Finding the Greatest Common Divisor by Repeated Subtractions

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INTRODUCTION

The greatest common divisor (GCD) of two or more positive integers is the largest integer that is a common divisor of the given integers – i.e. the largest integer that divides the given integers without leaving a remainder. There are many methods for finding the GCD of two or more integers. Some of the most commonly taught in schools are the methods of factor listing, prime factorisation, and common decomposition using the so-called ladder method. Each of these methods is briefly illustrated below for the pair of integers 18 and 24.

- FACTOR LISTING

  In this method the factors of each number are listed, and the greatest common factor is identified.

  Factors of 18: 1 ; 2 ; 3 ; **6** ; 9 ; 18

  Factors of 24: 1 ; 2 ; 3 ; 4 ; **6** ; 8 ; 12 ; 24

  Since 6 is the greatest common factor of 18 and 24, it follows that 6 must also be the greatest common divisor of 18 and 24. By direct inspection we thus have GCD (18, 24) = 6.

- PRIME FACTORISATION

  In this method, each of the numbers is prime factorised.

  $18 = \mathbf{2} \times \mathbf{3} \times 3$

  $24 = \mathbf{2} \times 2 \times 2 \times \mathbf{3}$

  Since 2 and 3 are the common prime factors of 18 and 24, we have GCD (18, 24) = 2 × 3 = 6.

- COMMON DECOMPOSITION USING THE LADDER METHOD

  In this method, each of the numbers is divided by the smallest common prime divisor. This process is continued until there is no further common prime divisor.

$$\begin{array}{r|cc} \mathbf{2} & 18 & 24 \\ \hline \mathbf{3} & 9 & 12 \\ \hline & 3 & 4 \end{array}$$

  Since 2 and 3 are the common divisors of 18 and 24, we have GCD (18, 24) = 2 × 3 = 6.

AN ALTERNATIVE METHOD FOR FINDING THE GCD

An alternative means of finding the GCD of two or more integers is a simple yet elegant Chinese method based on repeated subtractions. This method of finding the GCD dates back to ancient China and is mentioned for example in *Jiuzhang Suanshu* (also called *The Nine Chapters on the Mathematical Art*), an ancient Chinese mathematics text compiled by several generations of scholars and likely completed in the Han dynasty in the first century (Man, 2011). The method, according to Shen et al. (1999), works as follows:

> Subtract the smaller number from the greater. Repeat the process until the remainders are equal, which is the GCD of the given numbers.[1]

- EXAMPLE 1

  Suppose we want to calculate the GCD of 18 and 24. Begin by subtracting the smaller number from the larger number, i.e. subtract 18 from 24 to get 6. Now replace the larger number (i.e. 24) with the difference (i.e. 6) and repeat the process until the two numbers are equal. This repeated process is illustrated below:

$$
\begin{array}{l}
(24,18) \\
\quad \downarrow \; 24-18=6 \\
(6,18) \\
\quad \downarrow \; 18-6=12 \\
(6,12) \\
\quad \downarrow \; 12-6=6 \\
(6,6)
\end{array}
$$

  The GCD of 18 and 24 is thus 6, as previously obtained.

- EXAMPLE 2

  Suppose we want to calculate the GCD of 15 and 24. Again, we can apply repeated subtractions to find the answer as follows:

$$
\begin{array}{l}
(24,15) \\
\quad \downarrow \; 24-15=9 \\
(9,15) \\
\quad \downarrow \; 15-9=6 \\
(9,6) \\
\quad \downarrow \; 9-6=3 \\
(3,6) \\
\quad \downarrow \; 6-3=3 \\
(3,3)
\end{array}
$$

  The GCD of 15 and 24 is thus 3.

[1] Note that this method is related to a similar technique, often referred to as the *Euclidean Algorithm*, as described in Euclid's *Elements*.

WHY DOES THIS METHOD WORK?

The following well-known results from elementary number theory shed light on why the method of repeated subtractions works. If a, b and c are assumed to be integers, then:

- **Theorem 1:** If $a = b$, then GCD(a, b) = a
- **Theorem 2:** If a divides b and c, where $b < c$, then a divides $(c - b)$
- **Corollary:** If a = GCD(b, c) where $b < c$, then then a divides $(c - b)$

The method thus works because of the principle, which results from the above theorems and corollary, that the GCD of two numbers remains unchanged when the larger number is replaced by its difference with the smaller number.

EXTENSION

Although the general proof gets a little technical, it is worth noting that this method can be employed to find the GCD of more than two integers as well, as illustrated below for 24, 56 and 120. Work with the largest pair, each time replacing the larger of the two numbers by its difference with the smaller number:

$$
\begin{array}{l}
(120, 56, 24) \\
\quad \downarrow\ 120 - 56 = 64 \\
(64, 56, 24) \\
\quad \downarrow\ 64 - 56 = 8 \\
(8, 56, 24) \\
\quad \downarrow\ 56 - 24 = 32 \\
(8, 32, 24) \\
\quad \downarrow\ 32 - 24 = 8 \\
(8, 8, 24) \\
\quad \downarrow\ 24 - 8 = 16 \\
(8, 8, 16) \\
\quad \downarrow\ 16 - 8 = 8 \\
(8, 8, 8)
\end{array}
$$

The GCD of 24, 56 and 120 is thus 8.

CONCLUDING COMMENTS

The simplicity of this method lies in the fact that the only operation it requires is subtraction. For this reason it could readily be introduced to school students as an alternative method to those typically found in school textbooks. Furthermore, it is a useful way of introducing the idea of an *algorithm*, as well as highlighting an idea that dates back to ancient China.

REFERENCES

Man, Y. K. (2011). Chinese Mathematics. In Sarah J. Greenwald & Jill E. Thomley (Eds.), *The Encyclopedia of Mathematics and Society* (Volume I), pp.184-188, CA: Salem Press.

Shen, K. S., Crossley, J. N., & Lun, A. W. C. (1999). *The Nine Chapters on the Mathematical Art: Companion and Commentary*. Oxford: Oxford University Press.