'Our Brains Really Don't Like Abstract Ideas!' Using Representations Deliberately and Carefully to Help Students Access Abstract Mathematics

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THE CHALLENGE WITH ABSTRACT IDEAS

Our brains REALLY don't like abstract ideas! Instead, we much prefer learning and thinking through concrete examples[1]. This is why we look for concrete and practical examples to help to explain abstract concepts. This creates a challenge in mathematics, where much of the content to be learned is abstract in nature and without a direct link to real-world practice or experiences. This is why Maths teachers sometimes look for 'creative' ways to explain the relevance of the mathematics they are teaching.

The topic of *Factors* is a case in point here. Many students think about factors purely numerically, as two numbers that multiply together to give another number. When quizzed further, they often can't explain what factors 'look like'. This could explain why some students confuse factors and multiples: they struggle to understand how the two concepts are different because numerically they both involve multiplication.

Thankfully, representations and visual models provide the perfect tool to 'concretise' complex and abstract ideas. We spend the majority of our waking hours engaging with visual imagery and stimuli, and the human brain responds positively to information packaged in creative and visual ways. Representations and models provide succinct and organised summaries of information and relationships and, when combined with links to real-world contexts, serve as memory prompts for learned concepts – "can you remember the doughnut lesson, where we learned about using arrays to explore factors?" For these reasons, representations and models also reduce load on working memory and cognitive overload by chunking information into manageable chunks and limiting the amount of information to be processed.

In this article I illustrate the power of representations for helping students visualise abstract concepts like factors and multiples, and then unpick some of the deliberate strategies used in the design of these representations.

DOUGHNUTS AND PIZZAS AND FACTORS … YUM?

Consider the following tasks:

Sam has 30 doughnuts for a school fair.

He wants to arrange the doughnuts in a rectangular arrangement (with equal rows).

Draw pictures to show all of the possible ways that the doughnuts could be arranged.

Marcia has mini-pizzas on her table.

She has 42 pizzas that she wants to place in a rectangular arrangement (with equal rows).

Draw pictures to show all of the possible ways that the pizzas could be arranged.

FIGURE 1: Two tasks involving doughnuts and mini-pizzas.

[1] See, for example, Willingham, D. T. (2009). *Why don't children like school? A cognitive scientist answers questions about how the mind works and what it means for the classroom*. San Francisco: Jossey-Bass.

Possible arrangements for each food treat are:

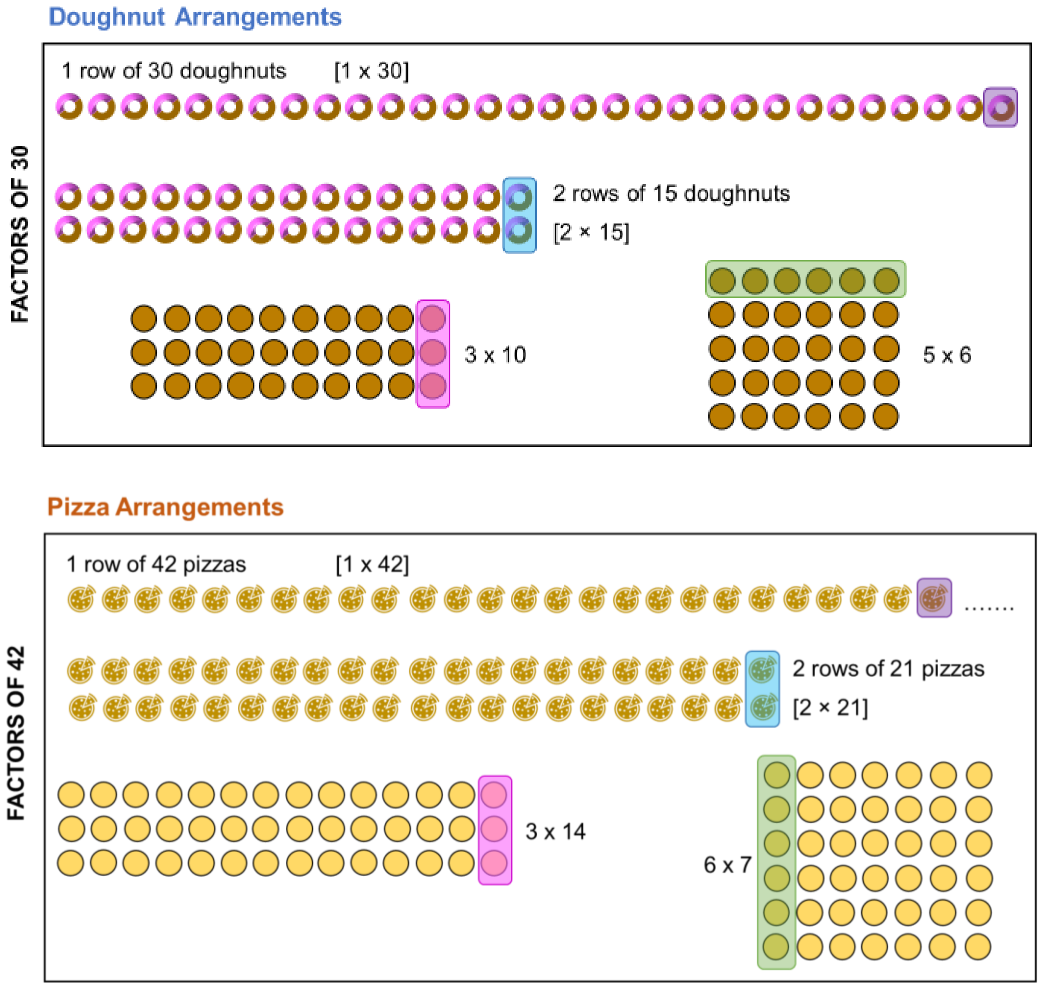

FIGURE 2: Array representations showing factors of 30 and 42.

The dimensions of each arrangement give a different multiplicative relationship that results in the number of dots (doughnuts or pizzas) that is the output of that relationship. Visually, *the <u>dimensions</u> of each rectangular arrangement give the factor pairs for the number represented by that arrangement*.

When picturing factors visually in this way, the *common factors* are now obvious – they are the dimensions of the rectangular arrangements for 30 and 42 that are equal. The *highest common factor* is also visually obvious – this is the biggest dimension of the rectangular arrangements for 30 and 42 that is equal.

This approach also helps to foreground why prime numbers will always only have two factors and why 1 is one of these factors:

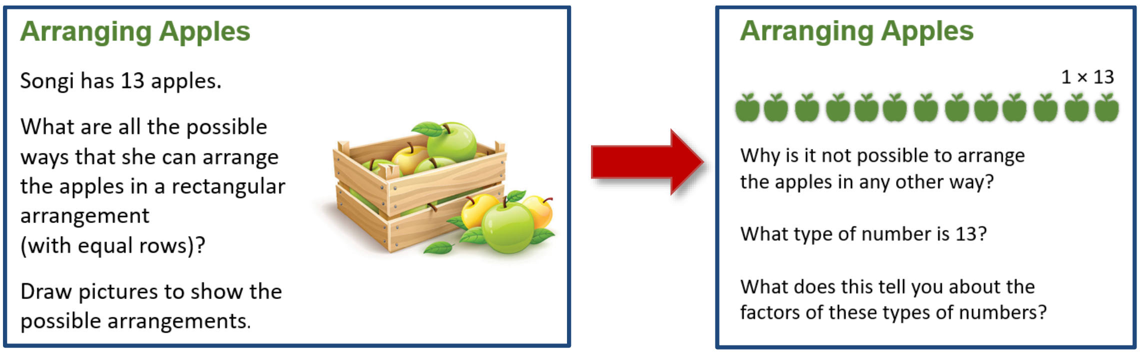

FIGURE 3: Foregrounding why prime numbers have only two factors.

We can use this same approach to explore the process of prime factorisation and show that no matter how the original arrangement or value is constructed as a multiplicative factor pair, the relationship can always be 'factorised' to the same combination of prime factors. This is why prime factors are so useful, because from the prime factors *all* possible factor pairs (arrangements) can be determined or reverse engineered.

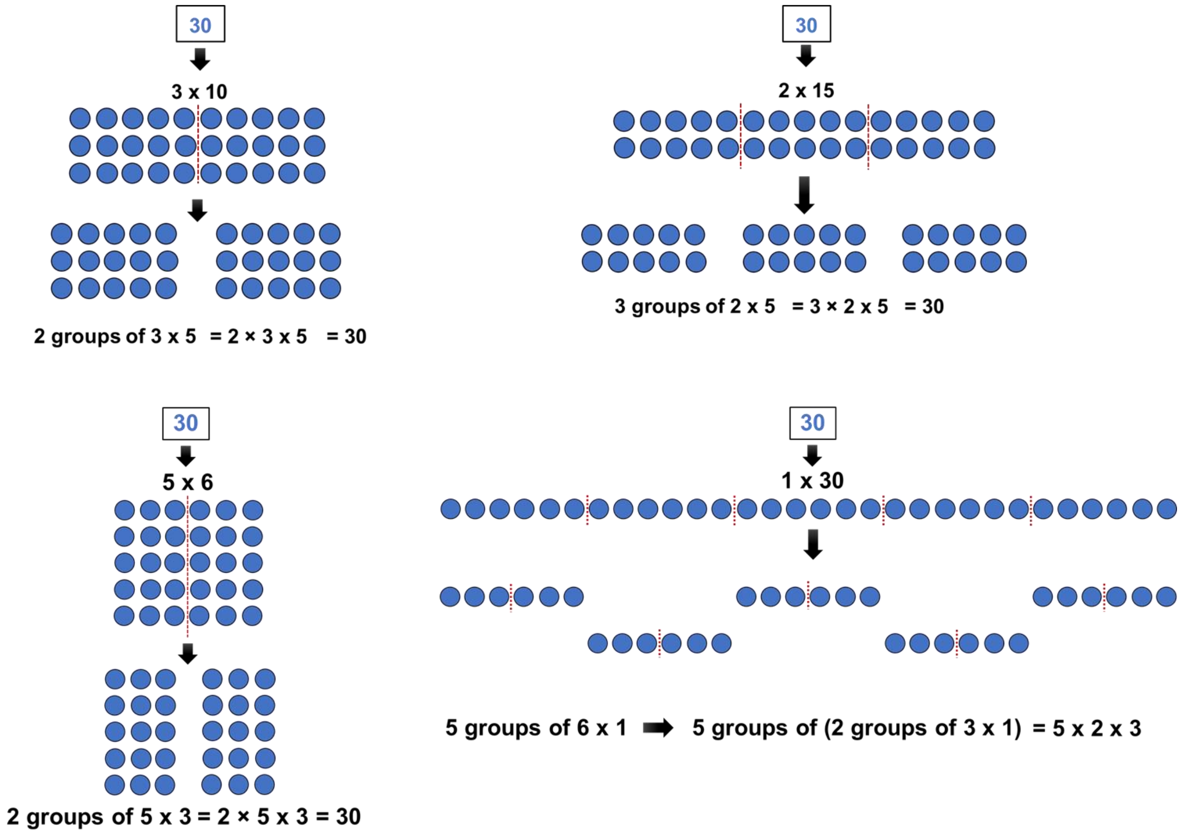

FIGURE 4: Arrangements showing prime factorisation for 30.

Finally, we can generalise and formalise the model by linking the array to a factor tree representation. This is useful because the factor tree format is a more efficient tool for structuring the prime factorisation process numerically and for generalising this process across situations.

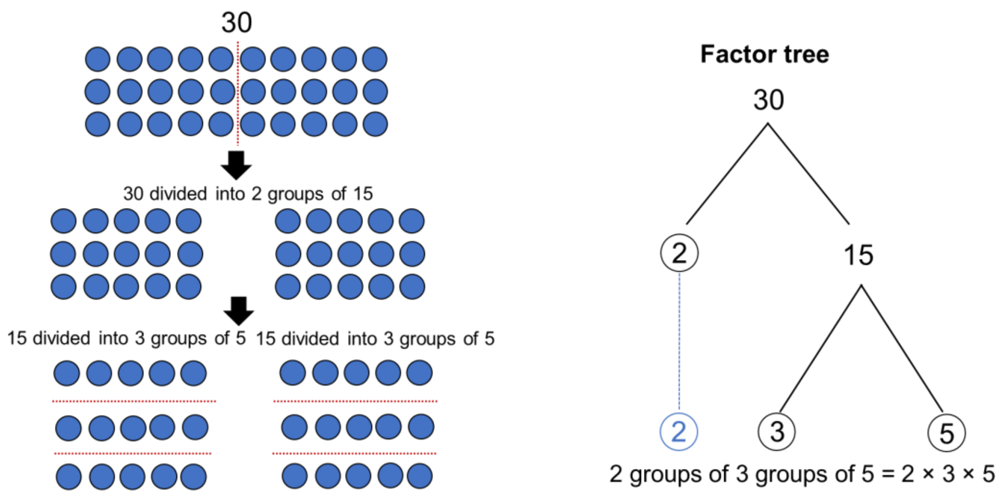

FIGURE 5: Prime factorisation on an array and a factor tree.

We can also use the same model to think about **multiples** and how multiples and factors are linked (but different):

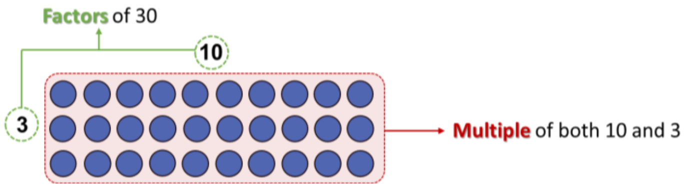

FIGURE 6: Factors and multiples.

Visually, if the factors are the dimensions of the array, then the multiple is the number of dots in the array: multiples are the outcome of a multiplicative relationship while factors are the components of the relationship. This approach shows clearly and visually why a number will always be a multiple of its factors. This approach also shows why a multiple of a number will always have that original number as one of its factors. For example, if the number 4 is represented as a 1×4 array, then the multiples of 4 will be the number of dots in all other arrays that have at least one dimension of 4.

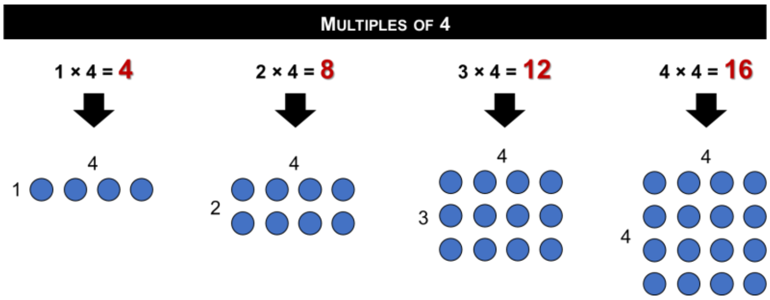

FIGURE 7: Array representation of the multiples of 4.

UNPACKING SOME DELIBERATE DESIGN STRATEGIES

As useful as representations are, they can also add another level of complexity to a learning process. Every new representation creates the potential for a different way of thinking about a concept, but also the potential for a new misunderstanding or misconception. For this reason we have to think carefully about which representations are preferable (and why), how to scaffold and sequence the development of representations, and how to link different representations of the same concept; and, we need to explain all of these decisions to our students. In short, we need to be *deliberate* about the representations we use and be *explicit* with students about our reasons for using these representations. This is how we give our students insight into how to use representations effectively to support their learning and understanding.

In the ideas discussed in this article there have been some careful and deliberate decisions about which representations to use and how to use those representations to, hopefully, have maximum impact on student understanding. These decisions are based on three key strategies.[2]

☑ **Using 'realistic' contexts to anchor understanding**

The tasks draw on contexts that students will be familiar with, can relate to, or imagine. These concrete and practical resources give students the opportunity to enact the mathematics they are working with and ground understanding of abstract concepts in a familiar experience. The contexts also give students a reference point to refer back to if they get confused when working in the abstract – for example, the context was reintroduced when dealing with arrangements for prime numbers. Contexts also provide a useful mental cue around which to organize or associate learning – the 'doughnut and pizza lesson' becomes synonymous for recalling factors and multiples.

[2] All three strategies stem from the Realistic Mathematics Education approach. See, for example, Van Den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*(1), 9-35.

☑ **Careful and deliberate choice of models**

Why use arrays and not a bar model or number line? The array is the most useful representation for this topic for two main reasons. First, it allows for easy construction and then deconstruction of different arrangements to demonstrate the prime factorization process. Second, the discrete dots in the array allow students to check that their arrangements contain the correct number of dots to match the original value. It is also vital to recognise the deliberate interplay between the choice of representation and choice of contexts. Once an appropriate representation was decided on, contexts were chosen that directly map to the representation structure so that there is a seamless transition from the models of the context to the models used for exploring mathematics: arrangements of round doughnuts and round pizzas look exactly like arrays.

☑ **The 'progressive formalisation of models' principle**

Two types of models are used. Initially, models of the specific activity of arranging doughnuts and pizzas are created by using pictures of doughnuts and pizzas and by referring to rows of doughnuts and pizzas. These models are then developed into more general array models that can be applied across a range of situations, problems and content topics (e.g. multiples) that may have nothing to do with doughnuts or pizzas, or even round objects. In the final stage the array model is developed into an even more symbolic and abstract model of the factor tree diagram which has no structural connection to the original doughnut and pizza scenarios. The 'models of' the pizza and doughnut scenarios are progressively formalised into more generalisable 'models for' exploring mathematical relationships across a range of contexts. This is a deliberate move from picture representations that are more closely tied to a specific scenario or problem, to symbolic representations that can be applied across a variety of content and problems. This deliberate formalisation of different formats of models helps students navigate a learning trajectory to abstract concepts and equips them with a small number of models that have applicability over a range of problem and content types.

BUT BE CAREFUL

As mentioned, every representation creates the potential for a different understanding, but also for new misunderstandings or misconceptions. So, every representation has advantages as well as limitations, and the use of the array model for exploring factors is no different. In particular, the array does not support understanding of why, for example, 5×6 and 6×5 are considered as the same factor pair for 30 when the array representations for each of these representations is different (one has 5 rows of 6 items and the other has 6 rows of 5 items). The challenge for teachers is to be aware of these limitations and to discuss and confront these directly with students to minimize confusion or misunderstanding.

A FINAL WORD

Representations provide the most amazing tool for visualising abstract ideas. However, they can also cause confusion and misunderstanding, particularly if the model we use is different to the visual model that students have of a concept. If we want students to understand the models we use, we need to explain our reasoning for these models to them. And, if we want them to draw models (accurately!) to support their thinking and working, then we need to set aside dedicated time for them to learn and practice how to do this.

ACKNOWLEDGEMENTS

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