Camouflaged Functions

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INTRODUCTION

As part of the South African school Mathematics syllabus, pupils are exposed to a number of different functions and their graphical representations. Pupils generally become fairly adept at identifying what type of graph a given function represents when the function is given in one of its standard, and hence familiar, forms. However, when the function is presented in a non-standard form pupils often struggle trying to make sense of the function.

As part of a 3-day Matric revision camp at the start of our final term I presented a session entitled "Hidden Functions & Other Camouflage". My intention in the session was to motivate pupils to think more laterally and more flexibly when engaging with functions and their graphs, particularly in relation to functions given in non-standard formats. In this article I share some of the ideas we explored.

EXPONENTIAL FUNCTIONS

The standard form of the exponential function, incorporating both vertical and horizontal translations, is:

$$f(x) = a.b^{x-p} + q \quad (b > 0\,; b \neq 1)$$

Pupils often struggle with exponents, and as a consequence often find engaging with the exponential function somewhat tricky. For this reason I began the session with an activity specifically focusing on exponential functions given in non-standard formats:

Which of the following functions are equivalent?

$y = \frac{9.3^x - 6}{3}$	$y = 3^x + 3^x + 3^x - 2$	$y = \left(\frac{1}{3}\right)^{-x-1} - 2$
$y = \frac{1}{9}.3^{x+3} - 2$	$y = \frac{3}{3^{-x}} - 2$	$y = 3(3^x + 1) - 5$

The purpose of the activity was not only to illustrate the variety of forms an exponential function can take, but to tease out any problems pupils might still be having with simplifying and manipulating exponential expressions. After a short while most pupils were able to simplify each function to $y = 3^{x+1} - 2$, thereby establishing that, despite seeming quite different at first glance, all six functions are in fact equivalent. This was a useful exercise in sensitising pupils to look more carefully at non-standard formats and to engage with the manipulation of exponential expressions.

As a follow up activity one could present pupils with two exponential functions, e.g. $f(x) = 2(2^x - 2)$ and $g(x) = \frac{1}{2}\left(\left(\frac{1}{2}\right)^x - 8\right)$ and ask them to establish what series of transformations would transform the graph of f to that of g.

Based on the first activity pupils should be able to manipulate f into the form $f(x) = 2^{x+1} - 4$. Manipulating g requires a little more care, but with a bit of prompting pupils should hopefully arrive at the form $g(x) = 2^{-x-1} - 4$. By direct comparison of the two functions it then becomes clear that the graph of f can be transformed into that of g by a reflection across the y-axis followed by a translation of 2 units to the left. If pupils then go on to draw the two graphs they should also notice that the graphs are reflections of one another across the line $x = -1$.

HYPERBOLAS

The next activity focused on hyperbolas. The standard form of the hyperbola function, incorporating both vertical and horizontal translations, is:

$$f(x) = \frac{a}{x - p} + q$$

Once again the purpose of the activity was to heighten pupils' sensitivity to non-standard forms of the hyperbola function. I began by presenting pupils with the function $y = \frac{3}{x-2} + 4$ and asking them to rewrite the expression on the right as a single fraction. This was a relatively simple matter to accomplish:

$$y = 4 + \frac{3}{x-2}$$

$$y = \frac{4(x-2) + 3}{x-2}$$

$$y = \frac{4x - 8 + 3}{x-2}$$

$$y = \frac{4x - 5}{x-2}$$

Note that in the above worked solution I have written the hyperbola function in the form $y = q + \frac{a}{x-p}$. This was done deliberately to mediate the discussion that followed. I asked pupils to look carefully at the form $y = \frac{4x-5}{x-2}$ and to think critically about reversing the process they had just carried out to rewrite it in the form $y = 4 + \frac{3}{x-2}$. In particular I wanted pupils to think carefully about the transformation of the numerator from $4x - 5$ to the algebraically equivalent expression $4x - 8 + 3$ and to rationalise why this was appropriate. After a bit of discussion most pupils were happy that the transformation was sensible given that $4x - 8$ was a multiple of the denominator $x - 2$. We then established the following protocol for the reverse process.

- Compare the coefficient of the x-term in the numerator to that of the x-term in the denominator.
- Use this comparison to rewrite the numerator as a multiple of the denominator, compensating for the adjustment by adding or subtracting an extra constant term.
- Factorise out a common factor from the first two terms.
- Split into two component fractions, simplify and rearrange.

I then asked pupils to carry out this reverse process on the function $y = \frac{2x+6}{x+5}$.

$$y = \frac{2x + 6}{x + 5}$$

$$y = \frac{2x + 10 - 4}{x + 5}$$

$$y = \frac{2(x + 5) - 4}{x + 5}$$

$$y = \frac{-4}{x + 5} + 2$$

An alternative approach to that shown above would be to use long division, and one could argue that long division would in fact be a quicker method. The two techniques of course complement one another since it is the same underlying process being used in both. However, as an algebraic technique I find the above process rather useful – particularly if pupils are able to establish it themselves through discussion and 'reverse synthesis'.

I then asked pupils to complete the following task using insights gleaned from the previous discussion:

Re-write each of the following functions in the standard hyperbola form $y = \frac{a}{x - p} + q$.

$$y = \frac{3}{2 - x} + 1 \qquad (y + 1)(x + 2) = -6$$

$$y = \frac{4x - 10}{x - 3} \qquad xy - 3y - 5x + 11 = 0$$

Reflection & concluding comments

The session on 'camouflaged functions' proved to be very helpful in terms of sensitizing pupils to non-standard forms that functions, particularly hyperbolas and exponential functions, can take. The notion of 'disguising' a standard form was particularly useful, and as a follow up activity I recommend getting pupils to take a few simple functions in standard form, disguising them through careful algebraic manipulation, and then getting their classmates to unravel the camouflage and restore each of the functions to a recognizable standard form. Not only will this encourage pupils to think more flexibly, but it is a useful means of teasing out problems that pupils may still have in terms of algebraic manipulation and simplification.