The Tangent to a Cubic Function and its Further Point of Intersection

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INTRODUCTION

In this article I consider the tangent to a cubic function at various points on the graph, and devise a simple method for calculating where the tangent will once again intersect the graph. In addition, I will illustrate that should no further intersection take place, then the tangent must exist at the point of inflection of the graph.

THE FUNDAMENTAL THEOREM OF ALGEBRA

Before looking at cubic functions specifically, let us first remind ourselves of the fundamental theorem of algebra. This effectively states that every polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$ and $n \geq 1$, will have at least one root in the complex number system. A corollary to this theorem states that the equation $P(x) = 0$ of degree n will have exactly n roots in the complex number system. These roots may be real or non-real, with equal roots being separately counted. Since non-real or complex roots always exist in pairs, for odd n, $P(x) = 0$ will always yield an odd number of real roots. Therefore, while an equation with even n may not have any real roots at all, one with odd n will always have at least one real root.

THE CUBIC FUNCTION

The cubic function is of the form $f(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$. The equation $f(x) = g(x)$, where g is the straight line function $g(x) = mx + e$, is of degree 3, which is odd, and must therefore yield either one or three real roots. Effectively then, the graphs of the two functions must intersect each another either once or three times. A double root exists at a point of tangency of the graphs of f and g, and hence, save for the tangent occurring at the point of inflection of f, the tangent *must* necessarily intersect the cubic graph at a second point.

THE TANGENT TO A CUBIC FUNCTION AND ITS FURTHER POINT OF INTERSECTION

Let us now consider the tangent to a general cubic function, $f(x) = ax^3 + bx^2 + cx + d$, at the point $T(x_1\,; f(x_1))$. The instantaneous gradient of the cubic function is $f'(x) = 3ax^2 + 2bx + c$. We can now use the formula $y - y_1 = m(x - x_1)$ to write the equation of the tangent as:

$$y - (ax_1{}^3 + bx_1{}^2 + cx_1 + d) = (3ax_1{}^2 + 2bx_1 + c)(x - x_1)$$

This simplifies to:

$$y = 3ax_1{}^2 x + 2bx_1 x + cx - 2ax_1{}^3 - bx_1{}^2 + d$$

We can now calculate the further point of intersection of the tangent and cubic graphs by equating the above equation of the tangent with the original cubic function:

$$3ax_1{}^2 x + 2bx_1 x + cx - 2ax_1{}^3 - bx_1{}^2 + d = ax^3 + bx^2 + cx + d$$

$$ax^3 + bx^2 - (3ax_1{}^2 + 2bx_1)x + 2ax_1{}^3 + bx_1{}^2 = 0$$

Now, since our point of tangency is $T\big(x_1\,;f(x_1)\big)$, we know that $x = x_1$ is a solution to this equation. We thus have $(x - x_1)$ as a factor, and by inspection we can complete the factorisation of this cubic equation as follows:

$$(x - x_1)\big(ax^2 + (ax_1 + b)x - (2a{x_1}^2 + bx_1)\big) = 0$$

$$(x - x_1)(x - x_1)(ax + 2ax_1 + b) = 0$$

This has solutions $x = x_1$ (twice) or $x = -2x_1 - \frac{b}{a}$.

The further point of intersection, P, between the function and the tangent is thus given by

$$P\left(-2x_1 - \frac{b}{a}\ ;\ f\left(-2x_1 - \frac{b}{a}\right)\right)$$

where x_1 is the x-coordinate of the point of tangency.

By way of example, let us consider the function $f(x) = 2x^3 + x^2 - 8x - 4$ which is shown below in Figure 1 (not drawn to scale).

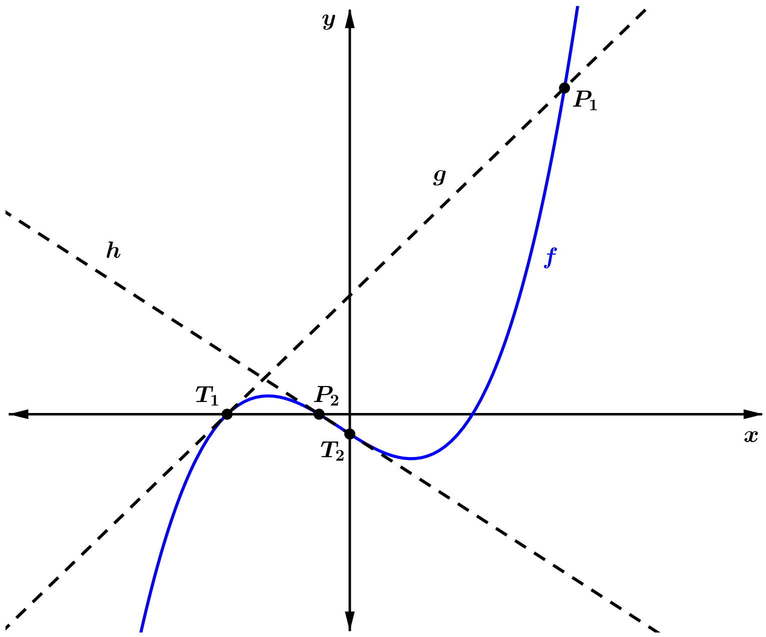

FIGURE 1: The graph of $f(x) = 2x^3 + x^2 - 8x - 4$

Let us consider tangents to f at $T_1(-2\,;0)$ and $T_2(0\,;\,-4)$. These tangents are respectively represented by g and h as illustrated in Figure 1. The points of further intersection of these tangents with the graph of f are P_1 and P_2 respectively. As established earlier, the x-value of the further point of intersection is given by $x = -2x_1 - \frac{b}{a}$. We thus have $x_{P_1} = -2(-2) - \frac{1}{2} = \frac{7}{2}$ leading to $P_1\left(\frac{7}{2};66\right)$.

We can cross-check result this by determining the equation of the tangent g and equating it to f. Since $f'(x) = 6x^2 + 2x - 8$, the gradient of the tangent g at the point T_1 is $f'(-2) = 12$. From this we can establish the equation of the tangent to be $g(x) = 12x + 24$. Equating this with the original cubic gives:

$$2x^3 + x^2 - 8x - 4 = 12x + 24 \ \rightarrow\ (x + 2)(x + 2)(2x - 7) = 0$$

This has solutions $x = -2$ (the point of tangency) or $x = \frac{7}{2}$, in agreement with the calculation above.

Similarly, $x_{P_2} = -2(0) - \frac{1}{2} = -\frac{1}{2}$ leading to $P_2\left(-\frac{1}{2}\,;0\right)$. The gradient of the tangent h at the point T_2 is $f'(0) = -8$. From this we can establish the equation of this particular tangent to be $h(x) = -8x - 4$. Equating this with the original cubic gives:

$$2x^3 + x^2 - 8x - 4 = -8x - 4 \;\; \rightarrow \;\; x^2(2x+1) = 0$$

This has solutions $x = 0$ (the point of tangency) or $x = -\frac{1}{2}$, in agreement with the calculation above.

THE TANGENT TO A CUBIC FUNCTION AT ITS POINT OF INFLECTION

The point of inflection of a cubic graph occurs where $f''(x) = 0$. Since $f''(x) = 12x + 2$ this means the point of inflection occurs at $x = -\frac{1}{6}$. Using our established fact that the further point of intersection of a cubic function and its tangent occurs at $x = -2x_1 - \frac{b}{a}$, we see that $x = -2\left(-\frac{1}{6}\right) - \frac{1}{2} = -\frac{1}{6}$. This confirms that there is no further intersection of the tangent and f, and there exists a triple root at $x = -\frac{1}{6}$. This may be further demonstrated by forcing the third root to be equal to the double root of the tangent:

$$-2x_1 - \frac{b}{a} = x_1 \;\; \rightarrow \;\; 3x_1 = -\frac{b}{a} \;\; \rightarrow \;\; x_1 = -\frac{b}{3a}$$

which is indeed the x-coordinate of the point of inflection of a cubic function. It is important to note that the tangent to a cubic function at its point of inflection is the only tangent which crosses the graph of the cubic function.

THE TANGENT TO A CUBIC FUNCTION OF THE FORM $k(x) = ax^3 + cx + d$

In this form the x-value of the further point of intersection of the tangent with the cubic graph collapses to $x = -2x_1$ since $b = 0$. By way of example, let us consider the function $k(x) = -4x^3 + 3x - 1$ and determine the further point of intersection of the tangent to k at the point where $x = 1$. Using $x = -2x_1$ we have $x = -2(1) = -2$, from which the further point of intersection is $P(-2\,;25)$.

Using the conventional approach, $k(1) = -2$ and $k'(1) = -9$. From this we can establish that the equation of the tangent is $y = -9x + 7$. Equating this with the cubic function k gives:

$$-4x^3 + 3x - 1 = -9x + 7 \;\; \rightarrow \;\; (x-1)(x-1)(x+2) = 0$$

from which $x = 1$ (the point of tangency) or $x = -2$, in agreement with the calculation above.

It is interesting to note that for cubic functions of this form, the point of inflection will always have the coordinates $(0\,;d)$, and the tangent at this point will be $y = cx + d$. In the above example the point of inflection is $(0\,;-1)$ and the tangent at this point is $y = 3x - 1$.

CONCLUDING COMMENTS

This article provides a simple method of calculating the coordinates of the further point of intersection of a tangent to a cubic function. It also illustrates that should the tangent occur at the point of inflection, no further intersection with the cubic function will occur, and the three real roots of the equation created by equating the function and the tangent will be equal.