Two Further Squares from Pythagorean Triples

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Consider the Pythagorean Triple 5, 12, 13. Have you ever noticed that if you add the even side to the hypotenuse you generate a perfect square (13 + 12 = 25)? Similarly, have you ever noticed that if you subtract the even side from the hypotenuse you also generate a perfect square (13 – 12 = 1)? Does this always hold true?

It is well known that there are an infinite number of positive whole number Pythagorean Triples a, b, c such that $a^2 + b^2 = c^2$. Euclid's formula for generating all primitive triples requires an arbitrary pair of positive integers m and n with $m > n$. The formula[1] states that the integers a, b, c form a Pythagorean Triple where:

$$a = m^2 - n^2$$
$$b = 2mn$$
$$c = m^2 + n^2$$

With b as the even side, then notice that:

- Hypotenuse plus even side: $c + b = m^2 + n^2 + 2mn = m^2 + 2mn + n^2 = (m+n)^2$
- Hypotenuse minus even side: $c - b = m^2 + n^2 - 2mn = m^2 - 2mn + n^2 = (m-n)^2$

This clearly shows that for all primitive Pythagorean Triples, the hypotenuse plus the even side is a perfect square, and the hypotenuse minus the even side is a perfect square[2].

A few examples:

a	b	c	c + *even*	c − *even*
3	4	5	$9 = 3^2$	$1 = 1^2$
5	12	13	$25 = 5^2$	$1 = 1^2$
8	15	17	$25 = 5^2$	$9 = 3^2$
12	35	37	$49 = 7^2$	$25 = 5^2$
20	21	29	$49 = 7^2$	$9 = 3^2$
133	156	205	$361 = 19^2$	$49 = 7^2$

[1] The triple generated by Euclid's formula is only primitive if m and n are coprime and not both odd.

[2] Note that a primitive Pythagorean Triple always comprises two odd numbers and an even number.