Cautionary Tales of Geometric Converses

Duncan Samson[1], Avi Sigler[2] & Moshe Stupel[2,3]

[1]St Andrew's College, Grahamstown

[2]Shaanan College, Haifa, Israel

[3]Gordon College, Haifa, Israel

d.samson@sacschool.com avibsigler@gmail.com stupel@bezeqint.net

INTRODUCTION

Within the South African school Mathematics syllabus, pupils are exposed to a number of geometric theorems along with their converses. By way of example:

- **Theorem:** The line drawn from the centre of a circle perpendicular to a chord bisects the chord.
  Converse: *The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.*

- **Theorem:** Angles subtended on a circle by a chord of the circle, on the same side of the chord, are equal.
  Converse: *If a line segment joining two points subtends equal angles at two other points on the same side of the line segment, then the four points are concyclic.*

- **Theorem:** The opposite angles of a cyclic quadrilateral are supplementary.
  Converse: *If any two opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.*

- **Theorem:** The angle between a tangent to a circle and a chord drawn from the point of contact is equal to the angle in the alternate segment.
  Converse: *The line drawn through the end-point of a chord of a circle forming an angle with the chord equal to the angle subtended by the chord in the alternate segment is a tangent to the circle.*

In each of these cases, both the theorem as well as its converse are true. Indeed, in the majority of mathematical theorems, the converse statement is also true. Very seldom at school level are pupils afforded the opportunity to engage with a theorem whose converse statement is false. For this reason, pupils often develop the misconception that, in general, the converse statement of a theorem is also true.

In this article we present three theorems in which the converse statement is false. In each case we present the initial theorem, formulate a possible converse statement, and then investigate a particular scenario which shows the converse statement to be false. Suggestions are made as to how these three cases could be explored in a classroom context.

THEOREM 1

Grade 12 pupils will be familiar with the proportional intercept theorem (a line drawn parallel to one side of a triangle divides the other two sides proportionally) along with its converse theorem (if a line divides two sides of a triangle in the same proportion then the line is parallel to the third side.

Pupils will also be familiar with the midpoint theorem (the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length) as a special case of the proportional intercept theorem, along with its converse theorem (the line segment passing through the midpoint of one side of a triangle parallel to another side bisects the third side).

The first theorem that we present follows from the above two theorems.

Theorem: Given triangle ABC such that AE = EC and DE ∥ BC, then DE = ½ BC.

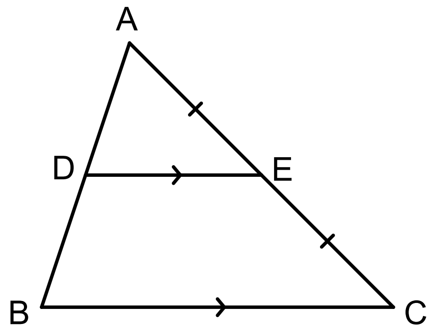

FIGURE 1: Triangle ABC with AE = EC and DE ∥ BC

Converse statement: Given triangle ABC such that AE = EC and DE = ½ BC, then DE ∥ BC.

While the veracity of the theorem follows directly from the proportional intercept theorem and midpoint theorem, the converse statement is false. Consider Figure 2 which shows triangle ABC with AE = EC and DE = ½ BC. A circle with centre E and radius ED is drawn. Note that the circle centred on point E intersects AB at two distinct points, D and D′ (Figure 3). Since ED = ½ BC and ED and ED′ are both radii of the circle, it follows that ED′ = ½ BC. And since ED and ED′ cannot both be parallel to BC, the converse statement is clearly false.

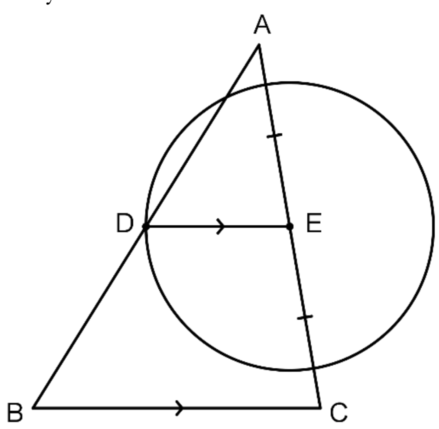

FIGURE 2

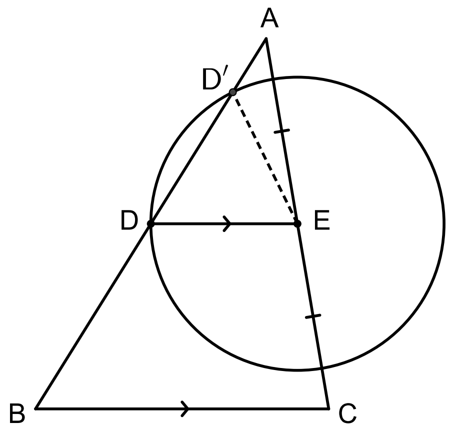

FIGURE 3

In the classroom, first present pupils with the theorem and briefly discuss it in relation to the proportional intercept theorem and the midpoint theorem. Next, present pupils with the converse statement and allow them time to explore it. Make it clear that the converse statement is false, so that its refutation becomes the purpose of the activity. Provide pupils with prompts in the form of questions, for example "Is it possible to draw another line from E to AB with the same length as DE?" or "How about constructing a circle centred on E with radius ED?" One could explore the context dynamically using GeoGebra, for example. In a dynamic environment one could ask further questions, e.g. what would happen if angle B was right-angled? What would happen if AE<DE, i.e. if point A was *inside* the circle centred on E?

THEOREM 2

The next theorem is similar to the previous one, but relates to a trapezium rather than a triangle.

Theorem: Given trapezium ABCD such that AB ∥ DC, AE = ED and EF ∥ AB & DC then EF = ½ (AB + DC).

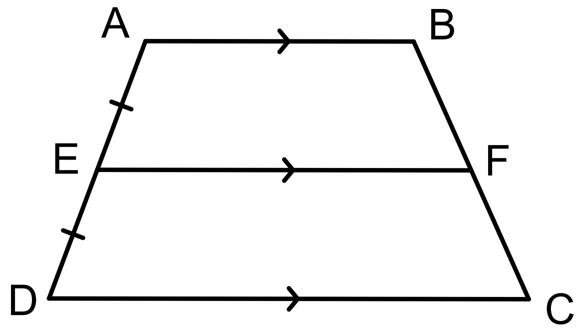

FIGURE 4: Trapezium ABCD with AB ∥ DC, AE = ED and EF ∥ AB & DC

The veracity of the theorem can be visualized quite readily. From the proportional intercept theorem it follows that BF = FC. If we now rotate trapezium ABFE about point F until B coincides with C, then the resulting parallelogram has equal horizontal side lengths DC + BA and 2 × EF, from which the result follows. However, the converse statement of the theorem is false.

Converse statement: Given trapezium ABCD such that AB ∥ DC, AE = ED and EF = ½ (AB + DC) then EF ∥ AB & DC.

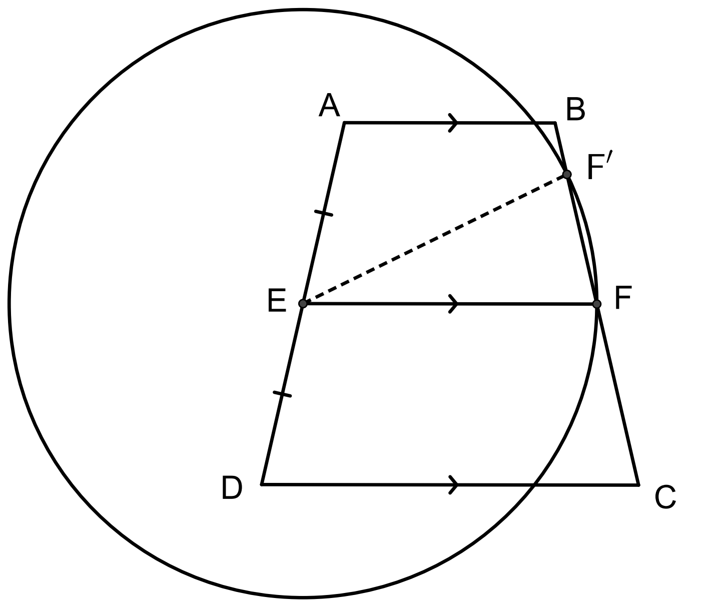

FIGURE 4: Refuting the converse statement

The converse statement can be refuted using a similar approach to that described in Theorem 1. All one needs to do is show that it is possible to construct a circle centred on E with radius EF that cuts side BC at two distinct points (Figure 4).

THEOREM 3

The next theorem relates to the isosceles triangle.

Theorem: Given isosceles triangle ABC with AB = AC and angle bisectors BD and CE intersecting at O, then EO = OD.

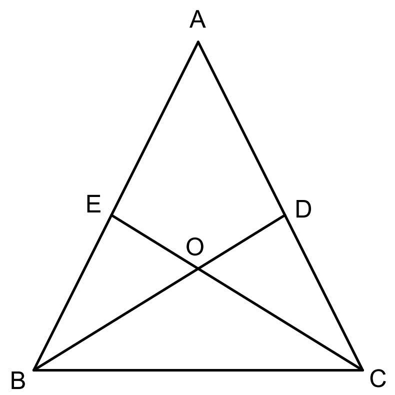

FIGURE 5: Isosceles triangle ABC with AB = AC and angle bisectors BD and CE intersecting at O

The theorem can readily be proved by showing that triangles EBO and DCO are congruent. However, the converse statement is once again false.

Converse statement: Given triangle ABC with angle bisectors BD and CE intersecting at O such that EO = OD, then AB = AC.

A specific counter-example that refutes the generality of the converse statement is the following. Draw triangle ABC with $\angle A = 60°$ and AB ≠ AC. Next, construct angle bisectors BD and CE intersecting at O. Note that in this particular construction it automatically follows that EO = OD.

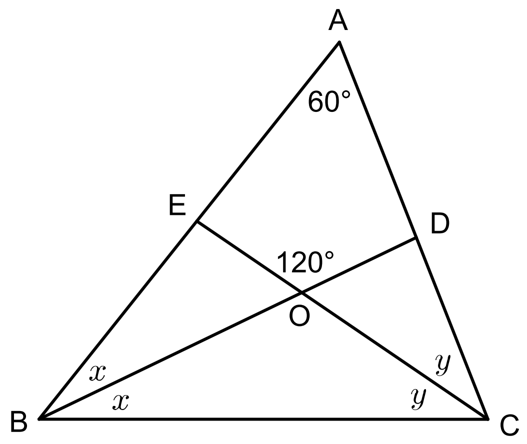

FIGURE 6

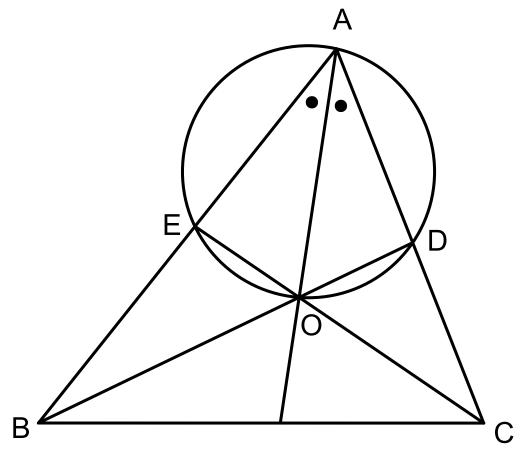

FIGURE 7

In Figure 6, if we let $\angle ABC = 2x$ and $\angle ACB = 2y$, then it follows that $2x + 2y = 120°$ and hence that $x + y = 60°$. From this it follows that $\angle BOC = 120°$ and hence that $\angle EOD = 120°$. Now, since the opposite angles of quadrilateral AEOD are supplementary, it follows that AEOD is cyclic. Furthermore, since O is the point of concurrency of the angle bisectors BD and CE, the bisector of $\angle A$ must also pass through O (Figure 7). We now have chords EO and OD subtending equal angles on the circle, and thus EO = OD. However, from our initial conditions we have AB ≠ AC, so our converse statement is false.

In the classroom, first present pupils with the theorem and ask them to prove it. Suggest the use of congruency if they get stuck and require a bit of prompting. Since the refutation of the converse statement requires knowledge of concurrency (which is no longer in the curriculum), this provides an excellent opportunity for a bit of extension work. Having explored the point of concurrency of the angle bisectors (i.e. the incentre) – and possibly some of the other points of concurrency as well – ask pupils to draw a non-isosceles triangle ABC with $\angle A = 60°$ and angle bisectors BD and CE intersecting at O. Now challenge them to prove that AEOD is a cyclic quadrilateral. Once this has been established, and using their knowledge of the incentre, ask pupils to try to prove that EO = OD.

As a further extension activity one could explore similar scenarios where BD and CE are either medians or altitudes as opposed to angle bisectors. In the case of medians and altitudes the converse statements are in fact true, i.e. AB = AC.

CONCLUDING COMMENTS

As we remarked in the introduction, school pupils are seldom afforded the opportunity to engage with geometric theorems whose converse statement is false. Many pupils thus leave school having developed the misconception that converse statements of geometric theorems are in general true. Such a misapprehension is not surprising given most pupils' school mathematical experience.

A careful exploration of geometric theorems in which the converse statement is false will not only enrich pupils' mathematical experience, it will also encourage them to be both critical and circumspect when considering the converse statement of a geometrical theorem. We believe the three theorems presented in this article provide a rich context for doing just that.

Not only do these theorems highlight the need for criticality when engaging with converse statements, they also allow for geometric exploration in a 'non-traditional' way. In the article we have described how classroom engagement could be structured around these theorems. Additionally, pupils could be encouraged to carry out dynamic explorations of the various scenarios using GeoGebra. Alternatively, perhaps these theorems, along with one or two others, could be worked into a guided portfolio item.

As an important final remark, note that in each of the three theorems discussed, the proposed converse statement isn't the only possible converse – a number of alternative formulations are also possible. Each of the three theorems has more than one premise, and as such a variety of converse statements are possible. By way of example, if we have a theorem of the form 'if p and q, then r', then possible converse statements could include 'if p and r, then q' or 'if r and q, then p' or simply 'if r, then p and q'. As an extension activity one could perhaps encourage pupils to formulate their own alternative converse statements and explore them.