Building Students’ Mathematical Proficiency: Connecting Mathematical Ideas Using the Tangram

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This article describes a classroom-tested approach that brings together different sub-domains of secondary school mathematics using the Tangram: Number Sense and Algebra (unpacking the idea of a square root and irrational numbers, understanding a reverse relationship between area and side length of a square); Geometry and Measurement (spatial reasoning, understanding properties of geometric shapes, measuring area and length).

The article also offers a holistic approach to the development of students’ mathematical proficiency. Too often, teachers treat major strands of mathematical proficiency (National Research Council, 2001) in isolation. The author evidenced many cases when teachers using good curriculum materials would overemphasize one strand - for instance, developing students’ conceptual understanding while paying little or no attention to building students’ strategic competence or other components of mathematical proficiency.

Development, selection and use of worthwhile connected activities in mathematics classrooms that link not only content sub-domains but also strands of mathematical proficiency is a challenging task for a middle school mathematics teacher. This paper illustrates one of the possible ways to face this challenge through the use of a set of connected activities using the Tangram. In the context of this paper, we consider the set of connected activities (SoCA) as a tool for building students’ mathematical proficiency across different mathematical sub-domains (number sense, algebra, geometry and measurement) and five strands of mathematical proficiency (conceptual understanding, procedural fluency, adaptive reasoning, productive disposition, and strategic competence). The SoCA also connects the main elements of mathematics classrooms: teaching, learning, and assessment.

Connectedness in mathematical classroom

National Council of Teachers of Mathematics Standards emphasize that “if curriculum and instruction focus on mathematics as a discipline of connected ideas, students learn to expect mathematical ideas to be related” (NCTM, 2000:275). Connectedness in the mathematics classroom is multifaceted (NCTM, 2000, 2006): across different mathematical ideas, across mathematics and other subjects, across multiple representations, etc. It is well documented that effective teachers have “a general intention to make connections among mathematical concepts and procedures, from simple and superficial connections between individual pieces of knowledge to complicated and underlying connections among different mathematical operations and sub domains. When reflected in teaching, this intention will prevent students’ learning from being fragmented. Instead of learning isolated topics, students will learn a unified body of knowledge” (Ma, 1999:122). The idea of connectedness in mathematics classrooms goes beyond the link between mathematical concepts and procedures. It extends to connection between teacher understanding of mathematical content and student learning: “If we are able to more profoundly connect our mathematical understanding to our growing understandings of how children learn mathematics, the fabric of our own teaching can become stronger” (Boaler and Humphreys, 2005:11). With regard to student learning, as students make connections and develop understanding of mathematical relationships “the fabric of their mathematical proficiency becomes ever more flexible and sturdy” (ibid).
Brief historical note

Tangram is one of the different types of rearrangement puzzles known in the history of humankind across the world. “The first known rearrangement puzzle dates to the third century B.C., was of Greek origin, and was called the ‘Loculus Archimedes’ or the ‘Stomachion’” (Slocum, 2004:11). The Stomachion puzzle had 14 pieces in the form of different polygons (mostly triangles and quadrilaterals) and it included two versions: rectangular and square puzzles. Since then, a variety of rearrangement puzzles were invented and used in different countries. In Japan, for example, a 7-piece rearrangement puzzle known as “Sei Shonagon’s Wisdom Plates” was used starting in the tenth century A.D. Later, in 1796, “The 14 Ingenious Pieces” puzzle was also invented in Japan. The main distinction of this puzzle compared to the previous ones was the inclusion of a circular piece along with other polygon pieces. Tangram is an old Chinese puzzle, one interpretation of which is known as “The Seven Clever Pieces”. According to some historians, the Tangram originated from the furniture set during the Song Dynasty (between 960–1279) and later became a set of wooden blocks for playing. The seven pieces are simple polygons: two small, one medium-size, and two large right isosceles triangles, a square, and a parallelogram. Even though several rearrangement puzzles were invented before and after Tangram, it remains the most popular rearrangement puzzle in the world.

Set of connected activities

The SoCA using the Tangram includes but is not limited to the following major classroom activities:

- introducing the Tangram: warm-up activity
- pre-assessment activity: constructing a square using 7 pieces
- learning about the Tangram pieces
- core activities: constructing squares using 1, 2, 3, 4, and 6 pieces
- student reflections and discussion
- review of pre-assessment activity and closure
- post-assessment activity: constructing a square using 5 pieces.

Let us briefly describe each of these SoCA activities with emphasis on the idea of connectedness across content sub-domains as well as across strands of students’ mathematical proficiency.

Introducing the Tangram

As a warm-up activity, a teacher may ask students to construct their own design using all 7 pieces of the Tangram. Students are usually very creative coming up with different designs of a house, a car, a person, a strange-looking tree, to name a few. The next step is to ask students to trace the borderline of the design they created and exchange sketches with a classmate so they can recreate each others’ designs. This warm-up activity, along with introducing students to the history of the Tangram, helps to build students’ positive productive disposition toward learning: it provides students with a chance to see the mathematics they are about to learn as being “sensible, useful, and worthwhile” with a belief that they can do it (National Research Council, 2001:116).

Pre-Assessment Activity

As an example of a pre-assessment activity, a teacher may ask students to construct a square using all 7 pieces. In order to be challenged by this task, let students work individually. In a university mathematics method class, more than half of the graduate students had difficulty completing this task in 15 minutes! Ask students to sketch the square they have constructed (not only the borderline but also the location of each piece in the square). Students will have the opportunity to share their sketches and strategies later during the pre-assessment review session.
Learning about the Tangram Pieces

A critical element of the SoCA is to engage students in learning and understanding properties of the Tangram pieces. A teacher may involve students in a whole class discussion posing the following questions or prompts:

- Look at each piece and tell what shape it is.
- Are some pieces congruent? Show pieces that are congruent and explain why they are congruent.
- Are some pieces similar? Select pieces that are similar and explain why they are similar.

The centerpiece of learning about the Tangram pieces is to fill out the table including measures of area and the side lengths of each piece. To have consistency with measurements let students consider the square piece as a unit square. An example of a completed table is shown in Figure 1. It is recommended that this stage of the SoCA be conducted as a whole class teacher-facilitated activity.

This element of the SoCA aims at developing students’ procedural fluency in carrying out procedures of recalling properties of different shapes as well as measuring areas and side lengths of different Tangram pieces accurately and efficiently. While completing the table students also have an opportunity to see relationships between different Tangram pieces. Ability to visualize connections across shapes (such as “a square is made of two congruent right triangles”) contributes to students’ conceptual understanding of properties and relationships between geometric shapes.

![Table]

**Figure 1.** Sample of student work completing the table

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Core Activities

The core activities continue to build on students’ conceptual understanding and procedural fluency through working with geometric shapes as well as applying and transferring properties of shapes to a new situation. However, the main goal of this stage is to start building students’ strategic competence. Students have an opportunity to solve a sequence of mathematical tasks constructing squares using 1, 2, 3, 4, and 6 pieces. Students are asked to sketch squares (including interior lines) and look for all possible solutions. At this stage of the SoCA, students are allowed to work in small groups and share their ideas and strategies with others. Students may be asked to come to the board and present their solutions to peers and justify their methods. Solutions to core activities for constructing squares with 1, 2, 3 and 4 pieces are shown in Figure 2 and Figure 3. The cases with 1 and 3 pieces have only one possible solution, whereas the case with 2 pieces has two solutions and the case with 4 pieces has 3 solutions.

![Figure 2. Squares made of 1, 2, and 3 Tangram pieces](image1)

![Figure 3. Squares made of 4 Tangram pieces](image2)

Student Reflections

After students complete the core activities and discuss strategies they used to construct squares with different numbers of Tangram pieces, a teacher may invite students to reflect on their learning. The reflection may include the following questions:

- What strategies and methods did you use to construct different squares? Write your response in the journal log.
- One arrangement in constructing a square is not possible. Which one is it and why is it not possible? Write your response in the journal log.

Based on observations of student reasoning during the core activities and students’ reflections, the author came up with the following categorization to reflect different levels of students’ strategic competence while they were involved with a set of the Tangram activities:

- Level 0: Non-strategic competence: random trial
- Level 1: Pre-strategic competence: attempt to strategize
- Level 2: Partial strategic competence
- Level 3: Strategic competence

The most frequently used strategy for students at Level 0 was “moving pieces around”. A sample of student reflection at Level 0 is shown in Figure 4.
An example of student strategic competence at Level 1 may be a “corners first” strategy where students attempt to construct a square starting from the corners (see Figure 5). At this level students try to use one of the basic properties of a square: all 4 interior angles of a square are congruent right angles.

Students’ strategic competence at Level 2 may be illustrated by the following strategy: using properties of the shapes, for example “duplicating triangles” where students attempt to construct a square based on the property that a diagonal of a square divides it into two congruent isosceles right triangles (see Figure 6).

Students’ Level 3 strategic competence may be characterized by using the relationship between area and length of side of the square where students construct a square based on the following two steps:

a) Identifying area of a hypothetical square that could be constructed using the given number of Tangram pieces;

b) Finding side length of the hypothetical square and constructing the square based on the calculated length of the side of the square.

An example of student reflections transitioning from Level 2 to Level 3 of strategic competence is presented in Figure 7.
Level 3 strategic competence is based on the idea of connected reversibility. Too often students use a direct approach to find the area of a square based on a known value for length of a side. Less frequently students are involved in reversing the process: finding the length of the side of a square based on a known value for its area. Moreover, these two processes are often considered in isolation. In the SoCA using the Tangram we treat these two processes in connection, which helps students to improve their strategic competence through reversibility of their mental actions (see Figure 8).

Review of Pre-Assessment Activity
After students’ reflection and whole class discussion on effective strategies, it may be helpful to review the case of constructing a square using 6 pieces and the pre-assessment activity before students take the post-assessment test. Following the Level 3 steps for strategic competence, a teacher may involve students in a whole class discussion on why it is impossible to construct a square with 6 pieces. According to the table (see Figure 1) there are 3 different values for areas of the Tangram pieces considering the fact that area of a square piece is a unit square:

- Two congruent small right isosceles triangles have areas equal to 0.5 unit squares
- A square, a parallelogram and medium size right isosceles triangle have areas equal to 1 unit square
- Two congruent large right isosceles triangles have areas equal to 2 unit squares.

Based on this information, we identify 3 types of hypothetical squares that could be constructed using 6 pieces: (1) a square with an area of 7.5 unit squares (total area of seven Tangram pieces that is equal to 8 unit squares “minus” one of the pieces of a small right triangle with the area of 0.5 unit squares); (2) a square with an area of 7 unit squares (total area of seven Tangram pieces that is equal to 8 unit squares
“minus” one of the pieces (either square, parallelogram, or one medium right triangle) with the area of 1 unit square); (3) a square with an area of 6 unit squares (total area of seven Tangram pieces that is equal to 8 unit squares “minus” one of the pieces of the large right triangle with the area of 2 unit squares). The length of the side of the first hypothetical square with area of 7.5 is equal to \(\sqrt{7.5}\). Accordingly, the side length of the second square should be \(\sqrt{7}\) and the third \(\sqrt{6}\). Notice that none of these side lengths of the 3 hypothetical squares can be composed using side lengths of Tangram pieces found in the table (see Figure 1): 1, \(\sqrt{2}\), 2, and \(2\sqrt{2}\). Therefore, it is impossible to construct a square using 6 pieces.

Similarly, a discussion on the pre-assessment review (constructing a square using all 7 pieces) can be based on the same strategy: since the area of the square-to-be-constructed is 8 unit squares, then its side length is equal to \(\sqrt{8}=2\sqrt{2}\). So, it is reasonable to start the construction using hypotenuses of large right triangles as side lengths for the new square (see Figure 9). At the same time, two large triangles make half of the square-to-be-constructed. This starting point helps students bring together the remaining Tangram pieces to complete the square.

**Figure 9. Square made of 7 Tangram pieces**

**Figure 10. Square made of 5 Tangram pieces**

**Post-Assessment Activity**

As a post-assessment activity, a teacher may assign students a task to construct a square using 5 pieces. Students are asked to work individually and sketch a square (not only the borderline, but also the location of each piece in the square).

Our observation showed that students at Level 3 of strategic competence were capable of identifying 5 pieces (excluding the two large right triangles) to construct a square with a hypothetical area of 4 unit squares. Knowing the area of the square-to-be-constructed, students may choose the side length of 2 that could be represented, for instance, by the hypotenuse of a medium size right triangle (see Figure 1). After these first two key steps, students usually finish the task very quickly (see Figure 10).

**Discussion**

As was mentioned earlier, the SoCA using the Tangram connects not only content sub-domains but also components of student mathematical proficiency. The Tangram-SoCA helps to build students’ comprehension of mathematical concepts, operations, and relations (conceptual understanding); develops students’ skills in carrying out procedures flexibly, accurately, efficiently, and appropriately (procedural fluency); improves students’ capacity for logical thought, reflection, explanation, and justification (adaptive reasoning); engages students into activities that makes mathematics sensible, useful, and worthwhile (productive disposition); and builds students’ ability to formulate, represent, and solve mathematical problems (strategic competence). The relationships between the main stages of the Tangram-SoCA and components of mathematical proficiency are shown in Table 1.

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Table 1. Building students’ mathematical proficiency through the SoCA-Tangram stages

<table>
<thead>
<tr>
<th>#</th>
<th>The SoCA-Tangram Stages</th>
<th>Mathematical proficiency</th>
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<tbody>
<tr>
<td>1</td>
<td>Warm Up Activity</td>
<td>CU +</td>
</tr>
<tr>
<td>2</td>
<td>Pre-Assessment Activity</td>
<td>PF + AR + PD + SC</td>
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<tr>
<td>3</td>
<td>Learning about the Tangram Pieces</td>
<td>+ + + +</td>
</tr>
<tr>
<td>4</td>
<td>Core Activities</td>
<td>+ + + +</td>
</tr>
<tr>
<td>5</td>
<td>Student Reflections and Discussion</td>
<td>+ +</td>
</tr>
<tr>
<td>6</td>
<td>Pre-Assessment Review and Closure</td>
<td>+ +</td>
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<tr>
<td>7</td>
<td>Post-Assessment Activity</td>
<td>+ +</td>
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</tbody>
</table>

Legend: CU – conceptual understanding; PF – procedural fluency; AR – adaptive reasoning; PD – productive disposition; SC – strategic competence

References


Proof Without Words: Parahexagon-Parallelogram Area Ratio

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Given a convex hexagon $ABCDEF$ with opposite sides equal and parallel, with $G$, $H$, $I$ and $J$ being the respective midpoints of sides $AB$, $BC$, $DE$ and $EF$, prove that area $ABCDEF = 2 \times $ area $GHIJ$. 