Mathematics Competitions – Year-based Problems

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INTRODUCTION & BACKGROUND

Local and international mathematics Olympiads and Competitions often make use of problems that incorporate the year of the competition. I have used problems of this nature in the training of pupils to compete in the International Mathematics Competition (IMC), and have also devised and submitted a number of original questions to the IMC committee for possible inclusion in the competition itself. Problems for elementary level, i.e. primary school pupils who are 13 years or younger at the date of competition, must avoid high level mathematics, and should rather be based on the application of logic and general mathematical principles. This article looks at a number of problems I submitted to the 2015 IMC held in China, and comments on their adaptability for future years. The reader is encouraged to attempt these problems and use them in the classroom, with suitable adaptation as desired.

PROBLEMS SUBMITTED TO THE 2015 IMC HELD IN CHINA

Problem 1

> If $M = (2-a)^2 + (0-b)^2 - (1-c)^2 + (5-d)^2$, what is the difference between the largest possible value of M and the smallest possible value of M where each of a, b, c, d is substituted by a digit of 2015 without repeat?

Note that this question makes use of the year of competition, 2015, in two different ways, and encourages logical thought in order to avoid time-wasting trial and error.

Solution: In order to maximise M, then b must be 5 and d must be 0. Thereafter it makes no difference whether a is 1 and c is 2 or vice versa. We therefore have:

$$M_{\max} = (2-1)^2 + (0-5)^2 - (1-2)^2 + (5-0)^2 = 50, \text{ or}$$

$$M_{\max} = (2-2)^2 + (0-5)^2 - (1-1)^2 + (5-0)^2 = 50$$

In order to minimise M, then c must be 5 and d must be the greatest value of those remaining, i.e. 2. Thereafter a must be 1 and b must be 0. We therefore have:

$$M_{\min} = (2-1)^2 + (0-0)^2 - (1-5)^2 + (5-2)^2 = -6$$

The final answer is thus $M_{\max} - M_{\min} = 50 - (-6) = 56$.

Note that this question can easily be adapted for the years 2016 to 2019.

Problem 2

Given the number 2015, and using each digit twice, the following operations are carried out: Multiply two digits, Add two digits, Subtract two digits, Divide two digits, and thereafter sum the totals of the four operations. Identical digits may not be used together in an operation, e.g. 2 + 2. What is the difference between the maximum and minimum possible sums?

Solution: For the maximum, multiply the two largest numbers allowed in combination, i.e. 2×5, and similarly add the two largest numbers allowed in combination, i.e. $2+5$. Divide 0 by the only possible remaining number, i.e. 1, which leaves us with $1-0$ as the final operation. We thus have:

$$Max=(2\times5)+(2+5)+(1-0)+(0\div1)=18$$

For the minimum, focusing on the multiplication and subtraction operations first, multiply 0 by 5 and subtract 5 from 1, thereafter divide 0 by 2 and add 1 and 2. Alternatively, divide 0 by 5, subtract 5 from 1, multiply 0 by 2 and add 1 and 2. Thus:

$$Min=(0\times5)+(1+2)+(1-5)+(0\div2)=-1 \quad \text{or} \quad Min=(0\times2)+(1+2)+(1-5)+(0\div5)=-1$$

The final answer is thus $Max-Min=18-(-1)=19$.

This question is also readily adaptable for the years 2016 to 2019.

Problem 3

Consider the number 2015………, where each digit, D, from and including the 4th digit, is created by the following calculation:

$$D_n=2D_{n-3}+D_{n-2}+D_{n-1}.$$

So $D_4=2\times D_1+D_2+D_3=2\times2+0+1=5$, the 4th digit of 2015……… Note that only the units digit of the answer to the calculation is used in creating the next digit of the number. For example, if D_n was calculated to be 16, then only the units digit 6 would be written down as the next number in the sequence. What is the value of the 2015th digit?

Solution: Using the calculation for D_n the digits are 2015639491877015563……… From this it becomes clear that there is a 12-digit repeating cycle commencing 015…. We can now determine the position of the 2015th digit in the 12-digit cycle as follows, keeping in mind that the initial digit 2 is not part of the 12-digit cycle:

$$\frac{2015-1}{12} = 167\ rem\ 10$$

The 2015th digit is thus the 10th digit of the repeating unit, i.e. 8.

A variation for 2017 would be to consider the number 2017………, where each digit, D, from and including the 4th digit, is created by the calculation $D_n=2D_{n-3}+D_{n-2}+3D_{n-1}$. The question could then be to determine the 2017th digit. The calculation results in the number 2017251275677017255……… which contains a 12-digit repeating cycle commencing 017. The 2017th digit is the final digit of the repeating unit, i.e. 7.

Problem 4

The 4-digit odd number $\overline{abcd}$ has unique digits. The sum of the digits equals 8 and the product of the digits equals 0. 2015 is one such number. How many different numbers can $\overline{abcd}$ represent?

Solution: Numbers commencing with 9, 8, 7 or 6 are all too large. We therefore only need to consider permutations of the last three digits of numbers commencing with 5, 4, 3, 2 and 1. Considering each possible case: 5012 has six permutations, two of which are odd. 4013 has six permutations, four of which are odd. 3014 has six permutations, two of which are odd. 2015 has six permutations, four of which are odd. 1025 has six permutations, two of which are odd. Finally, 1034 has six permutations, two of which are odd. There are thus 2 + 4 + 2 + 4 + 2 + 2 = 16 different numbers that $\overline{abcd}$ can represent.

A nice variation for 2016 is as follows: The 4-digit even number $\overline{abcd}$ has unique digits. The sum of the digits equals 9 and the product of the digits equals 0. 2016 is one such number. How many different numbers can $\overline{abcd}$ represent? (Answer is 34)

Problem 5

How many 4-digit odd numbers with unique digits can be arranged by using four digits from 012345 such that the resultant number is not greater than 2015?

Solution: We need only consider 1 _ _ 3 , 1 _ _ 5 , 2013 and 2015. 1 _ _ 3 and 1 _ _ 5 each yield twelve possible permutations. There are thus 12 + 12 + 1 + 1 = 26 such possible numbers.

The problem can readily be adapted for 2016: How many 4-digit even numbers with unique digits can be arranged by using four digits from 0123456 such that the resultant number is not greater than 2016? (Answer is 82)

Problem 6

The following problem was developed for, but not submitted to, the 2016 IMC held in Thailand:

What is the smallest perfect cube under 1000 satisfying $2^a + 0^b + 1^c + 6^d$, where a, b, c, d are discrete positive integers?

Solution: Since 2^a and 6^d are both even, and given that $0^b = 0$ and $1^c = 1$, it follows that the desired cube must be odd. The possible cubes are thus 27, 125, 343 or 729, and the equation we need to solve is $2^a + 6^d + 1 = x^3$. Now 6^d can only be 6, 36 or 216, and by checking the powers of 2, with a maximum of 512, it is easily established that $512 + 216 + 1 = 729$. The smallest perfect cube is thus 729.

CONCLUDING REMARKS

It is both interesting and challenging to develop unique problems of the above nature – a process that engages critical thought, logical reasoning, and problem solving ability. I encourage teachers not only to develop such problems themselves, but to encourage their pupils to do so likewise. This will certainly improve general mathematical ability, and also facilitate participation in mathematics Competitions and Olympiads.