Makhubo Constructs a 45° Angle

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INTRODUCTION

Mathematics is filled with wonderful surprises. This article describes an account of one such surprise from a recent workshop conducted by *Teachers Across Borders – Southern Africa* (TABSA) in Bloemfontein. TABSA is an international non-governmental organization. The week-long workshop in Bloemfontein was attended by local classroom teachers. Ntaopane Joseph Makhubo (the first author) was one of these local teachers, while Brad Uy (the second author) was one of the workshop facilitators.

SETTING THE SCENE

During the course of the workshop, participants were required to construct 60° and 30° angles. Makhubo did this by constructing two circles with equal radii in the Cartesian plane, as illustrated in Figure 1. The centres of the respective circles were at points A and B on the x-axis, with the circle centred on A passing through B and vice versa. With points A and B equidistant from the y-axis, with AB equal to the radius of each circle, the circles intersect on the y-axis at point K such that triangle KAB is equilateral. Angle $K\hat{A}B$ is thus 60°. To form a 30° angle Makhubo connected point K to point D creating isosceles triangle DAK with angle $K\hat{D}A$ being 30° (angle at centre equals twice the angle at the circumference). Note that the 30° angle could alternatively have been constructed through bisecting the 60° angle.

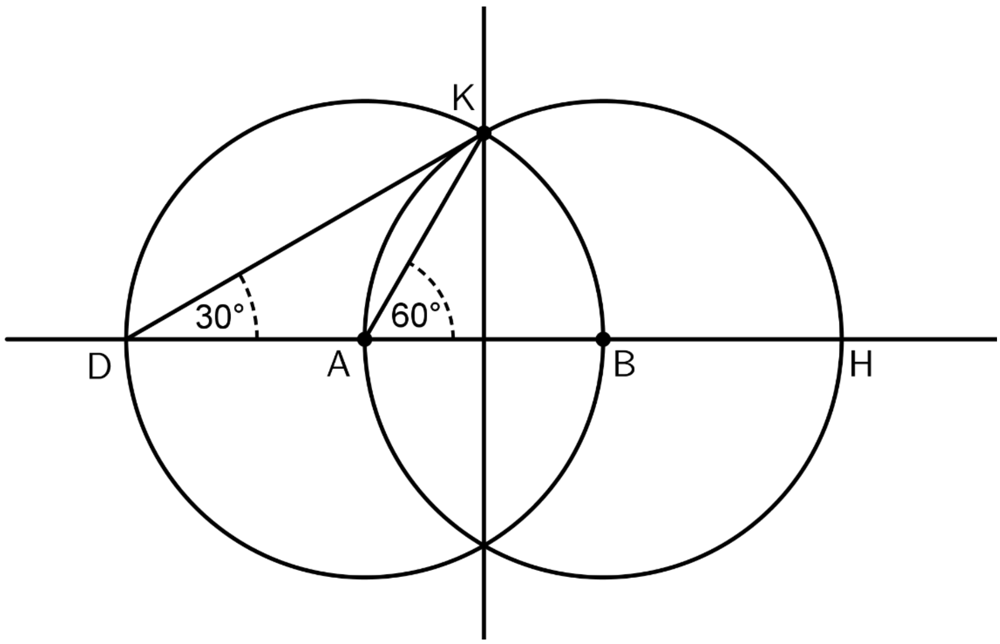

FIGURE 1: Constructing a 60° and 30° angle.

Following on from this construction, the following dialogue took place between Makhubo and Brad:

Makhubo: *So now that we have constructed a 30° angle and a 60° angle* [referring to Figure 1], *let us now construct a 45° angle. Since $K\hat{O}H$ is a right angle, we can bisect it to form a 45° angle. Simply open the compass to an arbitrary length and strike equal arcs from points H and K, and then draw a line through the origin and the point of intersection of the arcs* [Figure 2].

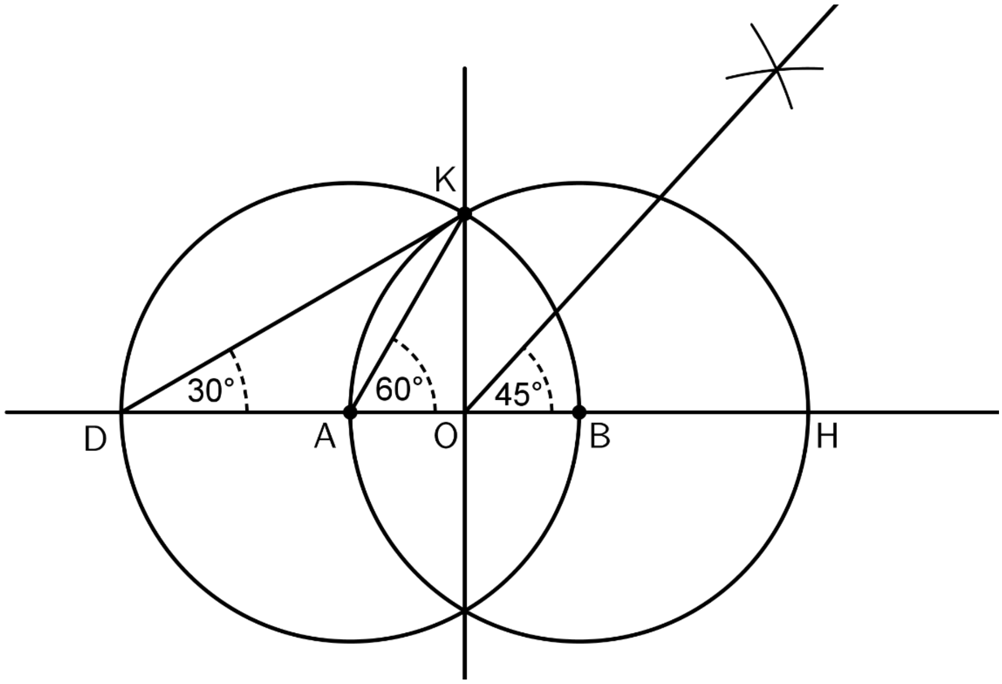

FIGURE 2: Constructing a 45° angle.

Brad: *Don't we need to use two points that are equidistant from the vertex of the right angle when we strike the two arcs?*

Makhubo: *Oh, that's a good point. Let's check.*

Makhubo then proceeded to bisect the right angle $K\hat{O}H$ using the conventional approach alluded to by Brad. Remarkably, the bisector thus formed passed through the point where the two earlier arcs intersected, and produced exactly the same angle.

Brad: *How can this be? I suspect it's just a coincidence, but it's certainly not one I've seen before. Interesting!*

INVESTIGATING FURTHER

Intrigued by this unexpected outcome, Brad decided to investigate further. If we choose two points that are *not* equidistant from the vertex of a right angle, under what conditions will equal arcs drawn from these two points intersect on the angle bisector? Let us consider a general case, as illustrated in Figure 3, where one point is x units from the vertex on one side of the right angle, and the other point is y units on the other side. We are interested in the point P with coordinates $(a; a)$ which would lie on the line $y = x$, i.e. it would lie on the angle bisector of the right angle at O.

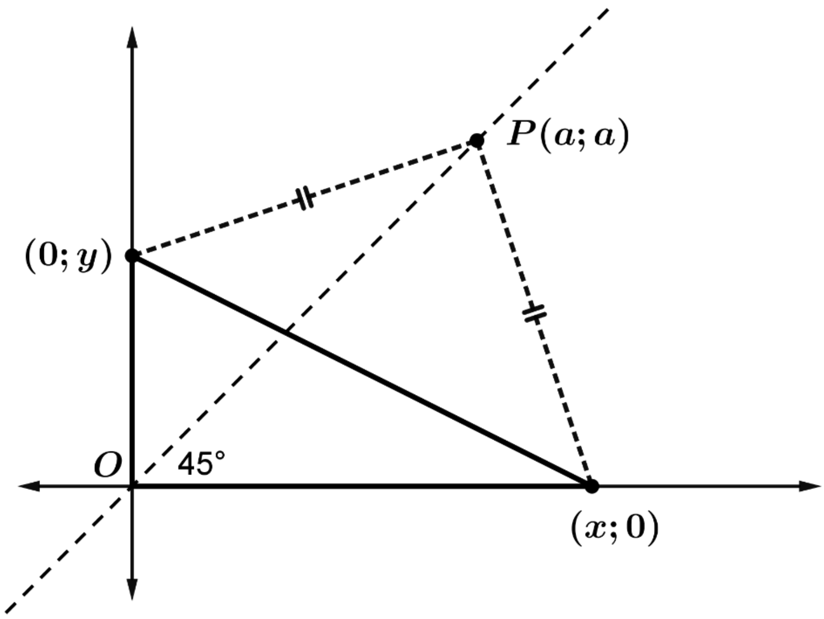

FIGURE 3: Finding the point on the bisector.

Since point P is equidistant from the the two vertices $(0;y)$ and $(x;0)$, we can use the distance formula to equate the two lengths:

$$\begin{aligned}
\sqrt{(a-0)^2+(a-y)^2} &= \sqrt{(a-x)^2+(a-0)^2} \\
a^2+a^2-2ay+y^2 &= a^2-2ax+x^2+a^2 \\
2ax-2ay &= x^2-y^2 \\
2a(x-y) &= (x+y)(x-y) \\
\therefore \quad a &= \frac{(x+y)}{2}
\end{aligned}$$

The crucial point along the angle bisector, determined by the intersection of the two arcs, thus has coordinates $\left(\frac{x+y}{2};\frac{x+y}{2}\right)$. The distance d between this point and either of the two chosen points, $(0;y)$ and $(x;0)$ is thus:

$$\begin{aligned}
d &= \sqrt{\left(\frac{x+y}{2}-x\right)^2+\left(\frac{x+y}{2}-0\right)^2} \\
&= \sqrt{\left(\frac{y-x}{2}\right)^2+\left(\frac{x+y}{2}\right)^2} \\
&= \sqrt{\frac{2x^2+2y^2}{4}} \\
&= \frac{\sqrt{2(x^2+y^2)}}{2}
\end{aligned}$$

THE DESIRED CONSTRUCTION

Knowing that the desired distance from each point is $\frac{\sqrt{2(x^2+y^2)}}{2}$ leads to a simple construction for a 45° angle:

Given a right angle at vertex O, select a point X on one side of the angle and a point Y on the other side as illustrated in Figure 4. Draw hypotenuse XY and construct the perpendicular bisector of XY passing through M, the midpoint of XY. With M as the centre, draw an arc with radius MX through the perpendicular bisector above the triangle. Label the point of intersection of this arc with the perpendicular bisector as point P. The line PO is now the bisector of the right angle and $P\hat{O}X = 45°$.

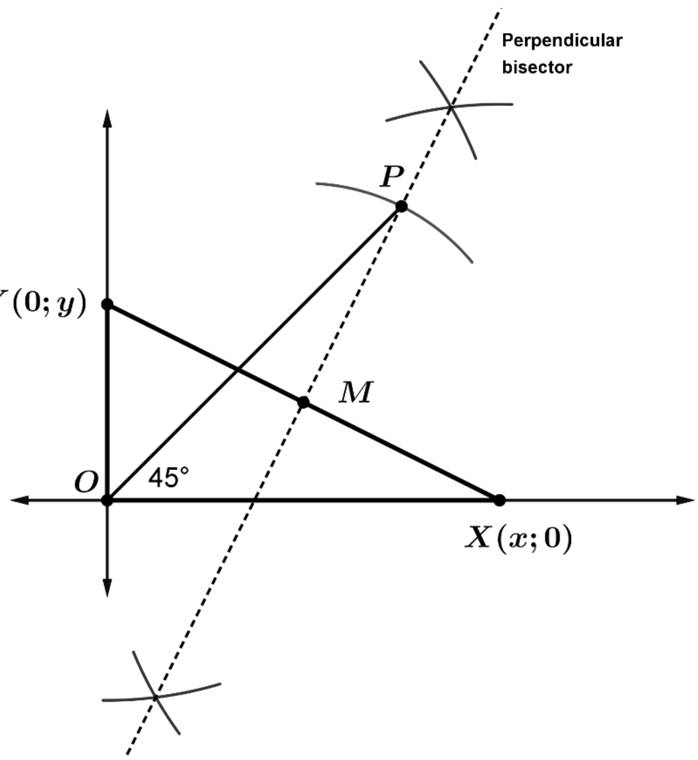

FIGURE 4: Constructing a 45° angle.

PROOF

We can prove that this construction does indeeed bisect the right angle very simply by noting that M is the centre of a circle that circumscribes cyclic quadrilateral OXPY. Since chord PX subtends an angle of 90° at M it follows that PX subtends an angle of 45° at O (angle at centre equals twice the angle at the circumference).

CONCLUDING COMMENT

In this article we have described how a serendipitous construction of a 45° angle led to a fascinating exploration which ultimately yielded an alternative construction process. The lesson here is that mathematics has the potential to surprise us if we listen to one another and share ideas. Had we simply dismissed the original arcs as being wrong we would have missed an opportunity to experience some of the beauty of mathematics.