Exploring the Difference of Two Squares

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One of the aspects of mathematics that I particularly enjoy is how a simple idea can often lead to a wealth of mathematical exploration. One such idea is the classic 'difference of two squares'. Within the school curriculum the difference of two squares is usually introduced as a specific form of factorisation. Learners usually become quite proficient at identifying the classic 'difference of two squares' structure and quickly learn how to factorise such expressions, for example $x^2 - 9$ factorises to $(x+3)(x-3)$ and $4x^2 - 1$ factorises to $(2x+1)(2x-1)$. However, despite the general proficiency exhibited with this kind of task, I suspect that for many learners the process is merely one of spotting a well-known algebraic structure and applying a practised algorithmic procedure with very little conceptual engagement with, or meaningful understanding of, the fundamental idea of the 'difference of two squares'. What I hope to share in this article is how by focusing on the underlying concept, rather than the algebraic process itself, one can open up a treasure trove of mathematical ideas, the exploration of which has the potential to create meaningful links between different areas of mathematics and thereby provide meaningful insight into the fundamental idea of the 'difference of two squares'.

AN INITIAL NUMERIC CHALLENGE

Let's begin by exploring the following simple question: "*How many natural numbers can be expressed as the difference of two squares?*" One method of exploring this question would be to pick a natural number and by a process of trial and error try to find a pair of perfect squares whose difference gives the chosen natural number. Perhaps a more systematic approach would be to list the first twenty or so perfect squares and to investigate potential patterns generated from their differences. Below are the first twenty perfect squares:

$$1 ; 4 ; 9 ; 16 ; 25 ; 36 ; 49 ; 64 ; 81 ; 100 ; 121 ; 144 ; 169 ; 196 ; 225 ; 256 ; 289 ; 324 ; 361 ; 400$$

In the above list I haven't included zero as a perfect square, but one could of course do so if one wanted. Notice that the difference between consecutive terms generates the sequence of odd numbers, namely $3, 5, 7, 9, 11 \ldots$ This sequence would of course have started with 1 if we had included zero in our sequence of perfect squares. To understand why the difference between consecutive terms always generates an odd number we could carry out a bit of algebraic exploration. Algebraically we can express the difference between consecutive perfect squares as $(n+1)^2 - n^2$ where $n \in \mathbb{N}$. This expression easily simplifies to $2n + 1$ which of course generates the sequence $3, 5, 7, 9, 11 \ldots$ for $n \in \mathbb{N}$. Although the algebra shows why the sequence of odd numbers is generated, greater insight can perhaps be gleaned from a more visual engagement with the scenario. Figure 1 shows the first five perfect squares represented pictorially.

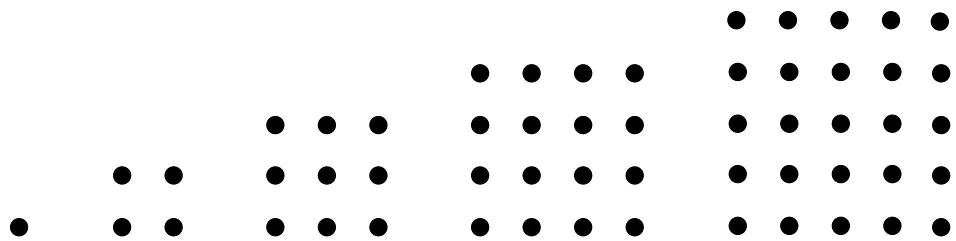

FIGURE 1: Visual representation of the first five perfect squares.

If we remove the preceding perfect square from each perfect square in the pictorial sequence, then it becomes visually clear why we always generate an odd number. Removing n^2 dots from a square array of $(n+1)^2$ dots always leaves an L-shaped structure comprising n dots directly above the removed square, n dots to the right of the removed square, and a single dot in the upper right-hand corner (Fig. 2). There are thus $2n+1$ dots in each L-shape, from which we obtain the sequence $3, 5, 7, 9, 11\ldots$ for $n \in \mathbb{N}$.

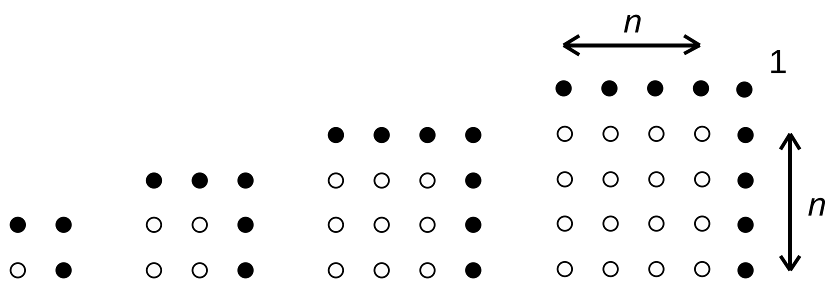

FIGURE 2: Visualising the difference between consecutive perfect squares.

Having investigated the difference between consecutive terms one might be tempted to investigate other differences. Notice for example that the difference between consecutive odd-numbered terms generates the multiples of eight: $9 - 1 = 8$; $25 - 9 = 16$; $49 - 25 = 24$; $81 - 49 = 32$ etc. This observation also lends itself rather nicely to a bit of algebraic exploration. Taking our original sequence of perfect squares and extracting only the odd-numbered terms generates the following sequence:

$$1 ; 9 ; 25 ; 49 ; 81 ; 121 ; 169 ; 225 ; 289 ; 361$$

Since this sequence represents the squares of the first n odd numbers, the n^{th} term of the sequence can be expressed algebraically as $T_n = (2n-1)^2$ for $n \in \mathbb{N}$. The difference between two consecutive terms of the sequence is thus:

$$\begin{aligned} T_{n+1} - T_n &= (2(n+1)-1)^2 - (2n-1)^2 \\ &= (2n+1)^2 - (2n-1)^2 \\ &= 4n^2 + 4n + 1 - (4n^2 - 4n + 1) \\ &= 8n \end{aligned}$$

The algebra thus neatly shows why the difference between consecutive terms generates multiples of eight.

Let's now return to the original question, "*How many natural numbers can be expressed as the difference of two squares?*", and approach it systematically by setting up a table (Figure 3). The upmost row shows the first sixteen counting numbers (x). The leftmost column also shows the first sixteen counting numbers (y). Each cell in the table represents $x^2 - y^2$. The table contains some interesting patterns. If we take the diagonal of zeros as being the first diagonal, then notice that the second diagonal is the sequence of consecutive odd numbers while the third diagonal gives the multiples of 4. The second diagonal represents $x^2 - y^2$ where y is one less than x. In other words it can be expressed as $(x^2 - (x-1)^2)$ which simplifies to $2x - 1$, i.e. the sequence of odd numbers. In a similar vein the third diagonal represents $x^2 - y^2$ where y is *two* less than x. In other words it can be expressed as $(x^2 - (x-2)^2)$ which simplifies to $4x - 4$, thereby explaining why every term in the third diagonal is a multiple of 4. Other diagonals can be explored similarly. In particular, notice that diagonals comprise either (i) only odd numbers or (ii) only even numbers that are a multiple of 4. Since we've established that all odd numbers can be expressed as a difference of two squares, and it would seem that all multiples of 4 can be expressed as a difference of two squares, the question arises as to whether there are any numbers that *can't* be expressed as the difference of two squares.

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
0	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225
1		0	3	8	15	24	35	48	63	80	99	120	143	168	195	224
2			0	5	12	21	32	45	60	77	96	117	140	165	192	221
3				0	7	16	27	40	55	72	91	112	135	160	187	216
4					0	9	20	33	48	65	84	105	128	153	180	209
5						0	11	24	39	56	75	96	119	144	171	200
6							0	13	28	45	64	85	108	133	160	189
7								0	15	32	51	72	95	120	147	176
8									0	17	36	57	80	105	132	161
9										0	19	40	63	88	115	144
10											0	21	44	69	96	125
11												0	23	48	75	104
12													0	25	52	81
13														0	27	56
14															0	29
15																0

FIGURE 3: Integers determined from the difference of two squares.

Closer inspection of Figure 3 reveals the absence of the numbers $2\,;6\,;10\,;14\,;18\,;22\ldots$ The numbers in this sequence are all twice an odd number – i.e. dividing every number in the sequence by 2 gives the sequence of consecutive odd numbers. Is it thus possible that the only numbers not expressible as the difference of two squares are those that are the product of an odd number with 2? Let's explore this conjecture a little more carefully. Given that the difference of two squares $x^2 - y^2$ can be expressed in factorised form as $(x+y)(x-y)$, it follows that only those numbers that can be expressed in the form $(x+y)(x-y)$ can be written as the difference of two squares. Notice that the sum of the two factors $(x+y)$ and $(x-y)$ is $2x$, i.e. the sum of the two factors is even. This means that the factors must either be both odd or both even. The only numbers for which this is not possible are those that are twice an odd number. By way of example, 14 (twice 7) can be expressed in factorised form as either 1×14 or 2×7. In both instances one factor is odd while the other is even. This reasoning also explains why some numbers can be expressed as the difference of two squares in more than one way. Let's take 40 as an example. We can express 40 as the product of two factors having the same parity in more than one way, e.g. 20×2 and 10×4. Since we now need to write 20×2 in the form $(x+y)(x-y)$, we have $x + y = 20$ and $x - y = 2$. Solving simultaneously gives $x = 11$ and $y = 9$ which means we can express 40 in the form $(11+9)(11-9)$ and hence in the form $11^2 - 9^2$. Using a similar approach we can write 10×4 in the form $(7+3)(7-3)$ and hence in the form $7^2 - 3^2$.

Interestingly, this observation should alert one to the fact that it is possible to express all prime numbers (with the exception of 2) as the difference of two squares. Since all prime numbers with the exception of 2 are odd, it follows that their only factors, i.e. 1 and the prime number itself, must necessarily both be odd and the prime number must therefore be expressible as the difference of two squares. One can use a similar line of reasoning to explain why it's possible to express all perfect squares and all perfect cubes as the difference of two squares. It is left to the reader to investigate this further.

A GRAPHICAL REPRESENTATION OF THE DIFFERENCE OF TWO SQUARES

We all know that $x^2 + y^2 = 9$ represents the graph of a circle centred on the origin with a radius of 3 units. But what about the graph represented by $x^2 - y^2 = 9$? One could of course simply make use of appropriate graphing software such as GeoGebra to plot the graph directly, but before we do that let's think about what the graph might look like. Let's begin by trying to find possible sets of co-ordinates $(x; y)$ that satisfy $x^2 - y^2 = 9$. Perhaps the two most obvious points would be $(3; 0)$ and $(-3; 0)$. Inspired by the 3-4-5 Pythagorean triple we might also quickly hit on the point $(5; 4)$ and its associated points $(-5; 4)$, $(5; -4)$ and $(-5; -4)$. If we extend our search beyond integer co-ordinates there are an infinite number of other possibilities, for example $(\sqrt{10}; 1)$, $(\sqrt{11}; \sqrt{2})$, $(\sqrt{12}; \sqrt{3})$, $(\sqrt{13}; 2)$ etc. And of course there are four variations for each point: $(+; +)$, $(+; -)$, $(-; +)$ and $(-; -)$. This observation itself tells us something useful about the symmetry of the graph of $x^2 - y^2 = 9$, namely that it has reflectional symmetry across the x-axis as well as the y-axis. Let's now turn our mind to the domain and range of the graph of $x^2 - y^2 = 9$. Clearly both x^2 and y^2 must be positive. The minimum value of y^2 is thus 0 (where y itself can take on any value), from which it follows that x^2 must be greater than or equal to 9. Solving the inequality $x^2 - 9 \geq 0$ gives $x \geq 3$ or $x \leq 3$. We now know that the graph of $x^2 - y^2 = 9$ has range $y \in \mathbb{R}$ and domain $x \in (-\infty; -3] \cup [3; \infty)$. After plotting a few points we should now have a reasonably good idea of what the graph would look like. We can confirm our conjecture by plotting the graph using graphing software (Figure 4).

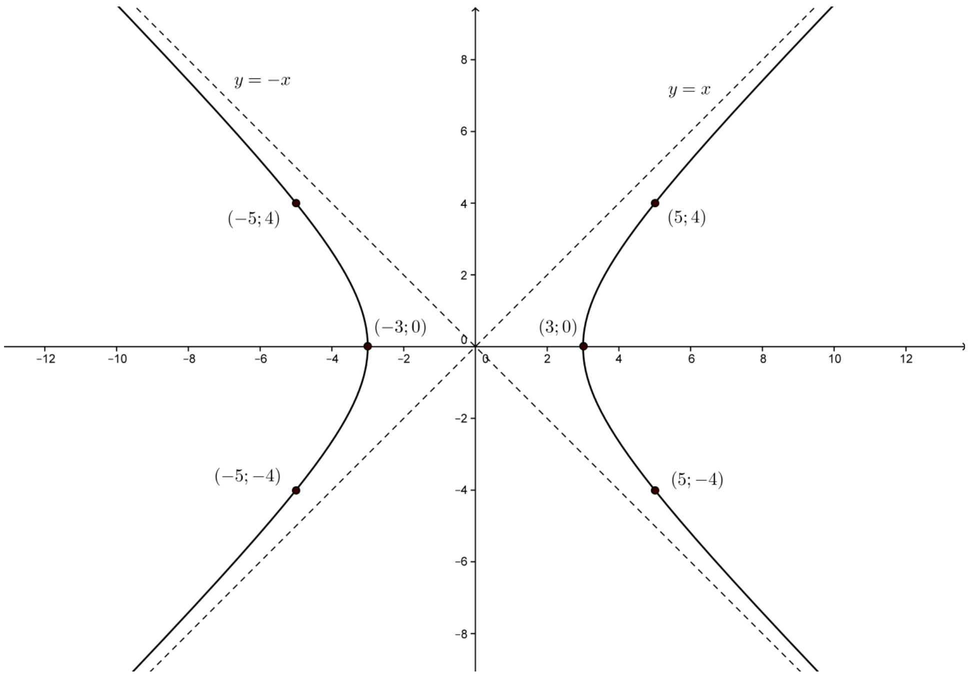

FIGURE 4: The graph of $x^2 - y^2 = 9$.

The graph of $x^2 - y^2 = 9$ takes the form of a hyperbola with asymptotes $y = x$ and $y = -x$. From here it should be a simple matter to visualise the graph of $y^2 - x^2 = 9$, i.e. the inverse graph.

USING AREA TO VISUALISE THE DIFFERENCE OF TWO SQUARES

Let us now explore the idea of the difference of two squares in a more literal way – i.e. by seeing each square as a geometrical shape. If we take a square piece of paper and cut off a square from its corner, then what remains of the original square piece is quite literally 'the difference of two squares'. If the original square piece of paper has side length x and the smaller square which is cut off has side length y, then the two squares have areas x^2 and y^2 respectively and the remaining piece will have area $x^2 - y^2$. Once the smaller square has been cut off from the corner of the larger square, the challenge is to determine the area of the remaining piece. One way of doing this is illustrated in Figure 5.

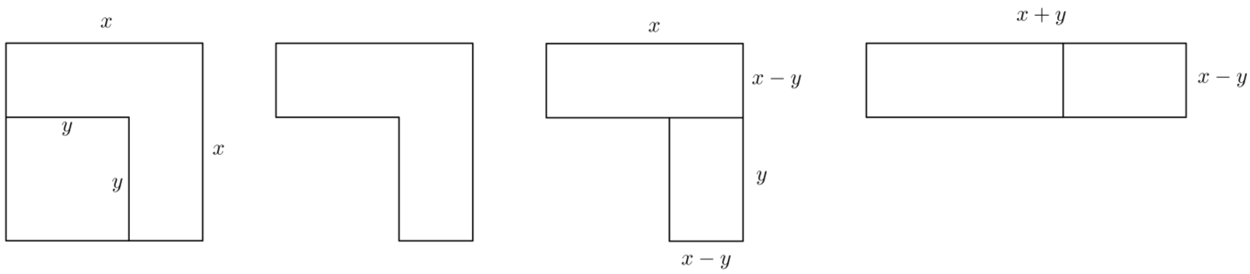

FIGURE 5: Dissecting the remaining piece into two rectangles.

Figure 5 shows how the remaining piece can be dissected into two rectangles which, after rearrangement, form a single rectangle with dimensions $(x + y)$ by $(x - y)$ and which thus has area $(x + y)(x - y)$. Once the smaller square has been cut off from the corner of the larger square, the remaining piece clearly has area $x^2 - y^2$. The rearrangement shown in Figure 5 thus gives visual insight into the algebraic equivalence of the two expressions $x^2 - y^2$ and $(x + y)(x - y)$. An alternative dissection, which involves one of the pieces being flipped over, is shown in Figure 6.

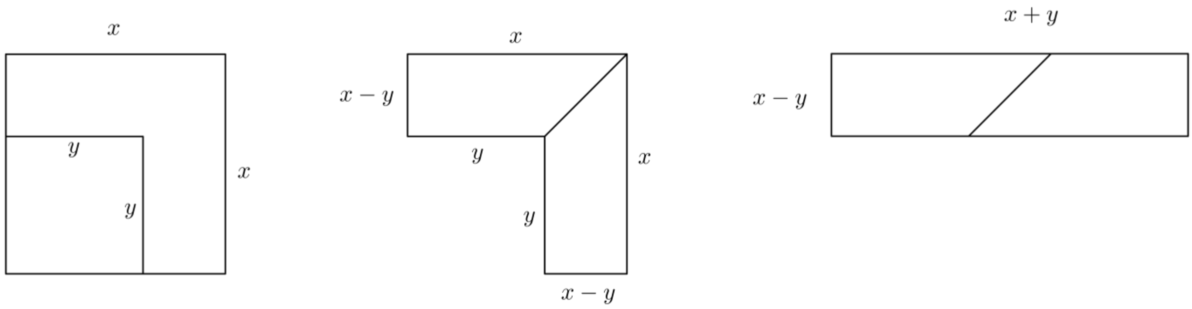

FIGURE 6: An alternative dissection.

Rather than cutting the smaller square from the corner of the larger square, what if we remove the smaller square from the centre of the larger square? Would we still be able to carry out a dissection of the remaining piece of paper to show that its area is $(x + y)(x - y)$? Figure 7 shows how this could be accomplished by sub-dividing the remaining piece of paper into four identical trapeziums which are then rearranged to form a parallelogram with base $(x + y)$ and perpendicular height $(x - y)$. Instead of trapeziums the remaining piece of paper could also be sub-divided into four identical rectangles which could then be rearranged into a larger rectangle with dimensions $(x + y)$ by $(x - y)$. It is left to the reader to explore this further.

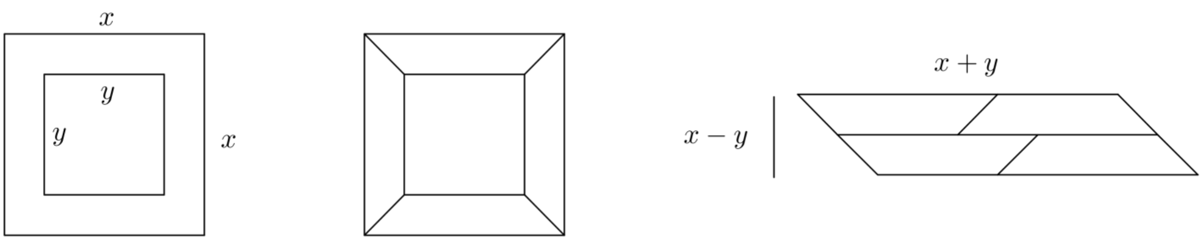

FIGURE 7: Dissecting the remaining piece into four identical trapeziums.

RATIONALISING DENOMINATORS USING THE DIFFERENCE OF TWO SQUARES

We know that $x^2 - 9$ factorises to $(x+3)(x-3)$. Similarly, $25 - 9$ factorises to $(5+3)(5-3)$. We can extend the idea for numbers that are not perfect squares. So, for example, $16 - 7$ can be written in the form $(4+\sqrt{7})(4-\sqrt{7})$. This broader idea of the difference of two squares is fundamental to the process of rationalising the denominators of fractions that have irrational denominators.

You will no doubt have noticed that natural display calculators never give answers with surds in the denominator of a fraction, i.e. fractions with irrational denominators. Let's take the example of $\frac{2}{5-\sqrt{3}}$. When this is entered into a natural display calculator it will automatically be converted into the form $\frac{5+\sqrt{3}}{11}$. The process of rationalising the denominator is illustrated below:

$$\begin{aligned} \frac{2}{5-\sqrt{3}} &= \frac{2}{5-\sqrt{3}} \times \frac{5+\sqrt{3}}{5+\sqrt{3}} \\ &= \frac{2(5+\sqrt{3})}{(5-\sqrt{3})(5+\sqrt{3})} \\ &= \frac{2(5+\sqrt{3})}{25-3} \\ &= \frac{5+\sqrt{3}}{11} \end{aligned}$$

Using the above process the original fraction, with irrational denominator $(5-\sqrt{3})$, has been converted to an equivalent fraction with a rational denominator of 11.

MENTAL ARITHMETIC USING THE DIFFERENCE OF TWO SQUARES

The difference of two squares can also be used as a rather nifty shortcut for mentally calculating products such as 43×37. Provided the two numbers being multiplied are equally spaced either side of a number whose square can be easily calculated, we can determine the product very easily as the difference of two squares. In our example, note that 43 and 37 are equally spaced either side of 40. We can now think of the product 43×37 as $(40+3)(40-3)$ which of course can be written as a difference of two squares, namely $40^2 - 3^2$. And since the square of 40 can easily be calculated mentally we can readily arrive at the answer of the original product: $43 \times 37 = 1600 - 9 = 1591$.

CONCLUDING COMMENTS

It was the purpose of this paper to show how a simple idea such as the 'difference of two squares' can lead to a wealth of mathematical exploration. What I hope I have demonstrated is how by engaging with the core idea of the 'difference of two squares', rather than simply focussing on factorising traditional binomial expressions, one is able to explore multiple representations of the 'difference of two squares' and thereby forge meaningful connections between different mathematical domains and develop conceptual insight into the core idea itself.

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