Investigating the Surface Area and Volume of Rectangular Prisms

Senzeni Mbedzi[1] & Duncan Samson[2]

[1]Mary Waters High School, Grahamstown

[2]FRF Mathematics Education Chair, Rhodes University, Grahamstown

[1]mbedzisenzeni@yahoo.com [2]d.samson@ru.ac.za

INTRODUCTION

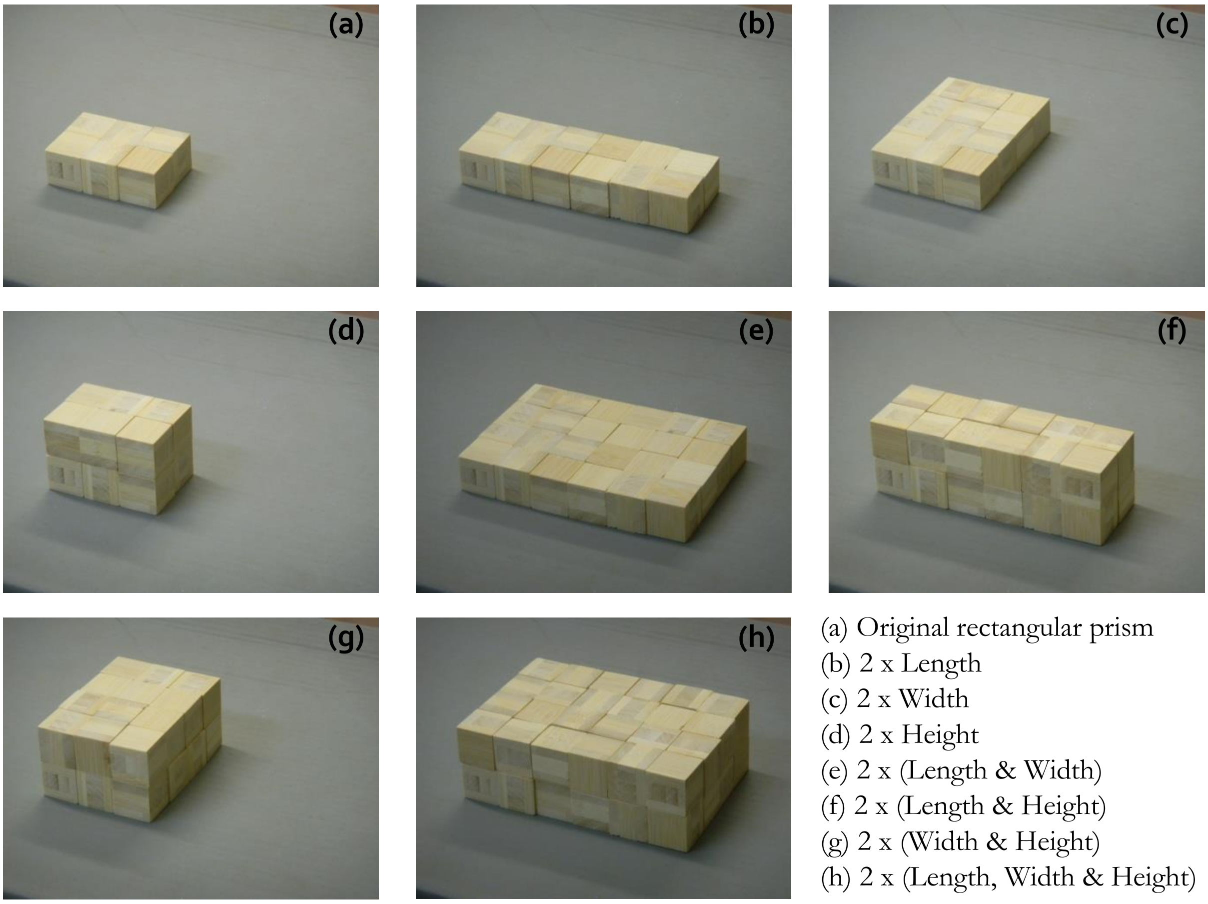
This paper describes a classroom investigation focusing on surface area and volume of rectangular prisms. The purpose of the activity is to investigate, with respect to the total surface area and volume, the effect of changing one, two or three dimensions of a rectangular prism by a given factor.

Learners were divided into small groups, and each group was provided with a pile of identical wooden cubes. Prior to beginning the activity it was agreed upon that areas and volumes would be measured in square units and cubic units respectively, where the unit in question was the side length of the wooden cubes. Each group was then tasked with deciding on the dimensions of their initial rectangular prism which they then built from the wooden cubes. Each group was also provided with a particular scaling factor (2, 3 or 4) as well as a worksheet on which to record their results.

By way of example, let us consider a scaling factor of 2 for an initial rectangular prism with length 3 units, width 2 units, and height 1 unit. The results of the investigation are summarised in Table 1, with the corresponding images shown in Figure 1.

TABLE 1: Changing one, two or three dimensions by a factor of 2.

	Figure	Length	Width	Height	TSA	Volume
Original rectangular prism	1(a)	3	2	1	22	6
2 x Length	1(b)	6	2	1	40	12
2 x Width	1(c)	3	4	1	38	12
2 x Height	1(d)	3	2	2	32	12
2 x (Length & Width)	1(e)	6	4	1	68	24
2 x (Length & Height)	1(f)	6	2	2	56	24
2 x (Width & Height)	1(g)	3	4	2	52	24
2 x (Length & Width & Height)	1(h)	6	4	2	88	48



(a) Original rectangular prism
(b) 2 x Length
(c) 2 x Width
(d) 2 x Height
(e) 2 x (Length & Width)
(f) 2 x (Length & Height)
(g) 2 x (Width & Height)
(h) 2 x (Length, Width & Height)

FIGURE 1: Changing one, two or three dimensions by a factor of 2.

After completion of the activity the individual groups reported back to the class and a general pattern was established from the results. Individual groups who had a scaling factor of 2 noticed that the volume of the prism increased by (i) a factor of 2 when only one dimension was changed, (ii) a factor of 4 when two dimensions were changed, and (iii) a factor of 8 when all three dimensions were changed. Groups who had a scaling factor of 3 noticed that the volume of the prism increased by (i) a factor of 3 when only one dimension was changed, (ii) a factor of 9 when two dimensions were changed, and (iii) a factor of 27 when all three dimensions were changed. For groups with a scaling factor of 4, the volume increased by a factor of 4, 16 and 64 respectively. From this we were able to establish the more general result, namely that given a scaling factor k, the volume of the original prism increased by (i) a factor of k when only one dimension was changed, (ii) a factor of k^2 when two dimensions were changed, and (iii) a factor of k^3 when all three dimensions were changed.

Trying to find a pattern for the total surface area was somewhat more problematic. When only one or two dimensions are changed, the increase in surface area is dependent on the specific length, width and height of the original prism. However, when all three dimensions are increased by a factor one could establish the general result that the total surface area increases by a factor of k^2. In Table 1 we can see that given a 3 by 2 by 1 rectangular prism and a scaling factor of 2, the original surface area of 22 square units increases to 88 square units, i.e. an increase by a factor of 2^2.

REFLECTION

Some aspects of this activity worked really well. Having each group choose their own dimensions for the initial rectangular prism means that different groups with the same scaling factor will be able to compare results and establish that the factor by which the volume increases is independent of the starting dimensions. Having different groups with different scaling factors is also very effective as it allows two different levels of generality to be established. The first level is noticing that all groups with the same scaling factor (but different starting dimensions) scale the volume in the same ratio. The second level of generalisation is noticing the general relationship between the volume of the scaled up prism and the original prism in terms of k, where k is the scale factor.

Another aspect of this investigation that worked well was the level of critical thinking needed to decide on the original starting dimensions. Although this might seem somewhat trivial, for groups who had been given a scaling factor of 3, and particularly those with a factor of 4, there were important practical issues to consider. For example, consider a group with a scaling factor of 4 deciding on starting dimensions of length 5 units, width 3 units, and height 2 units. By scaling all three dimensions by a factor of 4, the resultant prism would have length 20 units, width 12 units, and height 8 units. To build such a rectangular prism would require 1920 individual wooden cubes, which would be rather impractical. By asking groups to think carefully about their choice of starting dimensions – i.e. by asking them to think about whether they would have enough wooden cubes to build the various prisms – learners were already critically engaging with the notion of volume before physically building the structures.

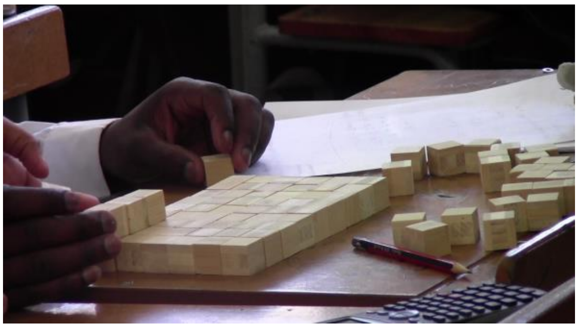
This investigation could of course just as easily be carried out without the use of physical manipulatives to build the various prisms. Completing a spreadsheet such as that shown in Table 1 could readily be accomplished simply by using the standard formulae for volume and surface area of rectangular prisms:

$$Volume = Length \times Width \times Height$$

$$Surface\ Area = 2 \times (Length \times Width + Width \times Height + Height \times Length)$$

However, there is certainly something to be said for the use of physical manipulatives. Firstly, as the learners build the different prisms by scaling up the various dimensions, patterns start to emerge in terms of the physical structures themselves. For example, if we consider the prisms shown in Figure 1, the starting dimensions are 3 by 2 by 1 (Figure 1a), giving a total of 6 wooden blocks (i.e. 6 cubic units). When the length is scaled up by a factor 2 the resulting prism (Figure 1b) comprises two of the original 6-block units, one next to the other. When the width is increased by a factor 2 the resulting prism (Figure 1c) likewise comprises two of the original 6-block units, one behind the other. A similar situation holds when the height is increased by a factor 2, the extra 6-block unit now being positioned on top of the original structure (Figure 1d). When two dimensions are changed (Figures 1e, f & g) the resultant structure clearly comprises four of the original 6-block units. Finally, when all three dimensions are increased by a factor of 2 the resultant rectangular prism can be seen as being comprised of eight of the original 6-block units. These observations give strong visual support to the numeric results.

Using physical blocks to build the various prisms rather than calculating volumes and areas using the standard formulae is also a very useful way of engaging learners with the idea of volume and surface area prior to establishing the standard formulae. The beauty of working with wooden cubes when investigating area and volume is that the total surface area can be calculated by simply counting the number of square faces on all six sides of the prism. Similarly, the volume can be calculated directly by counting the total number of wooden cubes in the structure. There are two extremely useful aspects to this process. Firstly, as learners calculate surface area and volume by counting the number of faces and cubes respectively, they should be encouraged to try to establish the most economical strategy for carrying out the counting process, thereby forging important conceptual links to the conventional formulae (Chiphambo, 2012). The second important aspect relates to the units of area and volume. By building the prisms from physical cubes (measuring 1 unit by 1 unit by 1 unit) it is quick to establish a meaningful definition of surface area and volume. Area can be defined as the number of identical square faces on the outer surface of the prism, while volume can be defined as the total number of wooden cubes making up the prism. This establishes a strong conceptual basis for talking about other measurements of area and volume such as cm^2 and cm^3 since one can immediately visualise building the prisms from wooden cubes measuring 1 cm by 1 cm by 1 cm.



CONCLUDING COMMENTS

This activity was designed to engage learners in a meaningful hands-on investigation through the use of simple physical manipulatives. The activity proved successful in engaging learners with the fundamental idea of area and volume. In addition, the investigation was a useful means of establishing generalities by comparing data generated from different groups. The use of physical manipulatives was also very powerful in terms of establishing visual patterns in the physical structure of the prisms themselves. Although a great number of wooden cubes are needed for this kind of activity, the pay-off in terms of conceptual development makes it an extremely worthwhile endeavour. Furthermore, as an investment for the classroom, wooden cubes represent a wonderfully versatile manipulative that can be used to support a wide variety of conceptual domains, including area and volume, symmetry, generalisation, strategy, logic, simple number theory, visual reasoning, and mathematical communication.

REFERENCES

Chiphambo, S. (2012). The role of physical manipulatives in teaching and learning measurement. *Learning and Teaching Mathematics, 13*, 3-5.

ACKNOWLEDGEMENT

This work is supported by the FRF Mathematics Education Chair hosted by Rhodes University, funded by the FirstRand Foundation Mathematics Education Chairs Initiative of the FirstRand Foundation, Rand Merchant Bank and the Department of Science and Technology.