From Counting Line Segments to Counting Rectangles

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INTRODUCTION

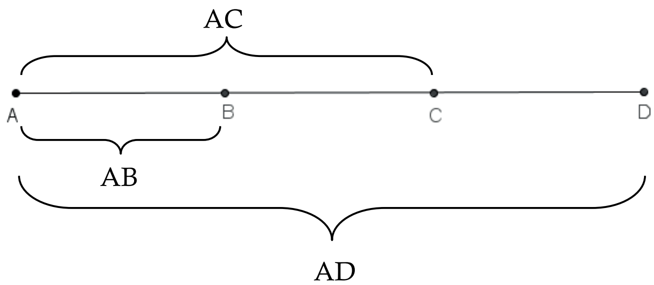
The problem of counting the total number of rectangles in a rectangular grid is periodically encountered as a problem solving activity in school mathematics and mathematical Olympiads. Here is an example:

"How many rectangles are there in the rectangular grid below?"

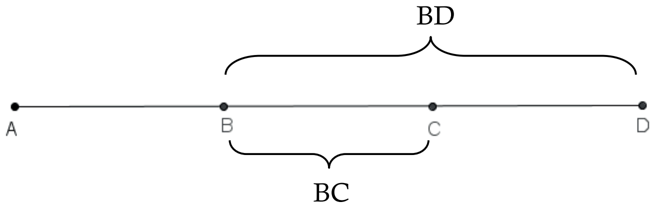
If the grid is relatively small one can carefully and systematically count the different sized rectangles. If the grid is large then one could perhaps explore smaller grids and try to establish a pattern. However, an alternative approach becomes apparent with the insight that the number of embedded rectangles in a rectangular grid is determined by the number of line segments on its length and width. Thus, as an alternative to counting the number of rectangles directly, one can simply count the number of line segments on the length and width of the rectangular grid. The total number of rectangles in the grid is simply the product of these two numbers. In this article I explore this particular strategy, and arrive at a general formula for solving this kind of problem.

COUNTING THE NUMBER OF LINE SEGMENTS

How can we count the number of line segments along the length and width of a rectangular grid? Consider a line segment AD with points B and C lying on it. If we choose A to be the left-most end-point then there are three possible line segments, AB, AC and AD.



If we choose B as the left-most end-point there are two further line segments – BC and BD.



Finally, if we choose C as the left-most end-point there is one further line segment, namely CD.

Hence, the total number of line segments on AD is 3 + 2 + 1 = 6. In general, if there are n division points (including the two end-points) on a line segment, then the total number of embedded line segments is $(n-1)+(n-2)+...+2+1$, which is equal to $n(n-1)/2$.

COUNTING RECTANGLES EMBEDDED IN A RECTANGULAR GRID

We can now use this technique on the original problem, i.e. finding the total number of rectangles embedded in a 3 by 2 rectangular grid.

The length has four division points (including the two end-points) and thus contains 3 + 2 + 1 = 6 line segments. The width has three division points (including the two end-points) and thus contains 2 + 1 = 3 line segments. The total number of embedded rectangles in the grid is thus 6 × 3 = 18.

A GENERAL FORMULA FOR COUNTING EMBEDDED RECTANGLES

We can now establish a general formula for determining the total number of rectangles embedded in a rectangular grid. For an m by n rectangular grid, the two sides would contain $m+1$ and $n+1$ division points respectively. The number of line segments on the length would thus be $m+(m-1)+...+2+1=m(m+1)/2$, while the number of line segments on the width would be $n+(n-1)+...+2+1=n(n+1)/2$. The total number of embedded rectangles would thus be:

$$\frac{m(m+1)}{2}\times\frac{n(n+1)}{2} = \frac{mn(m+1)(n+1)}{4}$$

USING COMBINATORICS

We have already established that for an m by n rectangular grid, the two sides would contain $m+1$ and $n+1$ division points respectively. Since each line segment is uniquely defined by two end points, for $m+1$ such points there would be ${}^{m+1}C_2$ ways of choosing two of these points. Similarly, for $n+1$ such points there would be ${}^{n+1}C_2$ ways of choosing two of the points. Thus, ${}^{m+1}C_2$ and ${}^{n+1}C_2$ represent the number of distinct line segments on the length and breadth respectively of an m by n rectangular grid. The total number of embedded rectangles in an m by n rectangular grid is thus ${}^{m+1}C_2 \times {}^{n+1}C_2$. Let us confirm that this gives us the same formula previously arrived at:

$$
\begin{aligned}
{}^{m+1}C_2 \times {}^{n+1}C_2 &= \frac{(m+1)!}{2!(m+1-2)!} \times \frac{(n+1)!}{2!(n+1-2)!} \\
&= \frac{(m+1)!}{2(m-1)!} \times \frac{(n+1)!}{2(n-1)!} \\
&= \frac{(m+1).m.(m-1)!}{2(m-1)!} \times \frac{(n+1).n.(n-1)!}{2(n-1)!} \\
&= \frac{(m+1).m}{2} \times \frac{(n+1).n}{2} \\
&= \frac{mn(m+1)(n+1)}{4}
\end{aligned}
$$

CONCLUDING REMARKS

Rather than systematically counting the rectangles directly, the insight that the number of embedded rectangles in a rectangular grid is determined by the number of line segments on its length and width allows us to develop a simple approach to solving the problem. We have been able to generalize this approach into a useful formula, namely that for an m by n rectangular grid there are $mn(m+1)(n-1)/4$ embedded rectangles. Although this formula is a useful way of quickly solving embedded rectangle problems by direct substitution of m and n, students should still have an understanding of why the formula works, and where it comes from, rather than simply using it blindly. For pupils who have learnt the basic concept of combinations, the alternative formula ${}^{m+1}C_2 \times {}^{n+1}C_2$ is also very useful.

REFERENCES

Biggs, N. (1990). *Discrete Mathematics* (Rev. ed.). Oxford: Oxford University Press.