TIMELINES AND ANNUITY FORMULAE
Pedagogical Content Tools in Financial Mathematics

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Financial mathematics is an elegant domain of mathematics. To fully understand how things fit together in the mathematics of finance, one needs to understand visually in both a relational and instrumental way. Financial mathematics provides a perfect example of how these two concepts introduced by Richard Skemp (1976) coexists in support of one another. In this article I will focus on using timelines and the annuity formulae as *pedagogical content tools* (Rasmussen & Marrongelle, 2006). A pedagogical content tool is a graph, an equation, a diagram or even a mathematical statement that the teacher uses to advance the understanding of some piece of mathematics. The focus in this article is the use of the timeline as a pedagogical content tool, so I assume that the reader is familiar with the financial mathematics formulae as well as the concepts of future (accumulated) value (F_V) and present (principal) value (P_V).

Timelines – a tool for understanding structure
Timelines in financial mathematics create the necessary structure that enhances both relational and instrumental understanding. The golden rule in problems where timelines are used is that all calculations need to converge to the same point on the timeline. The following example shows how useful the timeline is in building the problem structure and explicating what otherwise would have been hidden to the problem solver.

Example 1: R5 000 was invested into a fund that pays j% per annum compounded quarterly. After n quarters the fund stood at R6 736,76 and after a further 3 years, it stood at R10 535,91. Determine the values of j and n.

Solution: The timeline (Figure 1) represents the situation. The timeline clearly shows that the problem has two distinct components – a growth of n quarters for R5 000 to become R6 736,76. The second half of this investment runs for three years where the accumulated amount of R6 736,76 grows to R10 535,91 at the same rate.

Figure 1
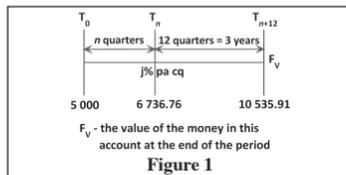
The timeline becomes the pedagogical content tool that creates the necessary structure that helps us decide on a strategy. The interest rate in our calculations will be the effective quarterly rate equivalent to a rate of j% per annum compounded quarterly, which is a nominal quarterly rate. The timeline shows that the final three years of this investment is the period where the rate j% per annum compounded quarterly, is the only unknown. Over the first n quarters both j and n are unknown, as is the case over the whole period of $n + 12$ quarters. We therefore will start by focusing on the last three years of this investment and we have a choice of working either in years or in quarters. Whichever period we choose to work with, the rate will be an effective rate per period. It is more time efficient here to work with the effective rate per quarter.

For the final three years:

$$F_V = P_V(1+i)^n \text{ where } i = \tfrac{j}{400}$$
$$\therefore 10535.91 = 6736.76(1+i)^{12}$$
$$\therefore (1+i)^{12} = 1.563943201$$
$$\therefore i = 0.037970669$$

Thus $j = 400 \times 0.037970669 = 15{,}19\%$ per annum compounded quarterly. This is the nominal quarterly rate. We need to work with an effective quarterly rate in order to calculate the value of n which is located in the first part of the timeline. We simply need to divide the rate by four to obtain the effective quarterly rate – that is the rate per quarter compounded quarterly. In our case we will simply work with the value of i as we calculated it in the above, as it is an effective quarterly rate.

To find n:

$$F_V = P_V(1+i)^n$$
$$\therefore 6736.76 = 5000(1.03797066897)^n$$
$$\therefore n = \tfrac{\log 1.347352}{\log 1.03797066897} = 8 \text{ quarters}$$

The timeline organises the information is such a way that the mathematical processes used to solve the problem, become visible. The next example highlights the choices that can be made when working with a timeline.

Example 2: R10 000 is originally invested for a period of ten years at an effective rate of 15% per annum. During the period of the investment, R2 000 was withdrawn at the end of the 15th month, R3 500 was invested at the end of the 27th month, and three years after the initial deposit, the interest rate was adjusted to 13% per annum compounded monthly. Determine the future value of this investment after ten years.

Solution: This example shows how the timeline transforms seamlessly to the structure of the problem that needs to be solved. Here the timeline as a pedagogical content tool effectively illustrates how we can move to any place in time as long as we work with the effective rates per period, and all calculations converge to the same point in time (on the timeline) – see Figure 2.

Figure 2
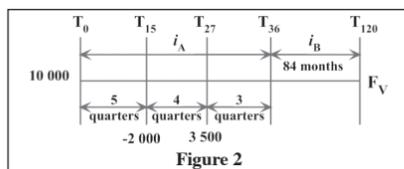
The effective quarterly rate i_A is obtained through the conversion formula:

$$\left(1+\tfrac{i_A}{4}\right)^4 = (1+i)$$
$$\therefore \tfrac{i_A}{4} = \sqrt[4]{1.15} - 1$$
$$\therefore i_A = 0.03555807634$$

The effective monthly rate i_B is given by the calculation $i_B = \tfrac{13}{1200} = 0.01083$. We can have the following equivalent scenarios from the timeline, which all lead to the same answer of $F_V = $ R40 893,47.

Moving to T_{36}: $\quad 10000(1+i_A)^{12} - 2000(1+i_A)^7 + 3500(1+i_A)^3 = F_V(1+i_B)^{-84}$

Moving to T_{120}: $\quad \{10000(1+i_A)^{12} - 2000(1+i_A)^7 + 3500(1+i_A)^3\}(1+i_B)^{84} = F_V$

Moving to T_0: $\quad 10000 - 2000(1+i_A)^{-5} + 3500(1+i_A)^{-9} = F_V(1+i_B)^{-84}(1+i_A)^{-12}$

Moving to T_{15}: $\quad 10000(1+i_A)^5 - 2000 + 3500(1+i_A)^{-4} = F_V(1+i_B)^{-84}(1+i_A)^{-7}$

Moving to T_{27}: $\quad 10000(1+i_A)^9 - 2000(1+i_A)^4 + 3500 = F_V(1+i_B)^{-84}(1+i_A)^{-3}$

Annuities and timelines
The standard annuity formulae that learners have to know are:

$$P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right] \quad \text{when working with the present value of an annuity, and}$$

$$F_V = x\left[\frac{(1+i)^n - 1}{i}\right] \quad \text{when working with the future value of an annuity.}$$

One very important aspect is to note that n in these formulae do not denote the period of the loan, but rather the number of payments made in the annuity structure.

These formulae are easy to remember and they provide useful *shortcuts* in the calculation phase. What needs to be understood is how the timelines relate to the formulae for the annuities.

Annuity Formula
Figure 3
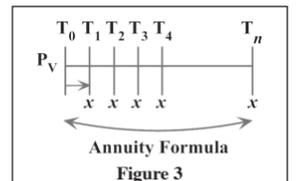
When working with the present value of an annuity (e.g. loans) one usually starts paying back at the end of the first period. So it is expected that there will be an *open period* before the payments, which amortises the loan, starts. The timeline is represented in Figure 3. At the beginning of the timeline, there is one period where *nothing happens*. This is the standard timeline for a scenario where the formula $P_V = x\left[\frac{1-(1+i)^{-n}}{i}\right]$ applies.

Any variation away from this happens *outside the annuity*, and is treated as normal compound interest using one of these formulas $P_V = F_V(1+i)^{-n}$ or $F_V = P_V(1+i)^n$. What follows are a few scenarios and how the present value annuity formula is adjusted.

Problem scenario 1: A loan is amortised by making monthly payments for 3 years into an account, starting at the end of the first month.
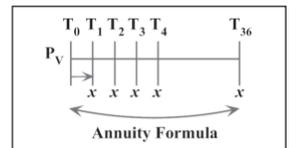
Annuity Formula

Formula: This is the standard formula with 36 payments in the annuity: $P_V = x\left[\frac{1-(1+i)^{-36}}{i}\right]$

Problem scenario 2: A loan is amortised over a three year period by making monthly payments starting six months after the loan is granted.

Deferred for 5 months · Annuity Formula
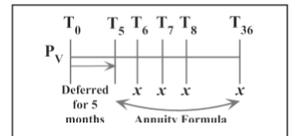
Formula: There are five months at the beginning where no payments are made. Payments start at the beginning of the sixth month. Thus a deferment of 5 months leaving the one space open for the annuity formula to apply. So 31 payments happen in the annuity:

$$P_V(1+i)^5 = x\left[\frac{1-(1+i)^{-31}}{i}\right]$$

Problem scenario 3: A loan is amortised by making 37 monthly payments of Rx starting immediately. Five additional payments of Ry are made at the end of each quarter for the full period of the loan.

Annuity for 36 payments of x
Annuity for 12 payments of y
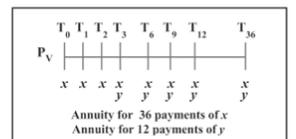
Formula: There are two separate annuities running simultaneously. Only 36 payments are in the annuity for x (the first payment is immediate and lies outside the annuity structure) and there are 12 payments in the annuity for y. i_A is the effective monthly rate and i_B is the effective quarterly rate.

$$P_V = x + x\left[\frac{1-(1+i_A)^{-36}}{i_A}\right] + y\left[\frac{1-(1+i_B)^{-12}}{i_B}\right]$$

For the future value formula, there is no space open. So it works on the notion that the annuity matures the moment the last payment is made.

To end
Using timelines as a pedagogical content tool provides the teacher with a didactical instrument that makes the structural aspects of financial mathematics visible in the class. It builds on both relational and instrumental aspects of the mathematical experience by using the structural aspects of financial mathematics.

To share your experience or to comment on this article, please feel free to email me at jacques.duplessis@wits.ac.za.

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