QED – The Beauty of Mathematical Proof

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> *"Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show."*
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> Bertrand Russell

I have titled this article 'QED', the abbreviation for the Latin phrase *quod erat demonstrandum* meaning 'that which was required to be proved'. This abbreviation is often used as the conclusion to a mathematical proof as an indication that it is complete. Mathematical proof is somewhat different to what we would normally consider proof. The critical distinction is that mathematical proof is absolute – once we have proved something mathematically, its infallibility is assured. In other areas of science, theories are proved by gathering experimental evidence and drawing conclusions based on this evidence. This kind of proof can be very reliable, but it is never absolute. In mathematics one may certainly use evidence as a guide, but one never relies on it.

A good example of why proof is so important is the Pólya Conjecture, which was devised by the Hungarian mathematician George Pólya. The conjecture has to do with prime numbers, i.e. numbers that are only divisible by 1 and themselves. The first few of these are 2, 3, 5, 7, 11 and so on. If a number is not prime then it can be written as a product of prime numbers. For example 6 can be written as the product of prime numbers as 2×3, while 8 can be written as $2 \times 2 \times 2$. This process is called prime factorisation, and since each number can only be written as a product of primes in one way, each number is said to have a unique prime factorisation. The Pólya conjecture has to do with these prime factorisations of numbers. What it's concerned with is whether a number has an *odd* or *even* number of prime factors. 6 has an even number of prime factors (two), namely 2 and 3, while 8 has an odd number of prime factors (three), namely 2 and 2 and 2. Note that for the purposes of this conjecture if a prime factor is repeated we count it each time. So we can split up the natural numbers into those numbers that have an even number of primes and those that have an odd number of primes in their factorisation. 1 is a bit tricky, but we put it into the even section because it has no prime factors, and 0 is even. 2 is a prime so it has one prime in its factorisation and it goes into the odd section. 3 is also a prime so it goes into the odd section. 4 has two numbers in its prime factorisation (2 and 2) so it goes in the even section. 5 is a prime and goes into the odd section – and so on as illustrated in the table alongside.

Pólya Conjecture

Odd	Even
2	1
3	4
5	6
7	
8	

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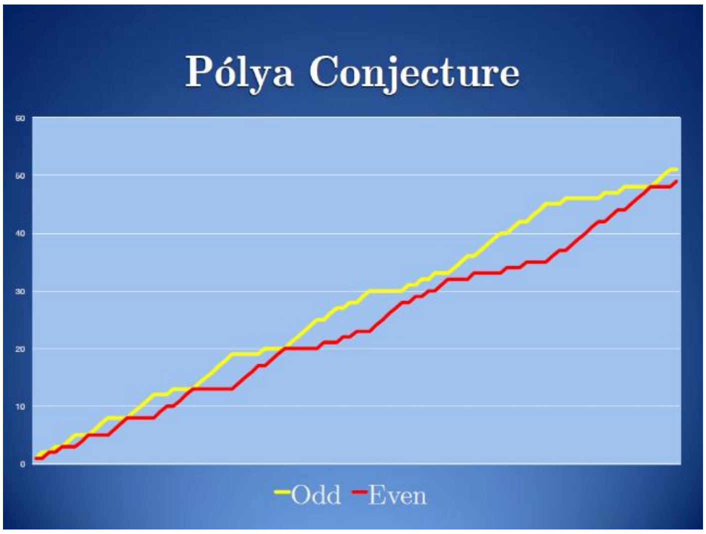
Ignoring the problematic case of 1, Pólya thought that as more numbers are added there are always at least as many in the odd section as in the even section. Extending the table on the previous page it can be shown that this remains true for the first 100 natural numbers.



The conjecture in fact remains true for the first 1000 natural numbers. Using a computer one can readily verify that it still holds true up to the first 1 000 000 natural numbers. If we continue systematically we find it still holds true for the first 10 000 000. By the time we get to the hundred millionth natural number the conjecture has still not failed. It is at this point that a regular scientist would probably be satisfied that it is always true. But the mathematician is *not* satisfied. And rightly so, because at 906 150 257 the conjecture fails! This is a good lesson about why one can never assume anything in mathematics. You can have all the evidence in the world, but mathematicians demand something more – absolute proof.

Proofs have become so important that you can even earn yourself a million dollars for proving certain hypotheses. The Riemann hypothesis, for example, is on a list of seven problems released by the Clay Mathematics Institute, for which they are offering 1 million dollars. Even though computers have checked that the hypothesis is true for the first 10 trillion cases, mathematicians will not be satisfied until it has been proven for certain. The Oxford mathematician, Andrew Wiles, says, "Mathematicians aren't satisfied because they know there are no solutions up to four million or four billion, they really want to know that there are no solutions up to infinity". Andrew Wiles is certainly one who knows about proof. Wiles is famous for a proof that was over 150 pages long and took him seven years to complete. The theorem he proved is the legendary 'Fermat's Last Theorem', and it took 358 years between the time it was conjectured to the time it was proven.

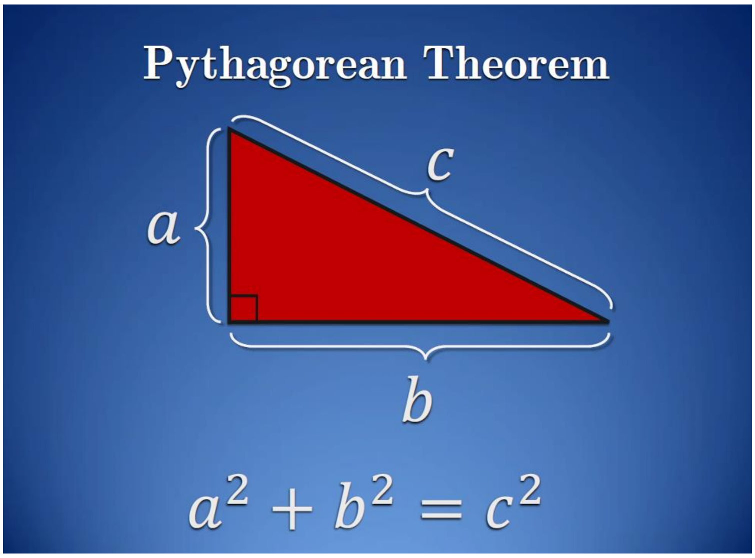
The man who conjectured the theorem was the French lawyer Pierre de Fermat. Whilst Fermat was a lawyer, he is most remembered for his mathematical research which he did in his free time as something of a hobby. Fermat had a copy of the ancient Greek book *Arithmetica* by Diophantus and he would scribble down his mathematical thoughts in the margin of this book. He wrote down many theorems but left most of them without proof. As time went on, mathematicians worked through all of these theorems and eventually managed to prove them all. All of them, that is, except for one. This theorem that evaded proof became known as 'Fermat's Last Theorem'. Despite it being so difficult to prove, the theorem itself is surprisingly easy to understand. In fact Andrew Wiles, who eventually proved it, first came across the theorem at the age of 10 in his local library. Wiles was amazed that this seemingly simple problem was so difficult to prove and it became his dream to be the one to prove it.

So what was the theorem? We are all familiar with the Pythagorean theorem, $a^2 + b^2 = c^2$. There are many solutions to this equation, for example, $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$ and, $8^2 + 15^2 = 17^2$. In fact there are infinitely many solutions to this equation. But Fermat asked the question, what if we changed it from squares to cubes? Are there still solutions to the equation $a^3 + b^3 = c^3$? And what about $a^4 + b^4 = c^4$? Are there solutions to this? Or, in general, what about the equation $a^n + b^n = c^n$ where n is any number greater than 2 – are there solutions to this equation? Fermat conjectured that there were no solutions whenever n is greater than 2. But what made this so frustrating for mathematicians was that Fermat claimed to have proved the conjecture. In the margin of his copy of *Arithmetica* he wrote, "I have discovered a truly remarkable proof of this theorem which this margin is too small to contain."

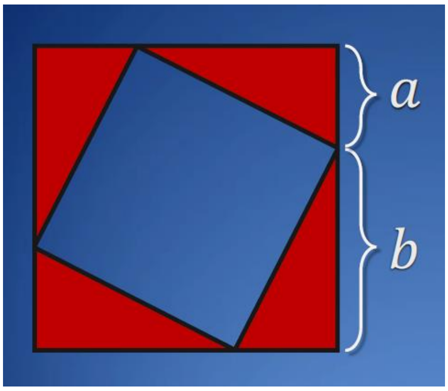
The methods Wiles used in his proof were not known in Fermat's time, and Wiles has stated that he doesn't believe Fermat had a proof, but rather that he thought he had a proof and hadn't noticed his mistake. And Wiles is definitely the expert on 'Fermat's Last Theorem'. He began working on it in 1986, and did so in complete isolation until 1993. He wanted to avoid the attention that he would get for working on the famous problem and so confided only in his wife until he was convinced that he had proven it. He gave a lecture on his proof in 1993 much to the excitement of the mathematical community, but later that year an error was found in the proof. It took him over a year to repair his proof, but eventually in 1995 he published a complete proof of the theorem, thus resolving the 358-year-old problem and earning the right to say QED.

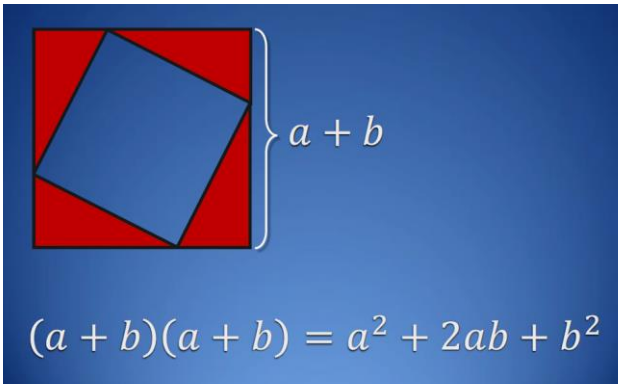
It's sad that in the high school Mathematics syllabus today, proofs play a very minor role. The few proofs that we learn can simply be memorised, and even in the AP Mathematics syllabus proof forms a relatively small component. The mathematics that we are taught in school is by no means useless, but we are only taught a few of the tools required for what I'll call 'real maths'. Even so, we never get around to *doing* the 'real maths'. To give an analogy, this is like learning about grammar, but never writing a story. You can't write a good story without good grammar, but grammar isn't interesting – the story is. In school one is typically taught the 'how' but not the 'why' of mathematics. I think this is perhaps why most people don't understand the idea of there being beauty in mathematics.

But mathematical proof is indeed just that – beautiful. Consider the Pythagorean theorem which states that the sum of the squares of the two shorter sides of a right-angled triangle is equal to the square of the longer side.

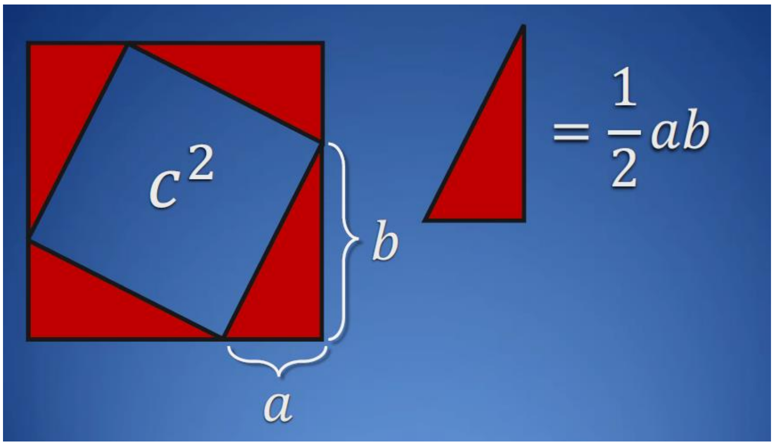


People often criticise mathematics for being too rigid and having no room for creativity, but in proofs this is not the case. There are, for example, well over 100 different proofs of the Pythagorean theorem. The proof which follows is, I think, particularly elegant. The first step is to make four copies of our right-angled triangle (with side lengths a, b and c) and arrange them in the form of a square. This is the key to the proof and is far from an obvious first step. Next we calculate the area of the square in two different ways. The first way is to calculate the area of the square directly. Since the two shorter side lengths of the right-angled triangle are a and b, the square has side length $a + b$. The area of the square is thus $(a + b)^2$ which we can expand algebraically to $a^2 + 2ab + b^2$.





The second way to calculate the area of the square is to add up the area of its individual component parts, i.e. the four triangles and the smaller square in the centre. Each triangle has an area of $\frac{1}{2}ab$ while the square in the centre has side length c and hence area c^2. The area of the large square is thus $4 \times \frac{1}{2}ab + c^2$ which we can simplify to $2ab + c^2$.



But now we have two different formulas for the area of our square. Since they both represent the same area they must be equal to each other, so $a^2 + 2ab + b^2 = 2ab + c^2$. Subtracting $2ab$ from both sides leaves us with $a^2 + b^2 = c^2$, and hence the Pythagorean theorem is proven. QED.

Let us now turn our attention to irrational numbers. Some familiar examples of these are π, e and $\sqrt{2}$. The decimal digits of these numbers carry on going not only forever, but without any pattern. The ancient Greeks used to think that all numbers were rational, which means that they can be expressed as a ratio of two integers. The Greeks held this belief until a man by the name of Hippasus suggested that $\sqrt{2}$ was actually irrational, i.e. that it could *not* be expressed in the form $\frac{a}{b}$ where a and b are both integers. This was more controversial than it may seem. The reason for this was that Hippasus was a student of Pythagoras. Pythagoras wasn't just a mathematician, but also a philosopher, and had a school of followers known as the Pythagoreans. Their philosophy became known as Pythagoreanism, and can be likened to what you'd get if you tried to turn mathematics into a religion. Their central belief was that 'all is number'. Another strongly held belief was that all things – astronomy, music, even virtue – could be described by, and understood through, rational numbers. Suggesting that there could be such a thing as an irrational number was thus blasphemous, something perhaps similar to Copernicus suggesting that the earth revolves around the sun. We don't know exactly what happened to Hippasus, but a popular account is that the Pythagoreans were on a sea voyage at the time and threw him overboard in order to keep the existence of irrational numbers a secret.

The fact that $\sqrt{2}$ is indeed irrational can be proved using a very elegant method of proof known as *reductio ad absurdum* or proof by contradiction. This technique is unfortunately not taught in high school. The idea behind this proof technique is that if you want to prove something true, assume it is false. If you can show that it being false results in a contradiction, or mathematical absurdity, then the assumption must have been incorrect and it must thus be true.

In our case, we want to prove that $\sqrt{2}$ is irrational. To do this we will assume that it is *rational*, and if we can show that this leads to a contradiction then it must in fact be *irrational*. If we assume $\sqrt{2}$ to be rational, then we must be able to represent it as a ratio of two integers in simplest form. Let's say $\sqrt{2} = \frac{a}{b}$ where a and b are both integers such that $\frac{a}{b}$ is in simplest form and cannot be reduced further (this is important for the proof so keep it in mind). Next we square both sides of our equation to get $2 = \frac{a^2}{b^2}$. If we multiply both sides by b^2 we get $2b^2 = a^2$. Since $2b^2$ has a factor of 2 it must be even, from which it follows that a^2 must also be even. But if a^2 is even then a must be even and we can thus write $a = 2m$ for some integer m. Our equation now becomes $2b^2 = (2m)^2$ which we can rewrite as $2b^2 = 4m^2$. Dividing both sides by 2 gives $b^2 = 2m^2$. But now we can use exactly the same reasoning as before. Since $2m^2$ is even, b^2 must be even and thus b must be even. We can now say that $b = 2n$ for some integer n. But here comes the contradiction. Remember that our initial specification was that $\frac{a}{b}$ is in simplest form. But we now have $\frac{a}{b} = \frac{2m}{2n}$, and this *can* be simplified further, to $\frac{a}{b} = \frac{m}{n}$. This contradiction means that our original assumption must have been wrong, so $\sqrt{2}$ is indeed irrational. QED.

I would like to conclude by responding to a criticism people often have about mathematics. "What's the point of it all?" or another favourite, "When am I going to use this in real life?" Now I could simply say that mathematics is important because it can be used in science, technology, engineering, finance, computing, and so on, and this is usually the standard answer. But, to be honest, it wouldn't matter to me if it didn't have these applications. Mathematics, to me, is a form of art – and *that* is the point of it. It may not be as accessible as other forms of art, but to those for whom it appeals it truly is beautiful.