Factorization remains a critical component of the South African school Mathematics curriculum. Factorization is used not only for simplifying algebraic expressions but also for determining the roots of equations and for determining the x-intercepts of graphs. However, factorization continues to be an area which learners find particularly problematic. In spite of having a basic conceptual understanding of a common factor, many learners have difficulty in getting to grips with the factorization of quadratic trinomials, procedurally as well as conceptually. Although there are many different approaches to factorizing quadratic trinomials, the one most often employed at schools involves time-consuming trial-and-error methods, a process which often leaves pupils unconvinced and anxious. This article engages with a number of different approaches to factorizing quadratic trinomials which teachers may find useful.

The method of “splitting the middle term”

The “splitting the middle term” method represents an attractive alternative. This method offers a structured procedural approach and supports the conceptual understanding of factorization as a process.

The basic method proceeds as follows: given a quadratic trinomial of the form $ax^2 + bx + c$, split the middle term $bx$ into two terms which when multiplied will be equal to the product of the other two terms of the quadratic trinomial. In other words, try to find two terms whose sum is $bx$ and whose product is $acx^2$. This will lead to the quadratic trinomial being able to be factorized pair-wise, from which the original trinomial can be expressed as the product of two factors.

The method in action

Let us consider the quadratic trinomial $6x^2 - 5x - 4$. First we determine the product $ax^2 \times c$, which in our case is $(6x^2)\times(-4) = -24x^2$. The next step involves re-writing the middle term of the trinomial, in this case $-5x$, as the SUM of two terms whose PRODUCT is $-24x^2$. The easiest way to accomplish this is to make a list of factor pairs as shown in Figure 1. Since the sum of the two terms needs to give a negative answer ($-5x$) we know that the larger of the two numbers must be negative:

<table>
<thead>
<tr>
<th>1</th>
<th>-24</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-12</td>
</tr>
<tr>
<td>3</td>
<td>-8</td>
</tr>
<tr>
<td>4</td>
<td>-6</td>
</tr>
</tbody>
</table>

Figure 1: Appropriate factor pairs for $-24$
The factor pair that adds to give the middle term \((-5x)\) of the trinomial is \(3x - 8\). We now have two terms, namely \(3x\) and \(-8x\), whose sum is \(-5x\) and whose product is \(-24x^2\). We can now re-write the original trinomial \((6x^2 - 5x - 4)\) as \(6x^2 - 8x + 3x - 4\). This will facilitate pair-wise factorization as illustrated below:

\[
\begin{align*}
6x^2 - 5x - 4 &= 6x^2 - 8x + 3x - 4 \\
&= 2x(3x - 4) + (3x - 4) \\
&= (3x - 4)(2x + 1)
\end{align*}
\]

**Direct factorization**

Rather than factorizing by pairing, it is possible to factorize directly into two binomials from \(6x^2 - 8x + 3x - 4\) by using the following approach. Firstly, since \(-8x\) and \(+3x\) are generated from the product of the outer pair and inner pair of terms in the binomial expansion, we can assign them each to a product pair:

\[
\begin{array}{c}
-8x \\
\hline
\begin{array}{c}
2x \\
+3x
\end{array}
\end{array}
\]

**Figure 2: Assigning product pairs**

To determine the first term in the first bracket all one needs to do is work out the highest common factor (HCF) of \(-8x\) and \(6x^2\) (the first term of the original trinomial). In this case the HCF is \(2x\), so this becomes the first term in the first bracket. Once this has been established then the remaining terms in the brackets can be determined by logical reasoning since \(6x^2\), \(-8x\), \(+3x\) and \(-4\) represent the products of the **Firsts, Outers, Inners and Lasts** respectively.

\[
\begin{array}{c}
-8x \\
\hline
\begin{array}{c}
2x \\
+3x
\end{array}
\end{array}
\]

**Figure 3: Steps 1 to 4**

**The box method**

A slight variation of the “direct factorization” method involves scaffolding the process by means of a \(2 \times 2\) box. Once the original trinomial has been written in the form \(6x^2 - 8x + 3x - 4\), create a \(2 \times 2\) box and place \(6x^2\) in the top left corner and \(-4\) in the bottom right corner. The remaining terms, \(-8x\) and \(+3x\), can be placed in either of the two remaining cells.

\[
\begin{array}{c|c}
6x^2 & +3x \\
\hline
-8x & -4
\end{array}
\]

**Figure 4: Positioning the terms in a \(2 \times 2\) box**
To proceed, determine the HCF for each row and each column, writing in each HCF as shown in Figure 5:

\[
\begin{array}{ccc}
3x & 6x^2 & +3x \\
-4 & -8x & -4 \\
2x & +1 & \\
\end{array}
\]

Figure 5: Determining the HCF for each row and each column

The sum of the factors of the columns and the sum of the factors of the rows give the terms in the two brackets of the factorized trinomial, i.e. \((3x-4)(2x+1)\). A word of caution: it is important to remember that before using the box method one should first take out any common factors from the original trinomial. As an experiment, try using the box method without first taking out a common factor and it should immediately become apparent why it is critical to do so!

We can now summarize the method of “splitting the middle term” by means of the following flowchart. Take note that the flowchart assumes that all common factors have been taken out of the original quadratic trinomial.

\[
ax^2 + bx + c
\]

Split the middle term \(bx\) into the form \(px + qx\) where \(p + q = b\) and \(pq = ac\)

\[
ax^2 + px + qx + c
\]

Factorise pair-wise Factorise directly into brackets Factorise using box method

The a-c method (an algorithmic approach)

The a-c method is an interesting alternative approach to factorizing complex quadratic trinomials (i.e. quadratic trinomials of the form \(ax^2 + bx + c\) where \(a \neq 1\)) and could be worked into an investigation or short portfolio item. To begin, determine the product \(a \times c\). Now change the original trinomial by (i) making the co-efficient of \(x^2\) (i.e. \(a\)) equal to 1, and (ii) changing the constant term (i.e. \(c\)) to the product \(a \times c\).

\[
6x^2 - 5x - 4 \rightarrow x^2 - 5x - 24
\]

The “complex” trinomial is thus converted to a simple trinomial which is much easier to factorize:

\[
x^2 - 5x - 24 \rightarrow (x-8)(x+3)
\]

One now multiplies the \(x\) in each of the two brackets by the original value of \(a\) - i.e. 6 in this particular case:

\[
(x-8)(x+3) \rightarrow (6x-8)(6x+3)
\]

Dividing each bracket by its HCF gives the final factorized expression of the original quadratic trinomial. Thus, dividing the first bracket by 2 and the second bracket by 3 gives \((3x-4)(2x+1)\).

To understand why this method works it is necessary to appreciate that it represents a “short-cut” to the technique shown in full on the following page.

Learning and Teaching Mathematics, Volume 11, 2011, pp. 18-21
Given the quadratic expression \(6x^2 - 5x - 4\), make the substitution \(k = 6x\) (or in general \(k = ax\)).

Thus, \(6x^2 - 5x - 4 = \frac{k^2 - 5k - 24}{6} = \frac{(k - 8)(k + 3)}{6} = \frac{(6x - 8)(6x + 3)}{6} = \frac{2(3x - 4)(3x + 1)}{6} = (3x - 4)(2x + 1)\)

A word of caution: as with the box method, it is important to remember that before using the a-c method one should first take out any common factors from the original trinomial. Once again, try using this method without first taking out a common factor and it should immediately become apparent why it is critical to do so!

Let us now consider a formal explanation for why the a-c method works in general. Suppose that we have a quadratic trinomial of the form \(ax^2 + bx + c\) that we wish to factorize into the form \((px + q)(ux + v)\). Multiplying out this factorized form gives \(pux^2 + (pv + qu)x + qv\) which, on comparison with the original trinomial, gives \(a = pu\), \(b = (pv + qu)\), and \(c = qv\). Let us now apply the a-c method to \(ax^2 + bx + c\).

Changing the original trinomial by making the co-efficient of \(x^2\) equal to 1 and changing the constant term to the product \(a \times c\) gives:

\[ax^2 + bx + c \rightarrow x^2 + bx + ac = x^2 + (pv + qu)x + (pu)(qv)
\]

\[= x^2 + (pv + qu)x + (pv)(qu)
\]

\[= (x + pv)(x + qu)
\]

Multiplying the \(x\) in each of the two brackets by the original value of \(a\) gives:

\[(ax + pv)(ax + qu)
\]

\[= (pux + pv)(pux + qu)
\]

Finally, dividing each bracket by its HCF gives the required factorized form, \((ux + v)(px + q)\).

**Concluding comments**

The intention of this paper was to provide alternative approaches to the factorization of trinomials – a section of the school curriculum with which learners often struggle. We have found that learners are far more confident about their factorization skills when using one or other of the methods presented here, and we encourage teachers to try out these approaches.

This work is based upon research supported by the FirstRand Foundation Mathematics Education Chairs Initiative of the FirstRand Foundation, Rand Merchant Bank and the Department of Science and Technology.

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