Exploring the Minimum Conditions for Congruency of Polygons

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INTRODUCTION

Polygons form an important component of Euclidean geometry within most school curricula. In the earliest grades pupils observe, describe and sort polygons according to their general characteristics. This then develops into a more sophisticated classification system based on properties and definitions in which various 'families' of polygons emerge. The general idea of congruency is then introduced, and is formally explored with a specific focus on triangles.

The formal exploration of congruency in polygons with more than three sides is seldom dealt with at school level. It is our contention that exploring congruency in polygons other than triangles is likely to enhance a more meaningful appreciation for the concept of congruency, and may prevent pupils from making erroneous generalisations. By way of example, pupils are familiar with the idea that two triangles are congruent when their sides are correspondingly equal. Pupils might thus fall prey to the misconception that two quadrilaterals are also congruent if their sides are correspondingly equal.

The purpose of this article is to present an exploration of congruency in polygons other than triangles. We first explore the necessary conditions to establish congruency in quadrilaterals, and then use an inductive process to establish a broader generalisation of congruency in polygons. It is hoped that such an exploration in the classroom will deepen pupils' appreciation for the concept of congruency.

STAGE 1: THE STORY OF TRIANGLES

Question: Do three distinct points (A, B and C) which are not collinear (i.e. do not lie on a single straight line) define a unique triangle?

Answer: Yes. Regardless of their order, the three points define a single closed path which comprises the line segments AB, AC and BC.

STAGE 2: THE STORY OF QUADRILATERALS

Question: Do the four points (A, B, C and D) shown in Figure 1 define a unique quadrilateral?

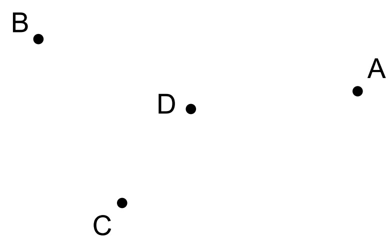

FIGURE 1

Answer: No, the four points do *not* define a unique quadrilateral. Figure 2 illustrates this with three possible quadrilaterals having A, B, C and D as vertices, namely quadrilaterals ADBC, ABCD and ABDC.

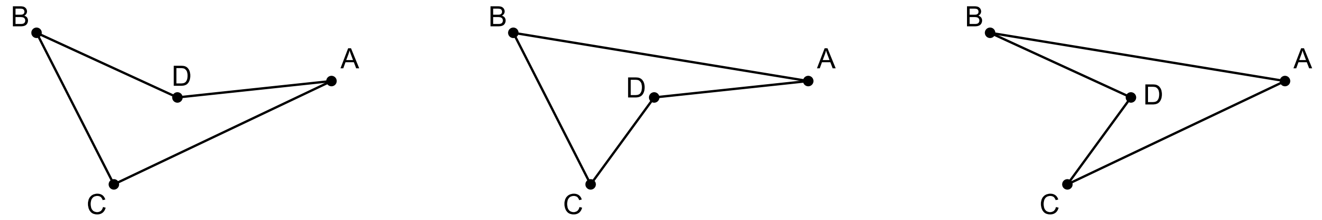

FIGURE 2

This leads to the important observation that in the case of quadrilaterals the *order* of the vertices takes on particular significance, since changing the order could result in *different* quadrilaterals.

STAGE 3: THE STORY OF TWO QUADRILATERALS

Question: Two quadrilaterals, ABCD and EFGH, are given. In quadrilateral ABCD, the diagonal AC along with the sides creates two triangles – ABC and ADC. In quadrilateral EFGH, the diagonal EG along with the sides creates two triangles – EFG and EHG. Suppose that triangles ABC and EFG are congruent to one another, and that triangles ADC and EHG are also congruent to one another. Given this scenario, does it necessarily follow that quadrilaterals ABCD and EFGH are congruent?

Answer: No. Figure 3 shows quadrilateral ABCD along with two versions of quadrilateral EFGH. In both versions, triangle ABC and EFG are congruent to one another, as are triangles ADC and EHG. However, it is clear that neither version of quadrilateral EFGH is congruent to ABCD.

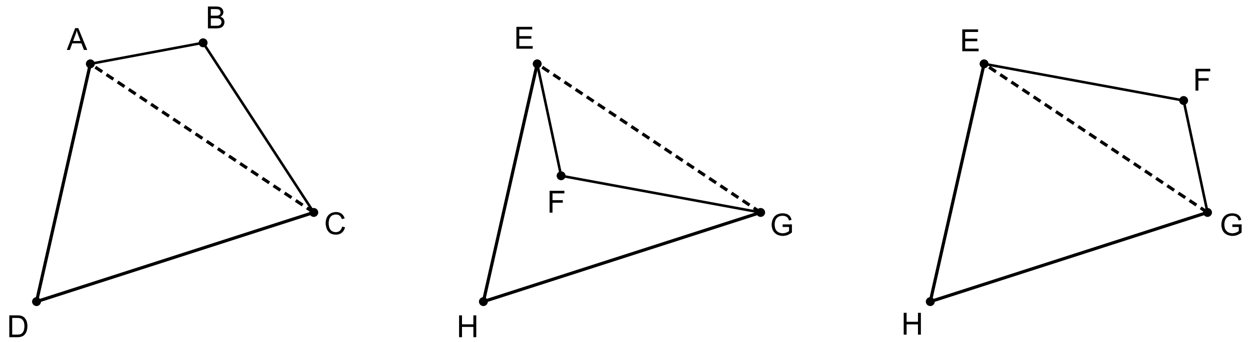

FIGURE 3

If we compare quadrilateral ABCD and convex quadrilateral EFGH, note that despite there being four correspondingly equal sides (AB = FG, BC = EF, AD = EH and CD = GH) and two correspondingly equal opposite angles ($\angle B = \angle F$ and $\angle D = \angle H$), the two quadrilaterals are still not congruent.

STAGE 4: THE STORY OF TWO HEXAGONS

Question: Compare the two hexagons shown in Figure 4. Are they congruent?

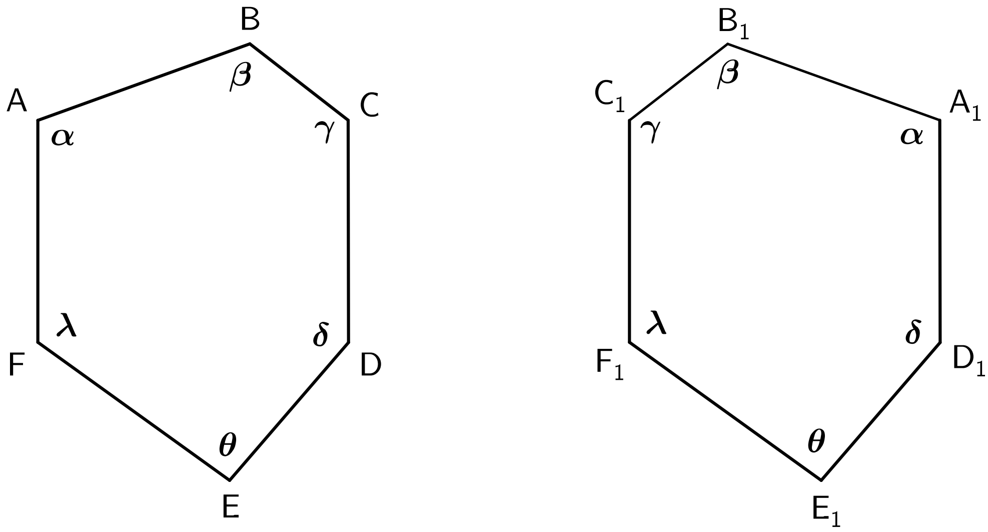

FIGURE 4

Answer: If we compare the two hexagons, ABCDEF and $A_1B_1C_1F_1E_1D_1$, then we notice that the sides are correspondingly equal:

$$AB = A_1B_1 \; ; BC = B_1C_1 \; ; CD = A_1D_1 \; ; DE = D_1E_1 \; ; EF = E_1F_1 \; ; FA = F_1C_1$$

and that the angles are correspondingly equal:

$$\angle ABC = \angle A_1B_1C_1 \;\; ; \;\; \angle BCD = \angle B_1C_1F_1 \;\; ; \;\; \angle CDE = \angle A_1D_1E_1$$

$$\angle DEF = \angle D_1E_1F_1 \;\; ; \;\; \angle EFA = \angle E_1F_1C_1 \;\; ; \;\; \angle FAB = \angle D_1A_1B_1$$

However, despite this, the two hexagons are *not congruent*, i.e. the above conditions are insufficient for congruency. Two polygons are congruent when it is possible to superimpose one polygon on top of the other (through rotation and mirror imaging if necessary) so that all sides and all interior angles match up precisely. These are the necessary conditions for congruency, and in such a situation there will be a correspondence between the vertices of the two polygons. Figure 4 clearly shows that just because two polygons have correspondingly equal angles and correspondingly equal sides, this combination of properties does not necessarily ensure the correct order of corresponding vertices. In the example shown in Figure 4, the hexagon which might have been congruent to ABCDEF is the hexagon with vertices in the specific order $A_1B_1C_1F_1E_1D_1$ and not $A_1B_1C_1D_1E_1F_1$.

For polygons with more than three sides, congruency requires correspondence of vertices which maintains the correct sequence of corresponding vertices.

STAGE 5: THE STORY OF TWO POLYGONS WITH n SIDES

A polygon with n sides also has n internal angles. We will refer to these features of a polygon, i.e. the sides and internal angles, as its '**main components**'. It is important to take cognisance of the following points:

- Polygons contain other components in addition to the main components, for example altitudes, angle bisectors, medians, perimeter, area and diagonals.
- In line with the definition of congruency, when two polygons with n sides are congruent with one another there will be a correspondence between their vertices such that the $2n$ main components of the polygons will be correspondingly equal.
- **In the case of triangles** – there are groups of three components (main or additional) where corresponding equality between components in these groups is a sufficient condition to establish congruency of the triangles. However, any combination of less than three correspondingly equal components is *insufficient* for establishing congruency (Patkin & Plaksin, 2011).
- **In the case of quadrilaterals** – Given two quadrilaterals, any combination of *less than five* correspondingly equal 'main components' is *insufficient* for establishing congruency. Consider the following examples which illustrate this important point. Figure 5 shows rectangle ABCD and square EFGH. Although the two shapes have four correspondingly equal angles, the shapes are *not congruent*.

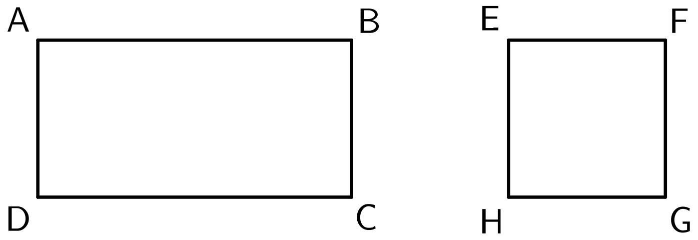

FIGURE 5

Figure 6 shows quadrilaterals ABCD and ABCE inscribed in a circle. Although the two shapes have two correspondingly equal angles ($\angle B$ is common and $\angle D = \angle E$) as well as two correspondingly equal sides (AB and BC are both common), the two quadrilaterals are *not congruent*.

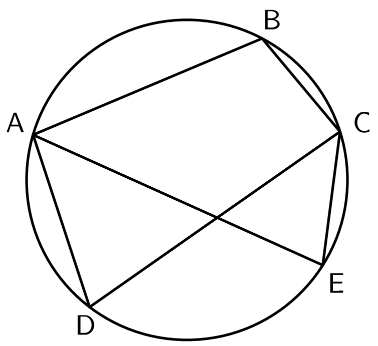

FIGURE 6

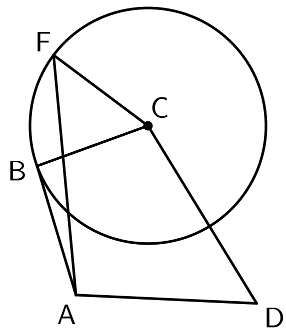

FIGURE 7

Figure 7 shows quadrilaterals ABCD and AFCD with C the centre of the circle and points B and F lying on the circle. Although the two quadrilaterals have three correspondingly equal sides (CB = CF, AD and CD are common) as well as one correspondingly equal angle ($\angle ADC$ is common), the two quadrilaterals are *not congruent*.

In order to prove congruency in two quadrilaterals, not only do we need *five* correspondingly equal 'main components', these also need to be in certain *specific combinations* (e.g. SASAS). By way of example, Figure 8 shows two trapeziums, ABCD and ABEF, which despite having four correspondingly equal angles and one correspondingly equal side (in the relative order ASAAA) are still *not congruent*. In similar vein, Figure 9 shows quadrilaterals ABCD and AFCD with C the centre of the circle and F and B lying on the circle. Although the two quadrilaterals have three correspondingly equal sides (CB = CF, AD and CD are common) as well as two correspondingly equal angles ($\angle D$ and $\angle A$ are common), giving the relative order SSASA, the two quadrilaterals are *not congruent*.

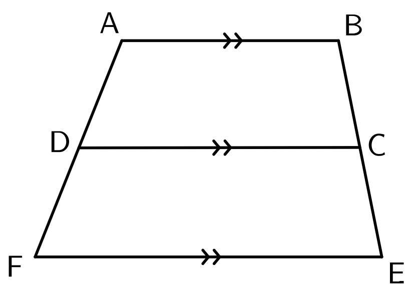

FIGURE 8

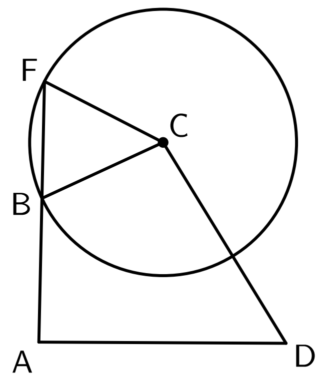

FIGURE 9

STAGE 6: THE INDUCTIVE PROCESS

By using an inductive process it is possible to show that when moving from a polygon with n sides to a polygon with $n + 1$ sides, the minimum number of correspondingly equal main components required to establish congruency between two polygons grows by 2. In general, to prove the congruency of two polygons with n sides requires a minimum of $2n - 3$ main components in equal correspondence (in a specific relative order).

CONCLUDING COMMENTS

The exploration described in this article focuses on the minimum conditions required to establish congruency in polygons as well as on the importance of maintaining the correct order of corresponding vertices. We have purposefully explored aspects of congruency beyond that which is traditionally explored in the classroom in the hopes of deepening pupils' appreciation for the concept of congruency, and sharpening their criticality when attempting to establish congruency in shapes other than triangles.

REFERENCES

Patkin, D., & Plaksin, O. (2011). Congruent triangles. Sufficient conditions and insufficient conditions. Suggested milestones for inquiry and discussion. *Research in Mathematical Education, 15*(4), 327-340.