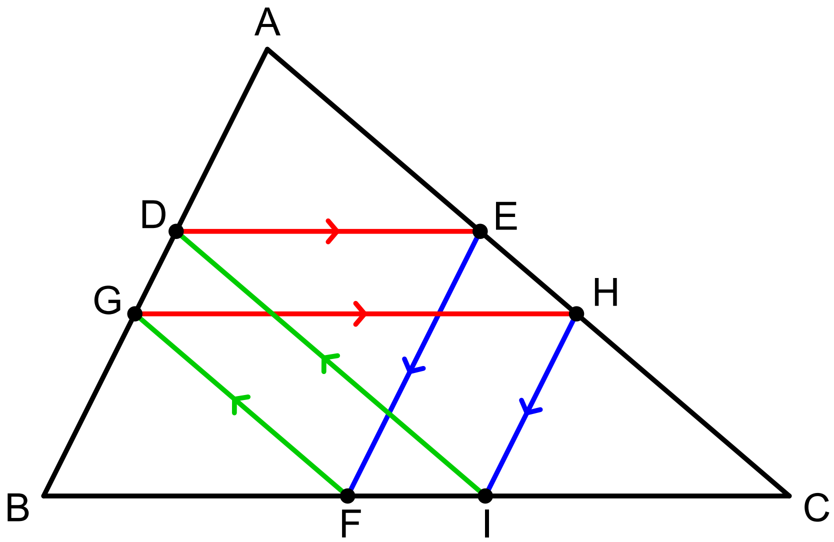

FIGURE 1: A triangular tour.

Now get each learner to measure the length of their triangular tour – i.e. the sum of the six inner lines. Next get them to measure the perimeter of the original triangle. Learners will be amazed to discover that the length of their triangular tour is exactly the same as the perimeter of the triangle! A dynamic applet to explore this result can be found here: https://www.geogebra.org/classic/n8vxejzp

Now draw learners' attention to the three sets of three parallel lines, e.g. DE, GH and BC. Learners will again be surprised to discover that DE + GH = BC. Get them to confirm that the same relationship holds for the other two sets of parallel lines, i.e. GF, DI, AC and HI, EF, AB. Now ask learners what they think would happen if their starting point D had been the midpoint of side AB. In this scenario the midpoint of side AB would connect directly to the midpoints of the other sides, resulting in a triangular tour exactly half the length of the perimeter of the original triangle, as illustrated in Figure 2.

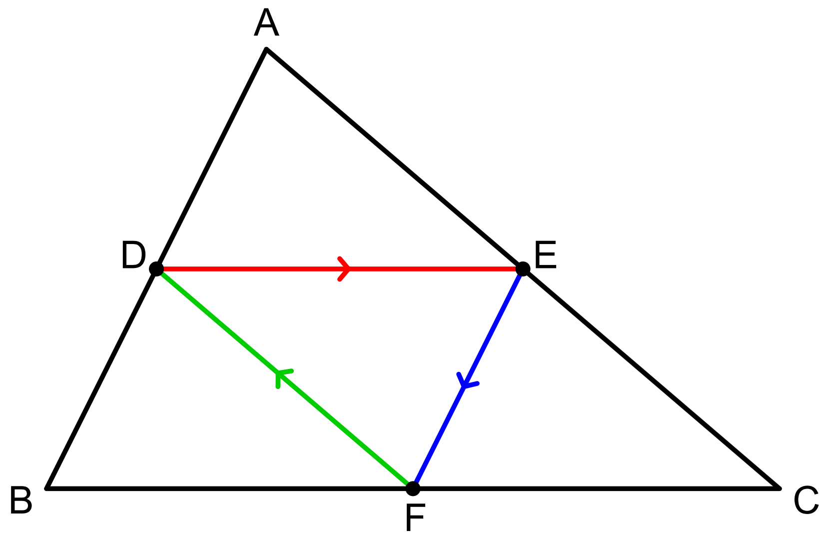

FIGURE 2: A triangular tour starting from the midpoint of side AB.

As a final challenge, get learners to find as many triangles, parallelograms and trapezia as they can in their diagram. Focusing on the parallelograms is one way of making sense of why the length of the triangular tour is exactly the same as the perimeter of the triangle (see Figure 3). It is left to the interested reader to explore this further.

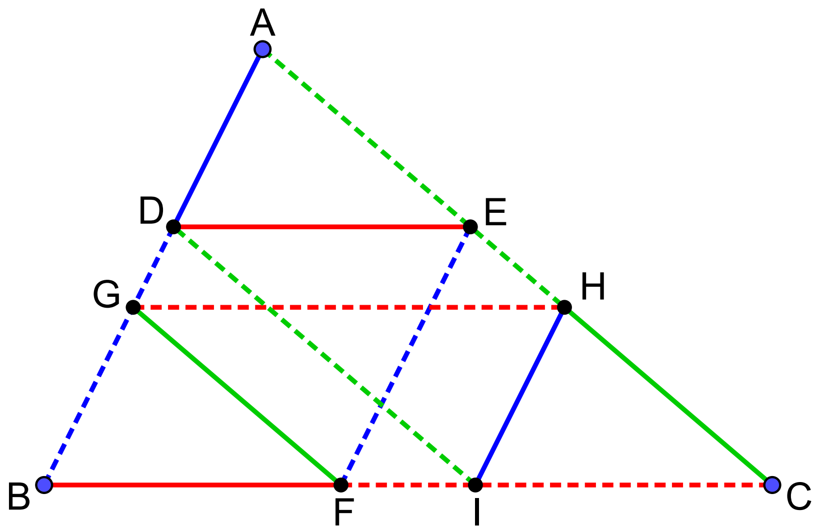

FIGURE 3: Focusing on the parallelograms.

An Interesting Locus Result

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INTRODUCTION

Senior high school pupils will be familiar with the "midpoint theorem" – the Euclidean geometry theorem that states that the line joining the midpoints of two sides of a triangle is parallel to the third side (and equal to half the length of the third side). This well-known theorem is illustrated in Figure 1.

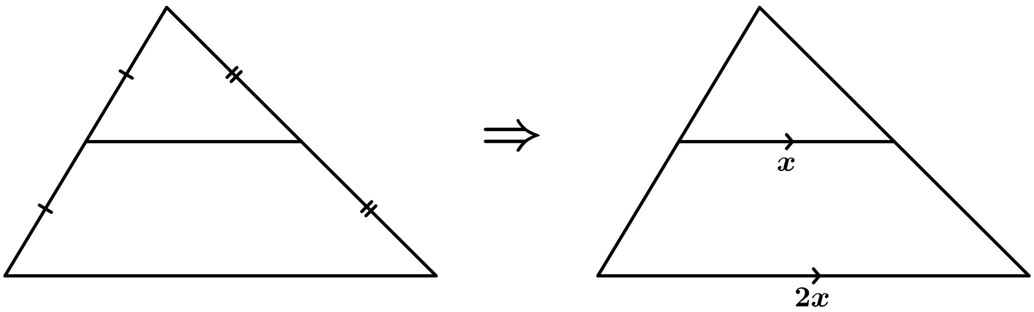

FIGURE 1: The midpoint theorem.

A somewhat less intuitive, but related scenario, is the following. Consider a straight line f and a fixed point A not on f. Create a series of straight-line segments by joining point A to a variety of different points on f as illustrated in Figure 2.

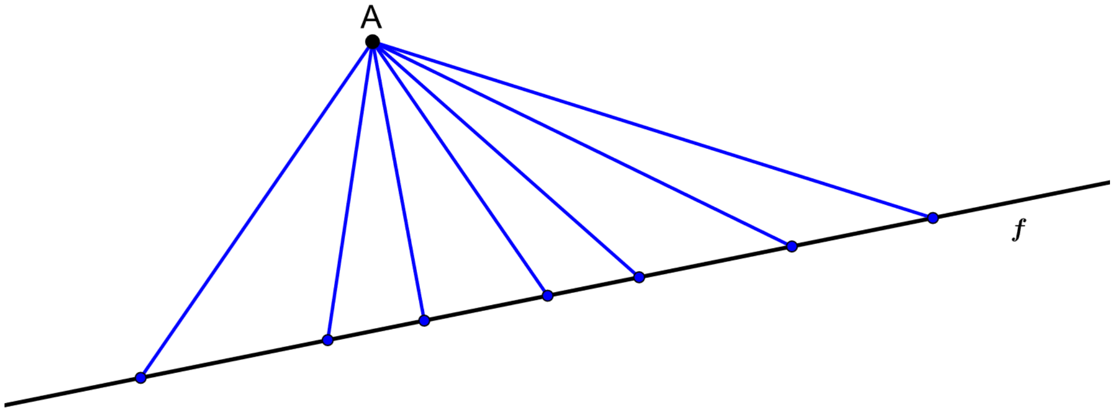

FIGURE 2: Straight-line segments from A to f.

Next construct the midpoint of each of these line segments. You may be surprised to find that not only are all of these midpoints collinear, but the straight line passing through them (g) is parallel to the original straight line f (Figure 3). Linking this scenario back to the midpoint theorem sheds light, from a geometric perspective, on why the midpoints are collinear as well as why the line passing through them is parallel to f.

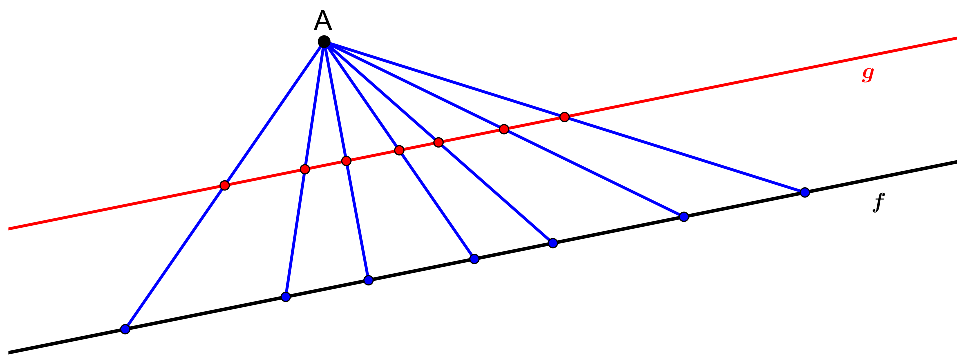

FIGURE 3: Collinear midpoints.

How could we go about determining the equation of g, the line passing through the midpoints? Let us take the equation of f to be $y_0 = mx_0 + c$, and the coordinates of A to be $(a; b)$. What we now need to determine is the equation representing the locus of all points $(x; y)$ lying on the midpoints of the line segments drawn from A to f.

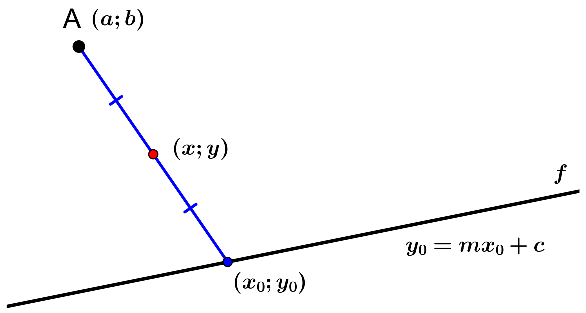

FIGURE 4: Determining the locus of points $(x; y)$ lying on the midpoints of the line segments.

Since $(x; y)$ is the midpoint of the line segment, we have:

$$x = \frac{a + x_0}{2} \quad \rightarrow \quad x_0 = 2x - a$$

$$y = \frac{b + y_0}{2} \quad \rightarrow \quad y_0 = 2y - b$$

We can now substitute the above expressions for x_0 and y_0 as follows:

$$y_0 = mx_0 + c$$

$$\therefore \quad 2y - b = m(2x - a) + c$$

$$\therefore \quad 2y = 2mx - ma + c + b$$

$$\therefore \quad y = mx + \frac{b + c - ma}{2}$$

This is the equation of the locus of all points $(x; y)$ lying on the midpoints of the line segments drawn from A to f. Note that the gradient of this line is also m, as it was for the original line f. This then confirms, from an algebraic perspective, not only that the midpoints lie on a straight line, but that this line is parallel to the original straight line.

EXTENDING THE IDEA

If this works for straight lines, would it also work for other functions? Let us consider a simple parabola such as $f(x) = \frac{1}{4}x^2$. As before, plot a fixed point A not on the parabola and create a series of straight-line segments from A to a variety of different points on f. Now construct the midpoint of each of these straight-line segments. Rather pleasingly, these midpoints also line up in the form of a parabola, irrespective of the position of the fixed point A (Figure 5).

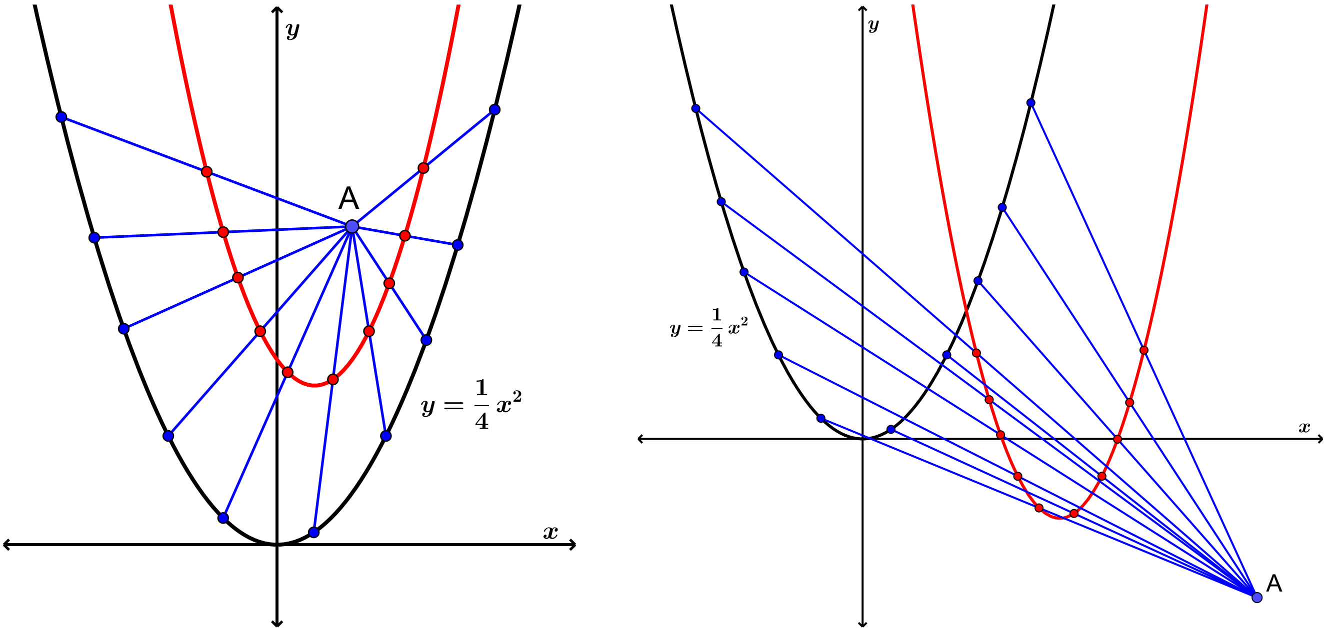

FIGURE 5: Joining the midpoints.

Let us now try to confirm algebraically that the midpoints do indeed line up on a parabolic curve. Additionally, let us try to establish what the relationship is between the original parabola and that represented by the locus of points lying on the midpoints of the line segments.

Let us take the equation of the original parabola to be $y_0 = kx_0{}^2$, and the coordinates of A to be $(a; b)$. What we now need to determine is the equation representing the locus of all points $(x; y)$ lying on the midpoints of the line segments drawn from A to $y_0 = kx_0{}^2$.

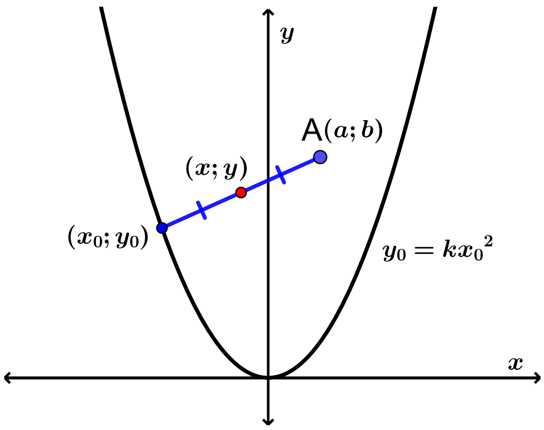

FIGURE 6: Determining the locus of points $(x; y)$ lying on the midpoints of the line segments.

Since $(x; y)$ is the midpoint of the line segment, we have:

$$x = \frac{a + x_0}{2} \quad \rightarrow \quad x_0 = 2x - a$$

$$y = \frac{b + y_0}{2} \quad \rightarrow \quad y_0 = 2y - b$$

We can now substitute the above expressions for x_0 and y_0 as follows:

$$y_0 = k{x_0}^2$$

$$\therefore \quad 2y - b = k(2x - a)^2$$

$$\therefore \quad 2y = k(2x - a)^2 + b$$

$$\therefore \quad 2y = k\left(2\left(x - \frac{a}{2}\right)\right)^2 + b$$

$$\therefore \quad y = 2k\left(x - \frac{a}{2}\right)^2 + \frac{b}{2}$$

This is the equation of the locus of all points $(x; y)$ lying on the midpoints of the line segments drawn from A to the parabola. The equation confirms that the locus of points is indeed parabolic. Note also that the axis of symmetry of the new parabola is parallel to the axis of symmetry of the original parabola. Additionally, note that the "stretch factor" of the new parabola is twice that of the original ($2k$ versus k in the original).

While we have only considered simple parabolas of the form $y = kx^2$, i.e. that have a turning point at the origin, the result would still hold true for parabolas containing a vertical and/or horizontal shift.

Concluding Comments

While the midpoint theorem is familiar to senior high school pupils, reimagining this idea from a slightly different perspective (i.e. exploring the locus of midpoints of a series of straight-line segments drawn from a fixed point to an arbitrary straight line) leads to an interesting, and perhaps less intuitive, result – namely that the locus of midpoints is a straight line parallel to the original straight line. We then took this basic idea and extended it to the case of a simple parabola with turning point at the origin and showed that in this case the locus of midpoints also forms a parabola with some interesting relationships to the original parabola. Similar locus observations occur in the case of the circle, ellipse and hyperbola, as well as other curves. Furthermore, while we have only considered the midpoints of the line segments, any fixed division of the straight-line segments would work.

One of the great joys of mathematics is how different concepts link, and how looking at a given scenario from different perspectives can lead to different insights and deeper understanding. With reference to Figure 5, we could reimagine the process as a dilation (reduction) of the parabola by a factor ½ from point A. From this transformation perspective it becomes clear why the process would work for any curve[1], as well as any fixed division of the straight-line segments.

[1] See for example the following webpage by Michael de Villiers: http://dynamicmathematicslearning.com/mystery-transform.html

A Curious Trigonometric Result

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There are essentially two different ways in which new mathematics is discovered or created. Sometimes we observe patterns in numbers or diagrams which can lead to conjectures and eventually proofs. This is often referred to as the *inductive* method since we generalize from a number of experimentally observed cases. However, sometimes we discover results purely by *deduction*, for example by logical reasoning without any prior experimentation. What follows is one such example.

The following curious trigonometric identity was discovered by the author around 2005 while applying the sine rule to the cosine rule. The result is a sort of 'angle version' of the familiar cosine rule:

$$a^2 = b^2 + c^2 - 2bc.\cos A$$

From the sine law $\left(\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\right)$ we have:

$$a = b\frac{\sin A}{\sin B} \quad \text{and} \quad c = b\frac{\sin C}{\sin B}$$

Substituting these values for a and c into the cosine rule given above, we obtain:

$$b^2\frac{\sin^2 A}{\sin^2 B} = b^2 + b^2\frac{\sin^2 C}{\sin^2 B} - 2b^2\frac{\sin C}{\sin B} \times \cos A$$

Multiplying through by $\frac{\sin^2 B}{b^2}$ we obtain:

$$\sin^2 A = \sin^2 B + \sin^2 C - 2\sin B\,.\sin C\,.\cos A$$

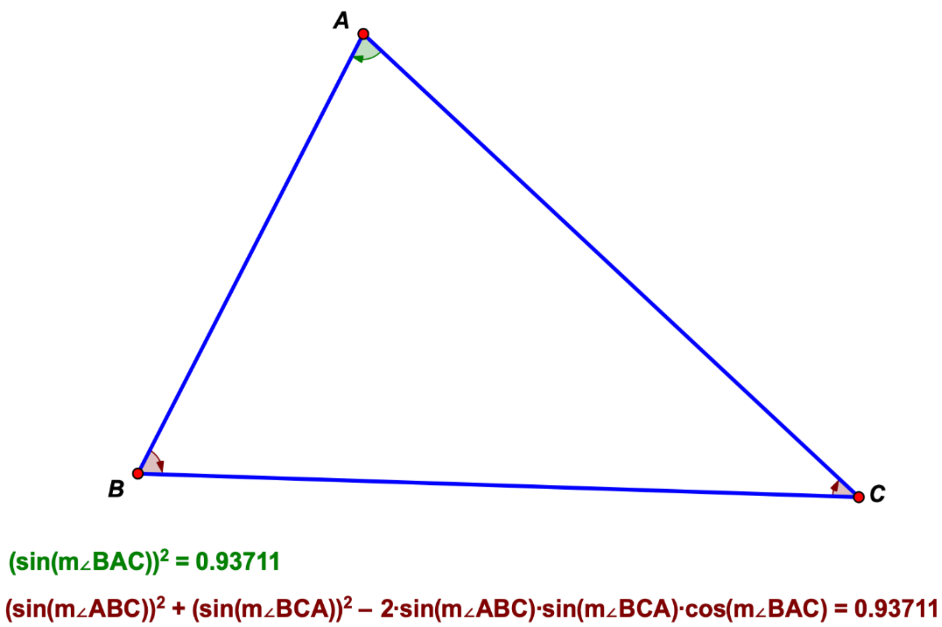

FIGURE 1

At first glance this result seemed so surprising to me that I felt compelled to use dynamic geometry software to quickly check that I had not accidentally made a logical error somewhere in my derivation (Figure 1). Readers are also invited to check the validity of this identity online at the following URL: https://dynamicmathematicslearning.com/cosine-anglerule.html

Note that when $\angle A = 90°$, this identity reduces to $1 = \sin^2 B + \sin^2 C$. In this special case it is, of course, equivalent to the more familiar identity $\sin^2 B + \cos^2 B = 1$.

As I later discovered, this curious trigonometric identity is not new or original. For example, it appears in Burton (1949) in his derivation of the Law of Cosines from the Sine Rule.

While the result seems to have little or no application, it could nevertheless be used in a high school classroom as an interesting, curious problem to challenge learners to come up with a proof. When this problem was posted in the Facebook Group *Romantics of Geometry* in May 2025, it prompted several solutions different from the one given above (these can be seen at the URL given earlier). The result thus definitely has some pedagogical potential as a multiple solution task for promoting and stimulating creativity (Leikin & Lev, 2007). Alternatively, it could easily be adapted as a learning activity to illustrate the discovery function of deductive reasoning to learners (De Villiers, 1990).

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Leikin, R. & Lev, M. (2007). Multiple solution tasks as a magnifying glass for observation of mathematical creativity. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 161-168. Seoul: PME.

Equations of the Form $\sin x + \cos x = m$

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INTRODUCTION

Care needs to be taken when solving trigonometrical equations of the form $\sin x + \cos x = m$. Having recently worked through a number of grade 12 worksheets, I have come across questions of this type where an incorrect solution is provided. This article considers both the "typically expected" method of solving such equations, together with an alternative method.

THE GENERAL SOLUTION

The following question appeared in an official 2024 matric mathematics practice document:

Determine the general solution of the equation: $\sin x + \cos x = \sqrt{\frac{3}{2}}$

The following solution was then provided:

$$x = 15^{\circ} + k.180^{\circ} \text{ or } x = 75^{\circ} + k.180^{\circ},\ k \in \mathbb{Z}$$

The reader is encouraged to solve this equation for themselves and see if they agree with the above solution.

THE "TYPICALLY EXPECTED" SOLUTION METHOD

The typical solution method involves squaring both sides and then using two standard trigonometric identities to manipulate the equation into a simpler form:

$$\sin x + \cos x = \sqrt{\frac{3}{2}}$$

$$(\sin x + \cos x)^2 = \frac{3}{2}$$

$$\sin^2 x + 2\sin x \cos x + \cos^2 x = 1{,}5$$

$$1 + \sin 2x = 1{,}5$$

$$\sin 2x = 0{,}5$$

From this we have: $2x = 30^{\circ} + k.360^{\circ}$ or $2x = 150^{\circ} + k.360^{\circ},\ k \in \mathbb{Z}$

$$x = 15^{\circ} + k.180^{\circ} \text{ or } x = 75^{\circ} + k.180^{\circ},\ k \in \mathbb{Z}$$

At fist glance this looks like a perfectly acceptable solution. However, if we consider various cases for different values of k then the situation is revealed to be a little more complex. Table 1 provides a list of values of $\sin x + \cos x$ based on the above solutions for various values of k $(-2\,;\,-1\,;0\,;1\,;2)$ and indicates whether or not each solution is valid.

k	x	$\sin x + \cos x$	VALID?	x	$\sin x + \cos x$	VALID?
-2	-345°	$\sqrt{\frac{3}{2}}$	Yes	-285°	$\sqrt{\frac{3}{2}}$	Yes
-1	-165°	$-\sqrt{\frac{3}{2}}$	No	-105°	$-\sqrt{\frac{3}{2}}$	No
0	15°	$\sqrt{\frac{3}{2}}$	Yes	75°	$\sqrt{\frac{3}{2}}$	Yes
1	195°	$-\sqrt{\frac{3}{2}}$	No	255°	$-\sqrt{\frac{3}{2}}$	No
2	375°	$\sqrt{\frac{3}{2}}$	Yes	435°	$\sqrt{\frac{3}{2}}$	Yes

TABLE 1: Values of $\sin x + \cos x$ for selected values of k.

Based on this table it would appear that the correct general solution is as follows:

$$x = 15^\circ + 2k.180^\circ \quad \text{or} \quad x = 75^\circ + 2k.180^\circ \ , \ k \in \mathbb{Z}$$

AN ALTERNATIVE SOLUTION METHOD

Multiplying all terms by $\frac{\sqrt{2}}{2}$ we get:

$$\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x = \frac{\sqrt{2}}{2} \times \sqrt{\frac{3}{2}}$$

Recognising this as the compound angle expansion of $\sin(x + 45°)$ we can thus write:

$$\sin(x + 45^\circ) = \frac{\sqrt{3}}{2}$$

From this we have the solution: $x + 45^\circ = 60^\circ + k.360^\circ$ or $x + 45^\circ = 120^\circ + k.360^\circ$, $k \in \mathbb{Z}$

$$x = 15^\circ + k.360^\circ \quad \text{or} \quad x = 75^\circ + k.360^\circ \ , \ k \in \mathbb{Z}$$

This solution is valid for *all* values of k and agrees with the suggested *adjusted* solution presented above.

Let us now visualise the solution set by sketching the graphs of $f(x) = \sin x + \cos x$ and $g(x) = \sqrt{\frac{3}{2}}$. Figure 2 shows these two graphs on the same set of axes for $x \in [-400^\circ ; 450^\circ]$.

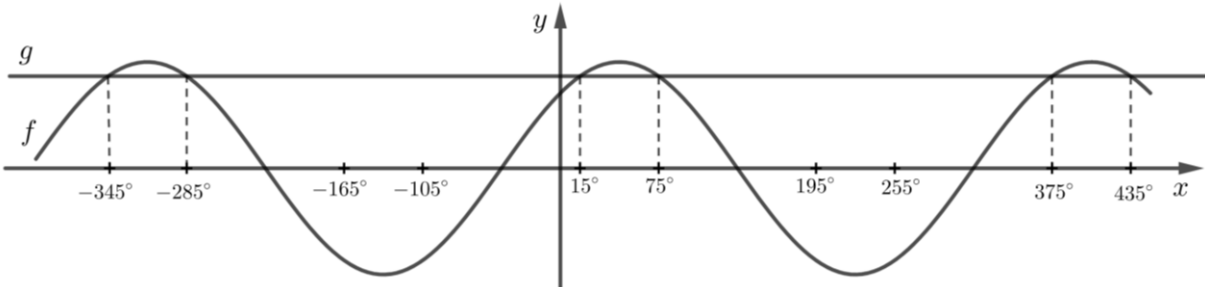

FIGURE 2: Graphs of $f(x) = \sin x + \cos x$ and $g(x) = \sqrt{\frac{3}{2}}$ for $x \in [-400^\circ ; 450^\circ]$.

Notice that all solutions occur at intervals of $\pm 360^\circ$ from the principal values of 15° and 75°. The shown intervals of $\pm 180^\circ$ clearly do not result in solutions. The reason for these spurious answers is due to the process of squaring both sides in the initial solution. This is similar to what may occur in the solution of surd equations where both sides of the equation are squared. If this method is used, *all answers must be checked* to ensure that they are indeed valid solutions.

It is important to note however, that when using the initial method of solving these equations, it is *not sufficient* to generalize that all valid solutions will be at the values determined by $2k.180^\circ$. By way of example, consider the general solutions of the equations (a) $\sin x + \cos x = \frac{1}{\sqrt{2}}$ and (b) $\sin x + \cos x = -1$. Squaring both sides and simplifying in each case yields the following apparent solutions:

$$\text{(a)}\ \ x = 105^\circ + k.180^\circ \quad \text{or} \quad x = 165^\circ + k.180^\circ \ ,\ k \in \mathbb{Z}$$

$$\text{(b)}\ \ x = k.180^\circ \quad \text{or} \quad x = 90^\circ + k.180^\circ \ ,\ k \in \mathbb{Z}$$

The alternative and immediately correct approach yields the following solutions:

$$\text{(a)}\ \ x = 345^\circ + k.360^\circ \quad \text{or} \quad x = 105^\circ + k.360^\circ \ ,\ k \in \mathbb{Z}$$

$$\text{(b)}\ \ x = 180^\circ + k.360^\circ \quad \text{or} \quad x = 270^\circ + k.360^\circ \ ,\ k \in \mathbb{Z}$$

Noting these complete alternative solutions, the solutions obtained from squaring both sides need to be adjusted as follows:

$$\text{(a)}\ \ x = 105^\circ + 2k.180^\circ \quad \text{or} \quad x = 165^\circ + (2k+1).180^\circ \ ,\ k \in \mathbb{Z}$$

$$\text{(b)}\ \ x = (2k+1).180^\circ \quad \text{or} \quad x = 90^\circ + (2k+1).180^\circ \ ,\ k \in \mathbb{Z}$$

Note that similar care needs to be taken with equations in the form $\sin x - \cos x = m$.

CONCLUDING THOUGHTS

Squaring both sides of these equations may be used in a number of interesting questions, for example to determine the maximum and minimum values of $\sin x \pm \cos x$. If we let $\sin x + \cos x = t$ then:

$$\sin 2x = t^2 - 1 \ \rightarrow\ -1 \le t^2 - 1 \le 1 \ \rightarrow\ 0 \le t^2 \le 2 \ \rightarrow\ -\sqrt{2} \le t \le \sqrt{2}$$

Thus $\sin x + \cos x$ has a minimum value of $-\sqrt{2}$ and a maximum value of $\sqrt{2}$. It may similarly be shown that $\sin x - \cos x$ shares the same range.

It is hoped that consideration of this article may assist in correcting errors when solving trigonometrical equations of the form $\sin x + \cos x = m$ and its variations. Consideration of both methods in the classroom will make for interesting and illuminating insight into the nuances of the two approaches.

Another Student Discovery: The Quasi-Circumcentre and Quasi-Incentre of a Quadrilateral

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INTRODUCTION

A very gifted and inspirational South African mathematics teacher, Tickey de Jager (1921-2008), who taught at Rondebosch Boys' High School in Cape Town, many years ago advocated the naming of classroom discoveries after the students who proposed or discovered them. While such student discoveries are seldom original or novel, such a practice is nevertheless very encouraging and motivating for students. From personal experience I have certainly found over the years that students and learners at any level tend to work harder and much longer on problems they have discovered or formulated for themselves. It gives them a sense of personal ownership and nourishes their desire to solve such problems.

To enable such student or learner discoveries requires the regular use of open-ended classroom investigations during which they can be encouraged to ask their own 'what-if' questions (Brown & Walter, 1990). While engaging learners in creative problem solving through, for example, mathematics competitions like the SA Mathematics Olympiad (SAMO) is great, even greater benefits can be achieved by also helping students become problem posers. This can already be stimulated at primary school level, encouraging learners to critically reflect on a solved problem (in the style of Polya, 1945), considering what they have learnt from different ways of solving the problem to varying the conditions of the problem, or trying to generalize, specialize or apply it to other contexts.

To quote from Mason et al. (2010, pp. 139-140): "*From their earliest years, children can develop confidence to question, challenge and reflect. But they must be encouraged and reinforced in this. Their curiosity needs nurturing, their investigative potential structuring, their confidence maintaining. … If you are in a position to affect the learning of others, note how frequently you create the opportunity for* ***them*** *to think, to articulate their own questions, to challenge conjectures and to reflect on what has or has not been established.*"

To achieve this requires a mind-set change of the teacher, seeing him or herself as less of an authoritarian expert in the classroom and more of a collaborative facilitator. It means acknowledging that one sometimes doesn't know the answer straight away or beforehand, but to have the willingness to listen to and acknowledge pupils' questions and to work collaboratively on solving them. While time pressures, curriculum constraints and other factors often dampen spontaneous explorations that may arise in the classroom, learner exploration and discovery should be cherished and valued.

RENATE'S THEOREM ABOUT THE QUASI-CIRCUMCENTRE OF A QUADRILATERAL

Here is a personal example of one of the discoveries made by a student in a classroom discussion in 2006 at Kennesaw State University, while I was teaching on a visiting professorship there. Students were busy with the Water Supply task from De Villiers (1999) of finding the best position to build a water reservoir for four towns of more or less equal size. An interactive online sketch based on this activity is available at:
http://dynamicmathematicslearning.com/water-supply-four-towns.html

While the four given towns in the initial problem formed a cyclic quadrilateral, and hence had a unique equidistant point (the centre of the circle), the students had discovered from the activity (to their surprise) that not all quadrilaterals were cyclic. This raised the natural question: what would be the best position to build a water reservoir for four towns if they did not form a cyclic quadrilateral?

An undergraduate student, Renate Lebleu Davis, then proposed the intersection of the diagonals of the quadrilateral formed by the adjacent perpendicular bisectors of the (non-cyclic) quadrilateral as a possible solution (see Figure 2). While this was not the optimal solution I had in mind[2], I encouraged the class to further explore the properties of that point. Using dynamic geometry, the class very quickly came up with the following conjecture:

"Given a non-cyclic quadrilateral $ABCD$, let K, L, M and N be the respective circumcentres of triangles ABD, ABC, BCD and CDA, then the intersection O of KM and LN is equidistant from opposite vertices A and C, as well as equidistant from opposite vertices B and D. (Call this point O the *quasi-circumcentre*[3] of $ABCD$)".

Alternative formulation: let K, L, M and N be the respective intersections of the perpendicular bisectors of the adjacent sides of $ABCD$. For example, let K be the intersection of the perpendicular bisectors of sides AD and AB, etc.

A dynamic online sketch showing Renate's theorem is also available for readers at:
http://dynamicmathematicslearning.com/quasi-circumcentre-quad.html

But why was the result true? While the dynamic construction convinced them of the truth of the result no matter how they dragged the configuration, even into a concave as well as a crossed configuration, this empirical, experimental confirmation did not explain *why* the result was true.

Given the conceptual groundwork that had already been laid with the introductory activity about a perpendicular bisector of a line segment as the locus (path) of all points equidistant from the endpoints of the line segment (see Figure 1), it did not take long for the students, with some guidance, to come up with the following explanatory proof.

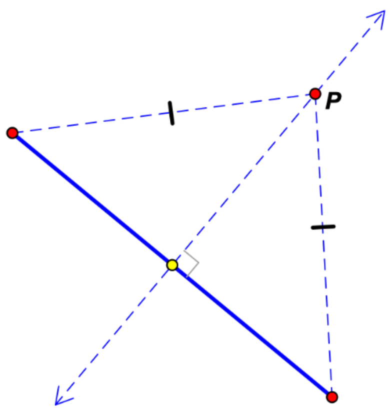

FIGURE 1: Perpendicular bisector as locus of equidistant points

[2] Mathematically, a more optimal solution can be obtained by minimizing the sum of the absolute values of all 6 differences between the four distances, which is equivalent to the least squares method.

[3] Initially we had chosen the name pseudo-circumcentre, but as it sometimes happens, later in the same year Myakishev (2006) proved the existence of a quasi-Euler line in relation to the same point, but calling it the quasi-circumcentre instead. This name seemed more appropriate, so we switched accordingly.

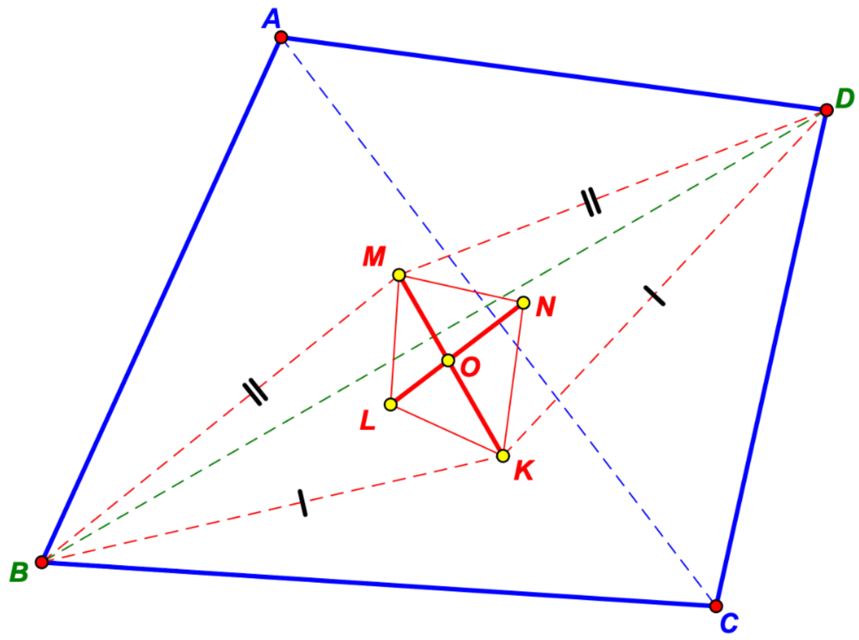

FIGURE 2: Quasi-circumcentre

Proof

Since both K and M lie on the perpendicular bisector of the diagonal BD, all points on the line KM are equidistant from B and D. Similarly, all points on the line LN are equidistant from A and C. Thus, the intersection O of lines KM and LN is equidistant from the two pairs of opposite vertices.

Renate's theorem was also later used in the Grade 11 Kennesaw State Mathematics Competition for High School students in 2007, as well as in the World InterCity Mathematics Competition for Junior High School students (up to Grade 9) in Durban in 2009. Noteworthy was that while the problem was experienced as one of most difficult ones for students participating in the Kennesaw competition, it was one of the easiest ones for the World Intercity competition with almost all primary and junior high school students from Asian countries getting full marks for it. None of the students from the South African team scored any marks for it. This clearly shows that – at least with respect to their mathematically talented learners – Asian countries appear to be engaging their students with much more in-depth, challenging geometry concepts and problems.

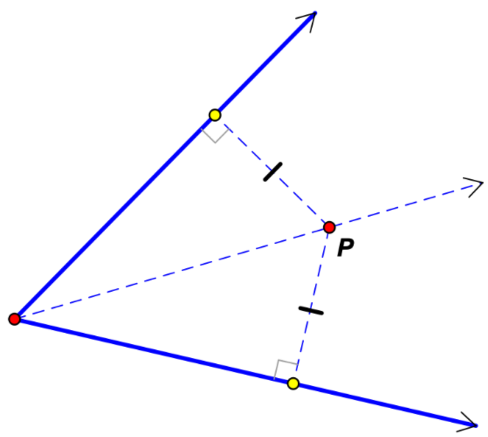

FIGURE 3: Angle bisector as locus of equidistant points

THE QUASI-INCENTRE OF A QUADRILATERAL

Since the angle bisector of an angle can, analogous to a perpendicular bisector, be seen as the locus of all points equidistant from the two rays forming the angle (see Figure 3), it was natural to next experimentally explore and formulate the following conjecture:

"Given a quadrilateral $ABCD$, construct the angle bisectors for each of the four angles as shown in Figure 4. Label E the intersection of the angle bisectors of angles A and B, label F the intersection of the angle bisectors of angles B and C, label G the intersection of the angle bisectors of angles C and D, and label H the intersection of the angle bisectors of angles D and A. Then the intersection I of EG and FH is equidistant from opposite sides AB and CD, as well as equidistant from opposite sides BC and DA. (Call this point I the *quasi-incentre* of $ABCD$)."

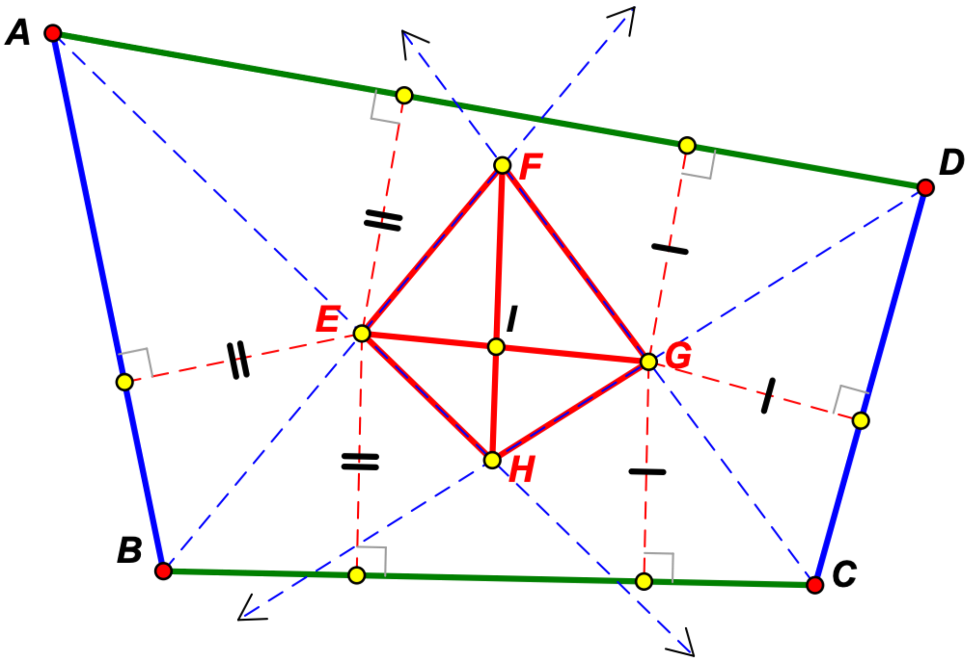

FIGURE 4: Quasi-incentre

Proof

Since E lies on both the angle bisectors of angles A and B, it is equidistant, by transitivity, from both AD and BC. Similarly, G is equidistant from AD and BC, and both H and F are equidistant from AB and CD. Hence, all points on the line EG are equidistant from AD and BC, and all points on the line FH are equidistant from AB and CD. Thus, the intersection I of EG and HF is equidistant from the two pairs of opposite sides.

A dynamic online sketch illustrating the quasi-incentre is also available at the same URL given earlier for the quasi-circumcentre.

CONCLUDING REMARKS

A recent study by Cai (2025, p. 164) reports that there has unfortunately been little progress in international curricula around the world to integrate problem posing into school mathematics, despite efforts by many over several decades. While it may sometimes feature in official policy statements, it has largely not penetrated the level of the intended curriculum – the prescribed curriculum, textbooks, and other learning materials – and even less at the level of the implemented (or attained) curriculum.

While Renate's theorem and its counterpart are not mathematically greatly significant, they appear to be fairly new and original. However, their value and significance for the students in the class was immeasurable – it gave them a sense of accomplishment and confidence in their own ability to discover and prove new mathematical results themselves. This little vignette also shows that it is possible for learners and students to be more active participants in classroom investigations.

It is therefore hoped that this little example will encourage other mathematics educators to create a fertile environment in their classes to provide similar opportunities for their learners and students to be creative and to ask and explore mathematical questions on their own or in collaboration with their teacher. Helping our learners and students become problem posers is as an important goal as developing their problem-solving skills. To quote Singer et al. (2013, p. 5): "*Problem posing is an old issue. What is new is the awareness that problem posing needs to pervade the education systems around the world, both as a means of instruction (…) and as an object of instruction* …"

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Suggestions to writers

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