A Dynamic Investigation of Geometric Properties with 'Proofs Without Words'

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INTRODUCTION

Dynamic geometry environments are a powerful way of engaging students in real-time mathematical exploration. Students are able to investigate mathematical properties through dynamic engagement by dragging objects and observing the effect immediately. Through this process it is possible not only to investigate geometric properties but to form conjectures and hypotheses relating to additional properties. Although it may be easy enough to establish that a conjecture is not true, we need to be a little more careful with establishing its veracity. Although the dynamic geometry environment can lead us to suspect that a conjecture is true, to verify that it is indeed true still requires a formal geometric proof. In this article we present a series of progressive tasks that are ideally suited to exploration in a dynamic geometry environment. The tasks are gradually developed through 'what if' question posing (Brown & Walter, 1990, 1993). Rather than getting students to attempt to prove various conjectures on their own (which they could of course do if they wanted), 'Proofs Without Words' (PWWs) are presented as a route to this verification process (Katz, Segal & Stupel, 2016; Nelsen, 2001; Sigler, Segal & Stupel, 2016). The idea is for students to engage with each PWW diagram, attempt to make sense of it, and then to articulate a formal geometric proof of the conjecture based on the PWW diagram.

TASK 1 – EQUAL SEGMENTS

CASE A: THE ACUTE-ANGLED TRIANGLE

An acute-angled triangle ABC with its circumcircle and orthocenter H is given. The chords AK, BL and CM all pass through the point H and intersect the triangle's sides at points D, E and F respectively. Given this setup note that HD = DK, HE = EL and HF = FM (Figure 1).

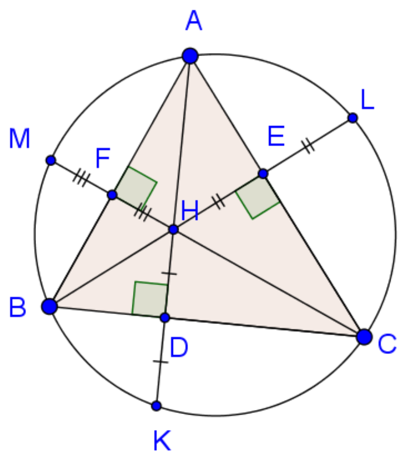

FIGURE 1: Equal segments created in a circle

Students should explore whether these pairs of line segments remain equal when the vertices of the triangle are moved, i.e. when the shape of the triangle is changed. For this we built a GeoGebra applet (Link 1) in which one can drag the triangle's vertices and thus change the lengths of its sides and the sizes of its angles while keeping track of the various segment lengths. Once students have engaged with the applet and found that the pairs of line segments remain equal, they should be presented with the PWW shown in Figure 2 which provides a proof of HD = DK (the proofs of HE = EL and HF = FM can be visualized in a similar manner). Students should use the PWW as a basis for articulating a formal geometric proof that HD = DK

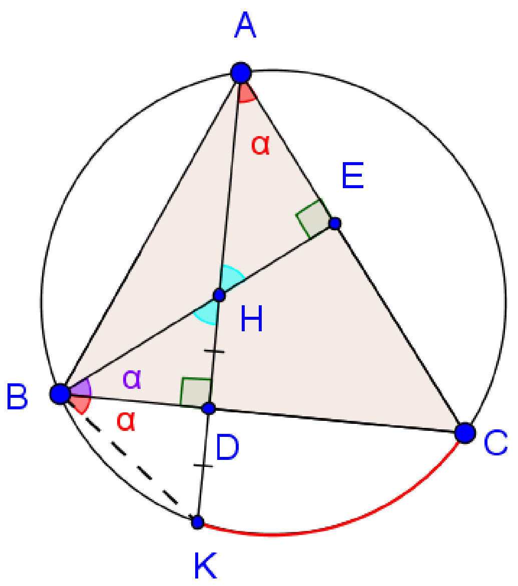

FIGURE 2: PWW for HD = DK

Figures 1 and 2 show the scenario for an acute-angled triangle. Would the pairs of line segments remain equal if the triangle was right-angled or obtuse?

CASE B: THE RIGHT-ANGLED TRIANGLE

The dynamic investigation shows that for a right-angle triangle, in which the altitudes meet at point A (which coincides with point H), we obtain a segment AK that is perpendicular to the diameter of the circle (Figure 3). In this case it is clear that HD = DK since a line from the centre perpendicular to a chord bisects the chord.

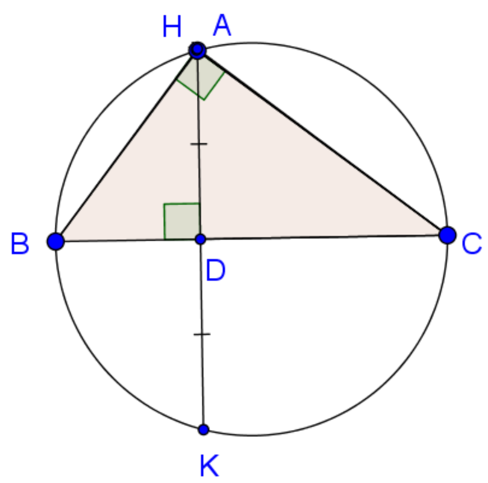

FIGURE 3: Chord AK bisected by diameter BC

CASE C: THE OBTUSE-ANGLED TRIANGLE

For an obtuse triangle the altitudes meet at a point exterior to the circumcircle (Figure 4). In this case we obtain that HD = DK, LE = EH and MF = FH.

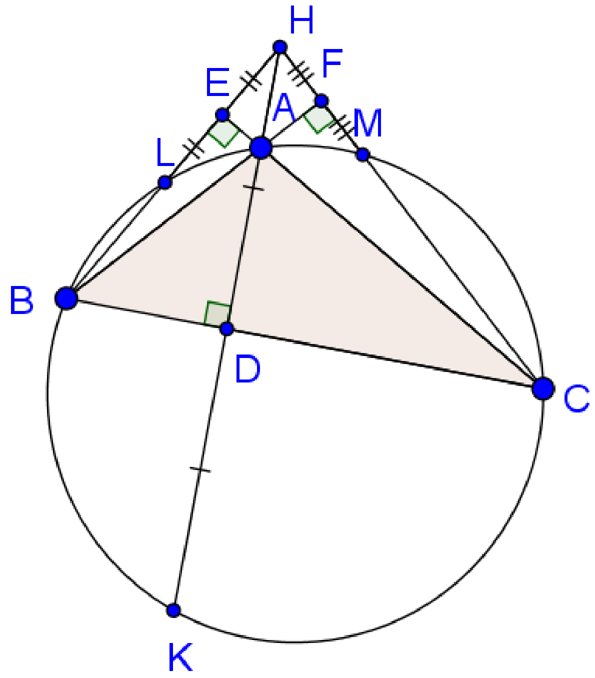

FIGURE 4: The case of the obtuse triangle

The PWWs for HD = DK and MF = FH are shown in Figure 5a and Figure 5b respectively. The proof of LE = EH is similar to that of MF = FH.

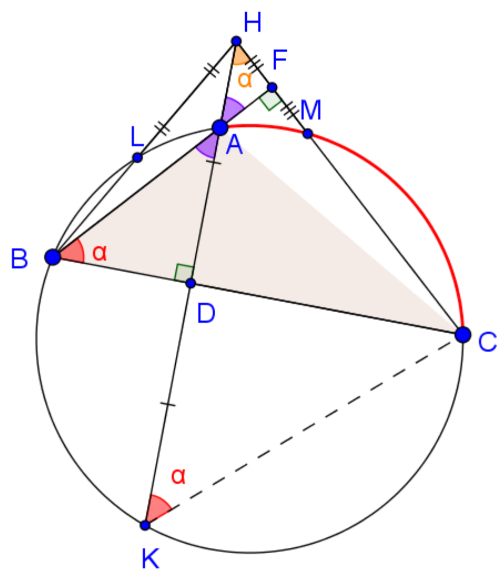

FIGURE 5a: PWW for HD = DK

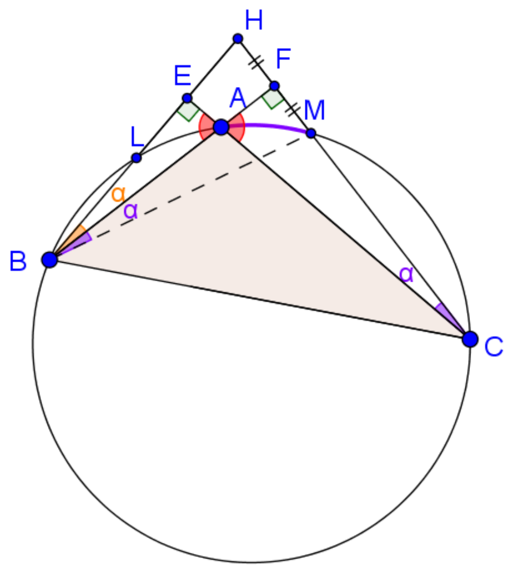

FIGURE 5b: PWW for MF = FH

As previously, students should engage with these PWWs and use them as a basis for verbalising formal geometric proofs.

TASK 2 – THE CREATION OF SIMILAR TRIANGLES AND ISOSCELES TRIANGLES

CASE A: SIMILAR TRIANGLES

Given a triangle ADB, a circle is drawn passing through vertices A and B such that vertex D is exterior to the circle. The sides AD and BD intersect the circle at points E and C respectively. Given this setup, note that triangles DAB and DCE are similar (Figure 6a). Students should explore this scenario using the prepared GeoGebra applet (Link 2) by dragging vertices B and D while keeping track of the appropriate angles. Once students have established that the triangles remain similar they should be presented with the PWW shown in Figure 6b which presents the case where points E and C are on the same arc in relation to chord AB. The reader can easily adjust the PWW for the cases where points E and C are on different arcs or where one of them coincides with the vertex (i.e. where one of the sides is a tangent to the circle).

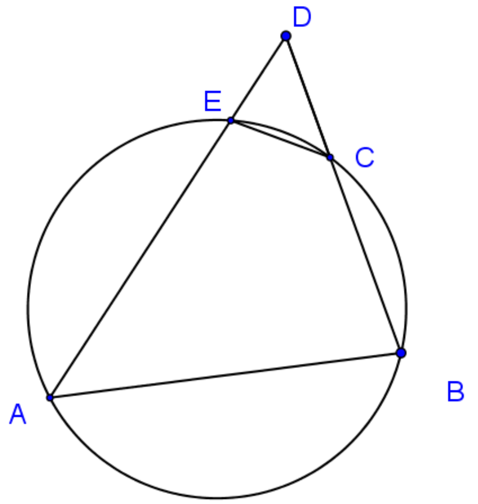

FIGURE 6a: Similar triangles DAB and DCE

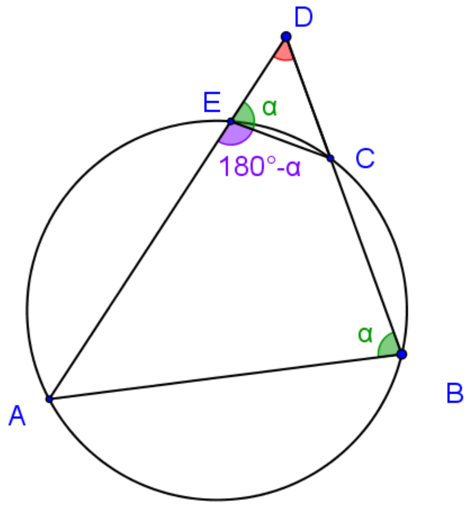

FIGURE 6b: PWW for $\Delta DAB|||\Delta DCE$

CASE B: ISOSCELES TRIANGLES

Given triangle ADB, what would happen if the circle through vertices A and B was constructed such that AB was a diameter of the circle and AC was the bisector of $\angle DAB$? Using the constructed GeoGebra applet (Link 3) one can find that in this case triangles DCE and DAB are not only similar but also isosceles (Figure 7).

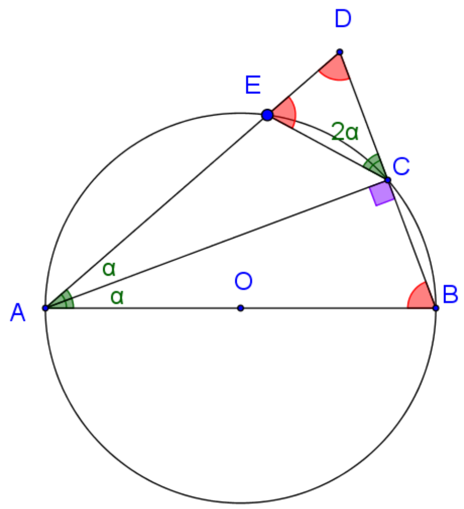

FIGURE 7: PWW for similar isosceles triangles DCE and DAB

TASK 3 – INTERSECTION OF ANGLE BISECTORS IN A QUADRILATERAL

This final task involves an exploration of the points of intersection of the four angle bisectors in a quadrilateral. Let A, B, C and D be the vertices of some convex quadrilateral and let the angle bisectors of ABCD intersect at points K, I, N and G. These four points form the vertices of a cyclic quadrilateral. The initial setup is illustrated in Figure 8 while a PWW is shown in Figure 9.

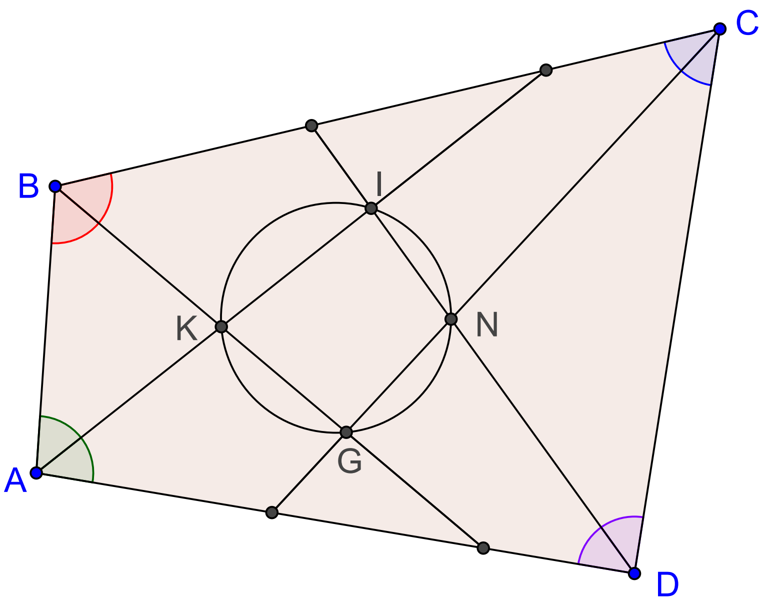

FIGURE 8: Cyclic quadrilateral KING

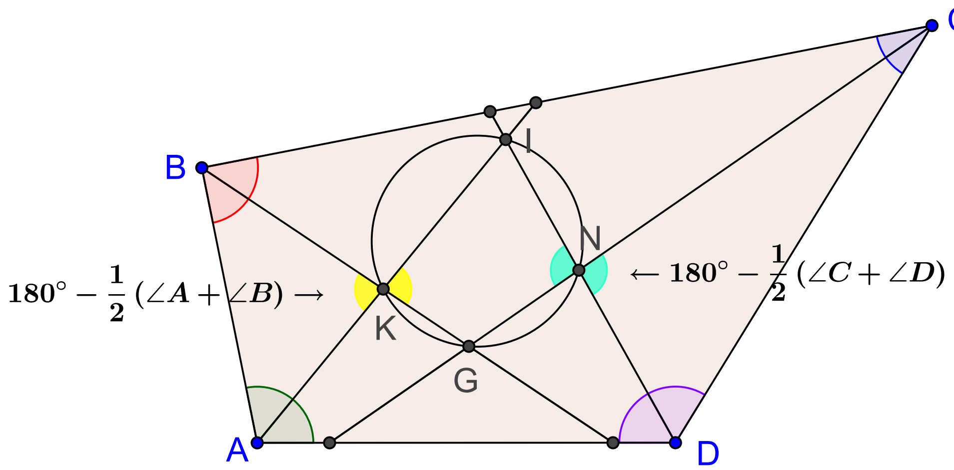

FIGURE 9: PWW for ∠IKG + ∠ING = 180°

Having explored the initial setup students should be encouraged to consider the following. Does the property persist for concave (as opposed to convex) quadrilaterals? What would happen if ABCD were a square, rhombus, kite or parallelogram? These questions can be explored dynamically using the prepared GeoGebra applet (Link 4).

It is left for the reader to explore the following observations:

- The property persists for a concave quadrilateral. In this case a crossed (i.e. self-intersecting) cyclic quadrilateral is obtained (Figure 10).
- When ABCD is a square, rhombus or kite, the quadrilateral KING degenerates into a single point.
- When ABCD is a parallelogram, the quadrilateral KING is a rectangle.

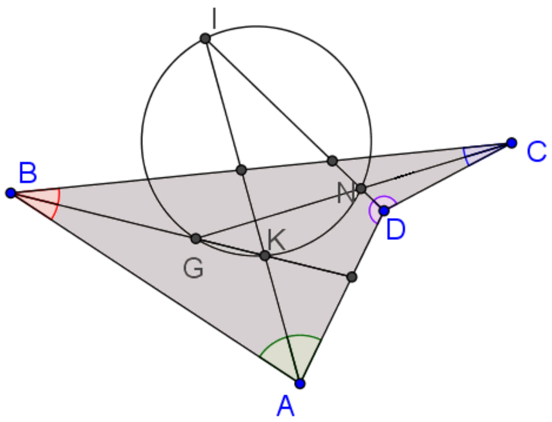

FIGURE 10: Concave quadrilateral ABCD with cyclic crossed quadrilateral KING

CONCLUDING COMMENTS

Dynamic geometry environments provide a wonderful opportunity for students to actively engage with geometric contexts. Such environments provide a dynamic way to develop visual thinking capability which is such an important component of mathematical problem solving. The incorporation of 'Proofs Without Words' adds an additional element by focusing students' attention on specific components of the diagram en route to a formal geometric proof.

ACKNOWLEDGEMENTS

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GEOGEBRA APPLET LINKS

Link 1: https://www.geogebra.org/m/n63E3sAC

Link 2: https://www.geogebra.org/m/MkJN3jY3

Link 3: https://www.geogebra.org/m/hraXruFc

Link 4: https://www.geogebra.org/m/w8u6e929