Exploring Learners' Geometric Understanding

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INTRODUCTION

Geometry plays an important role in the development of mathematical thinking. This article explores the use of a particular type of formative assessment that has the potential to provide insight into learners' geometric understanding and reasoning, interrogating both concept image as well as concept definition. Examples of responses to these assessment items are provided and briefly analysed in terms of learners' potential misconceptions. Suggestions are made as to how to move learners beyond such misconceptions.

CONCEPT IMAGE, CONCEPT DEFINITION & PROTOTYPE IMAGE

When engaging with learners' geometric understanding, a useful distinction to consider is that between 'concept image' and 'concept definition' (Tall & Vinner, 1981). A concept image can be thought of as a collection of mental representations and associated properties which grows and develops through a learner's engagement with examples and non-examples of a particular concept. By contrast, a concept definition is an articulated expression (either written or verbal) of the concept. This may be a formal definition, i.e. one that is accepted by the general mathematical community, or it can be a learner's personal reconstruction of the formal definition. The concept definition provides a criterion for distinguishing between examples and non-examples of a particular concept. When learners engage with a task, they typically ground themselves in their concept image rather than in the concept definition. Any discrepancy between the concept image and the concept definition will thus inevitably lead to errors and/or further misconceptions.

Another useful concept to consider when engaging with learners' possible misconceptions is that of a 'prototype image', the mental go-to image that a learner has of a particular concept. If, for example, a learner has a prototype image of a square with its base horizontal, a square in a different orientation (e.g. rotated through 45°) might not immediately be perceived as a square. Many errors stem from the judgment by a prototype image instead of a judgment by the concept definition.

RELATIONS OF INCLUSION BETWEEN SHAPES

While building geometric concepts and their interrelations, learners need to engage with the hierarchical classification of the various shapes/concepts. The relation of inclusion, which relates to a single shape being included in several categories, is characterised by asymmetry and transitivity. The asymmetry is demonstrated by the fact that a shape which is included in the larger family includes a shape from the smaller family, but not vice versa. Thus, every rhombus is a parallelogram but not every parallelogram is a rhombus. The transitivity is manifested by the fact that when the first shape is included in a second, and the second shape is included in a third, then the first shape is included in the third. For example, a square is a rectangle and a rectangle is a parallelogram. Hence, a square is a parallelogram. Understanding the hierarchical relations between concepts is an important part of the concept building process, and facilitates generalisations, connecting concepts and deductive reasoning.

USING DYNAMIC GEOMETRY SOFTWARE

Studies show that dynamic software-oriented learning enhances the structuring of geometric concepts and the development of inductive and deductive reasoning. Geometry software such as *GeoGebra* enables the dynamic manipulation of geometric constructions. Shapes, constructed in accordance with their defining properties, can be translated, stretched, and rotated while attention is focused on which features remain the same, and which change. Such dynamic geometry environments also enable the presentation of counter-examples for the purpose of refuting false hypotheses.

FORMATIVE ASSESSMENT USING TRUE/FALSE QUESTIONS

Now that we have considered a number of important background concepts we can move on to the main thrust of the article, namely the use of true/false questions as a type of formative assessment aimed at providing insight into learners' geometric understanding. For each true/false question, learners need to decide whether the given statement is either true or false, and provide a justification for their decision. The statements used in these true/false questions can be formulated for the purpose of defining concepts, managing incorrect concept images, changing prototype images, and promoting understanding of relations of inclusion between shapes. Through the analysis of learner responses, teachers should be able to obtain a picture of each learner's understanding, allowing them to plan future teaching with more directed emphasis. In order to illustrate the potential value of true/false questions, four examples are now illustrated. In each example a learner response is given which indicates an underlying misconception. The response is analysed, and remediation activities are suggested.

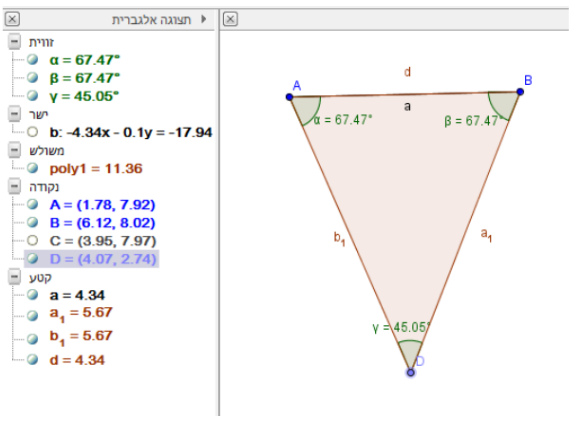
EXAMPLE 1

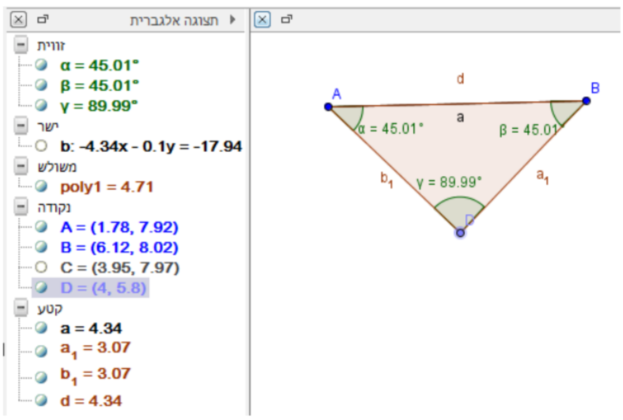
Statement: An isosceles triangle, one of whose angles is 45º, is also right-angled.

Learner response: True. The sum of the angles in a triangle equals 180º. Therefore by subtracting two base angles we obtain the apex angle which equals 90º.

Analysis: The learner has ignored the option that the given angle could also be the apex angle. The learner has an erroneous concept image related to the base angles.

Suggestion: Use dynamic geometry software to construct a translatable isosceles triangle (there are a number of different construction protocols one could potentially use). By shifting the vertices it becomes clear that two possible options exist for an isosceles triangle containing a 45° angle: a right-angled triangle in which the 45º angle is one of the two equal base angles (as asserted in the statement), but also a triangle in which the 45º angle is the apex angle, in which case it is not a right-angled triangle.

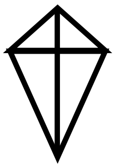




EXAMPLE 2

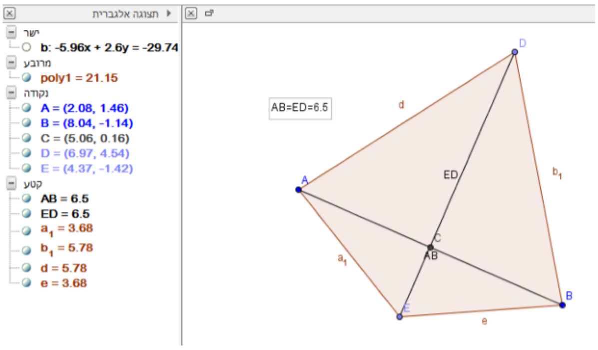
Statement: There is no kite whose diagonals are equal to each other.

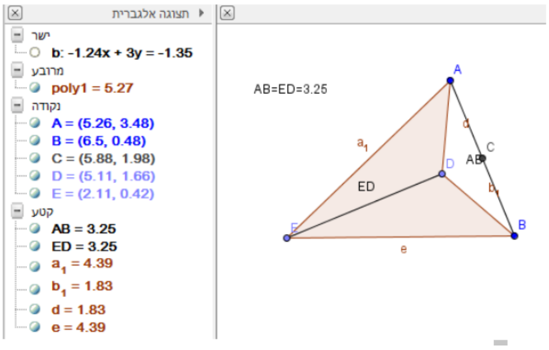
Learner response: True. The offered justification: drawing a typical kite whose diagonals are different.



Analysis: The learner is using an incorrect concept image or prototype image and has difficulty in understanding relations of inclusion.

Suggestion: Use dynamic geometry software to construct a kite using the following construction protocol: (i) draw segment AB, (ii) construct a perpendicular bisector to the segment, and create two points, E and D, on the perpendicular bisector, (iii) join E to D and hide the perpendicular bisector, (iv) form kite AEBD using line segments. The learner can now shift the vertices of the kite while observing the measurements of its diagonals. Shifting the vertices will reveal not only convex kites with equal diagonals, but also concave kites with this property. One can also illustrate in this manner that a square is also a kite with equal diagonals.





EXAMPLE 3

Statement: A quadrilateral with three right angles is a rectangle.

Learner response: False. It could also be a square.

Analysis: The learner has a difficulty in understanding the relations of inclusion (a rectangle includes a square) as well as with attributing a concept to several categories and referring to it by several names (the square can also be called a rectangle).

Suggestion: Using dynamic geometry software, construct a quadrilateral with three right angles. When shifting a vertex the structural integrity of the quadrilateral remains intact (i.e. the three right angles remain unchanged) yet the translations allow for a variety of rectangles to be formed, including a square. Since the construction has been carried out according to the given criteria (three right angles), shifting the vertices helps learners understand that a square is included in a rectangle, and that the most general quadrilateral obtained in accordance with the construction conditions is a rectangle.

EXAMPLE 4

Statement: A quadrilateral whose diagonals are perpendicular to each other is a rhombus.

Learner response: True. In a rhombus the diagonals are perpendicular to each other.

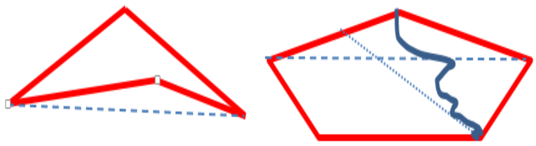
Analysis: The learner is assuming a converse relationship based on the properties of a rhombus, as well as having difficulty in understanding the relations of inclusion (a quadrilateral whose diagonals are perpendicular to each other *includes* the rhombus).

Suggestion: Construct a quadrilateral through a construction protocol based on perpendicular diagonals. Shifting the vertices results in a variety of quadrilaterals whose diagonals are perpendicular, among them a square, rhombus, and an isosceles trapezium.

RECOMMENDATIONS FOR CONCEPT BUILDING

When teaching a geometric concept, I have found the following particularly helpful in terms of building meaningful concept images in learners:

- Get learners to draw shapes in different orientations so as to avoid becoming "locked" in a single prototype image.
- Get learners to provide examples as well as non-examples for each concept. For example, when defining a diagonal in a polygon, draw various examples as well as non-examples and ask learners to identify which are diagonals and which are not.



- Get learners to draw shapes according to specified requirements. For example, 'draw a quadrilateral which has only two right angles and which is not an isosceles trapezium'. Or 'draw a quadrilateral whose diagonals are perpendicular to each other and which is neither a rhombus nor a kite'.
- Get learners to explore statements whose inverse statement is false. Dynamic geometry software lends itself beautifully to this.

CONCLUDING COMMENTS

In the process of learning geometrical concepts, learners often form a faulty or incomplete concept image which in turn leads to further misunderstanding. This article shows how the use of true/false questions has the potential to provide useful insight into learners' geometric understanding and reasoning, and how dynamic geometry software could be used to challenge and strengthen each learner's concept image for a given geometric shape.

REFERENCES

Tall, D.O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.