Using Visual Models to Solve Problems and *Explore Relationships in Mathematics Lessons*

Marc North

School of Education, University of Nottingham, UK

marc.north@nottingham.ac.uk

INTRODUCTION

Visual models and representations are a key part of the learning and teaching of Mathematics, and in previous issues of LTM (No. 24 and No. 28) I have explored the importance of using deliberately and carefully crafted representations in Mathematics lessons. In this article I elaborate on this focus by exploring how representations can be used to both solve problems *and* explore relationships and mathematical structures. It may sound strange to suggest that teachers don't already do both of these, but in my own experiences of working with both Primary, Secondary and Further Education Mathematics teachers, most often representations are used primarily as a means to an end for finding answers to problems rather than as a tool for exploring and investigating relationships.[1]

The discussion below starts with a brief outline of key theoretical ideas about different formats of representations and different purposes for using representations. These ideas are then illustrated via a practical classroom Mathematics task.

WHY BOTHER WITH MODELS AND REPRESENTATIONS?

Visual models provide a useful tool for 'concretising' complex and abstract ideas. The human brain responds positively to information packaged in creative and visual ways, which is why throughout our daily lives we are constantly bombarded with visual imagery and stimuli. It is often easier to remember information presented in picture form than as a string of words, and visual models provide succinct and organised summaries of information – which, some argue, helps to reduce cognitive load (see, for example, Willingham, 2010). Visual models can also demonstrate relationships between different items and, when shown dynamically, can show how those relationships change and evolve.

Bruner (1966) identified three different modes of mental representation that model the stages of learning and reflect the ways in which humans store and encode knowledge and information in memory: (1) *enactive* mode ('based on action') involves the encoding and storage of information through direct manipulation of objects; (2) *iconic* mode ('based on images') involves an internal representation of external objects visually in the form of a mental image or icon; and (3) *symbolic* mode ('based on language') is when information is able to be stored in the form of a code, symbol or through language. Like pieces of a puzzle, each of these modes provides a different way of thinking about and engaging with a concept or experience, and it is the collective of all three that gives a more complete view and understanding ('seeing the whole picture'). These ideas have been operationalised into classroom practice in different ways, with the underlying principle being the importance of using different formats of representations (and identifying the connections between these) to offer students access to mathematical concepts via different perspectives.

---

[1] It is not difficult to understand a possible reason for this … exams! Exams commonly assess students' abilities to select and engage successfully with a specific method (often pre-determined by the examiner and specified in the marking memorandum) for solving a problem. The backfill effect of exams impacts on teachers placing greater priority on using models and representations to find answers rather than as tool for exploring mathematical relationships and comparing methods and ways of working.

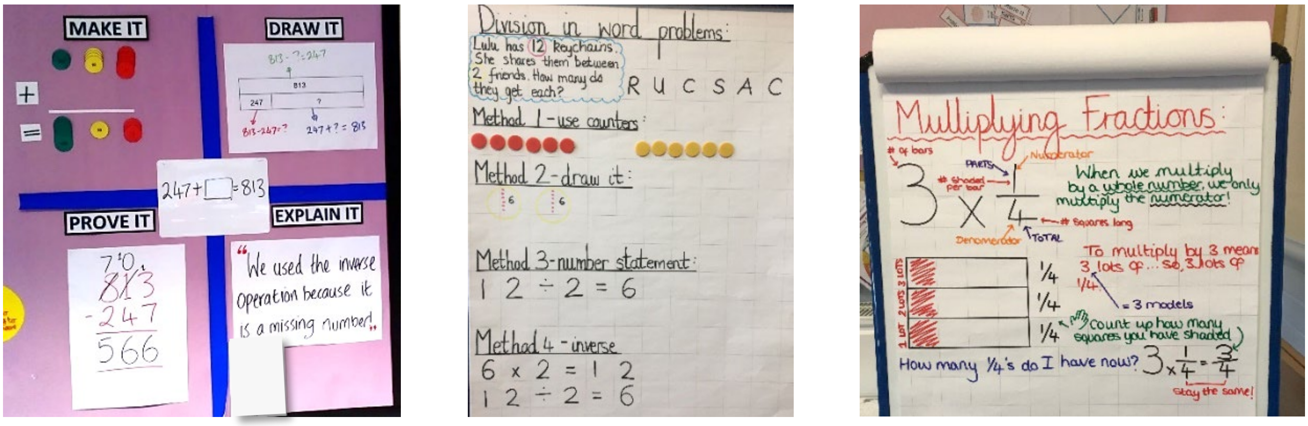


FIGURE 1: Use of different formats of representations to support mathematical understanding.

Alongside considering different representation, it is also useful to consider the *purposes* for using different types of representations. Three key features of the Realistic Mathematics Education (RME) approach (Van den Heuvel-Panhuizen & Drijvers, 2014) are helpful here: (1) use of realistic contexts; (2) different purposes for models; and (3) the 'progressive formalisation' principle.

Contextualising mathematical content in contexts that students can imagine and relate to ('realistic' contexts) provides students with an anchor in which to ground understanding of abstract content and a reference point to refer back to when expanding their thinking about abstract ideas. From here, *models of* a problem can be developed to represent and solve the problem. These models bear a close connection to the problem situation at hand – for example, using a picture of a pizza to represent a fraction of a whole. These models can then be developed further and generalized into *models for* representing, describing, and investigating mathematical structures and relationships over a range of problem situations and even content topics. These 'models for' are powerful precisely because they allow us to investigate mathematical relationships and explore different ways of working. From a RME perspective, when using representations it is essential to choose models and representations that can easily be developed from models of a specific local situation to models for describing more general and abstract relationships. This *progressive formalisation* of models helps students navigate a learning trajectory to abstract concepts and equips them with a small number of models that have applicability over a range of problem and content types.

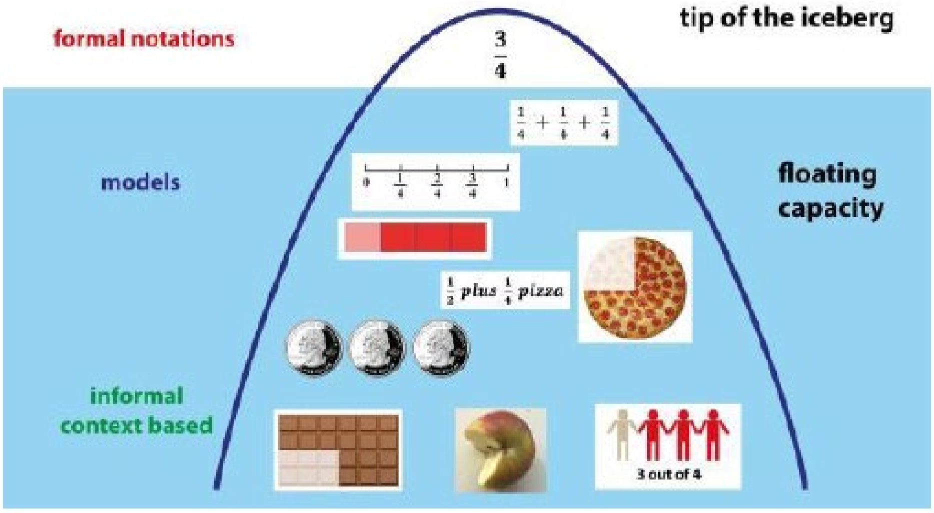


FIGURE 2: Progressive Formalisation of Models (Hough, Solomon, Dickinson and Gough, 2017).

By carefully constructing learning sequences that draw on both *models of* and *models for*, we create opportunities for both solving problems and for exploring, comparing, and investigating mathematical relationships and structures. The discussion below illustrates this approach.

USING CAREFULLY CHOSEN MODELS TO INVESTIGATE MATHEMATICAL RELATIONSHIPS

Consider the problem shown in Figure 3 (and note that the problem is contextualised in a 'realistic' context):

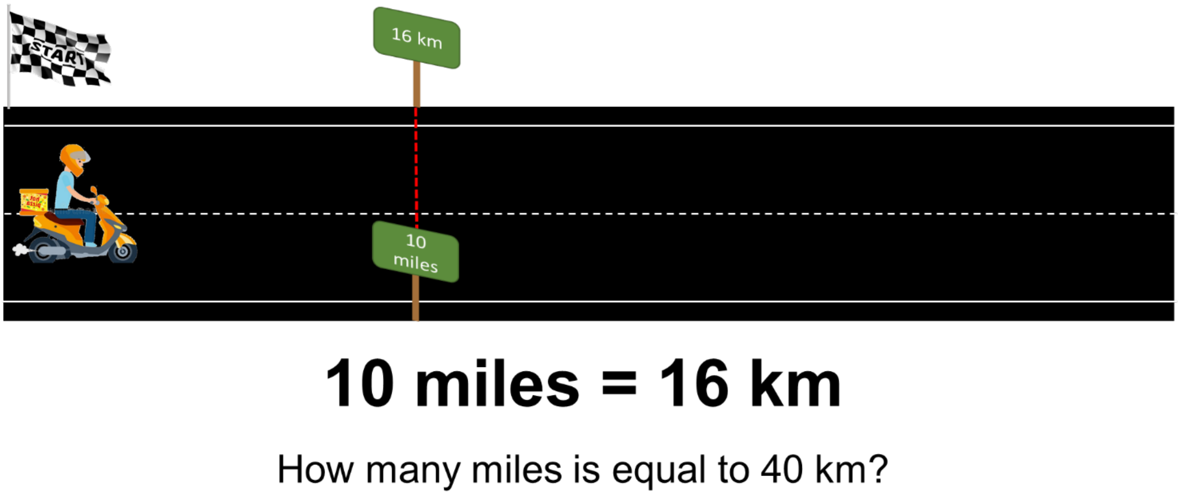


FIGURE 3: A conversion rate problem.

There are a number of numerical methods for solving this problem, some of which are shown in Figure 4.

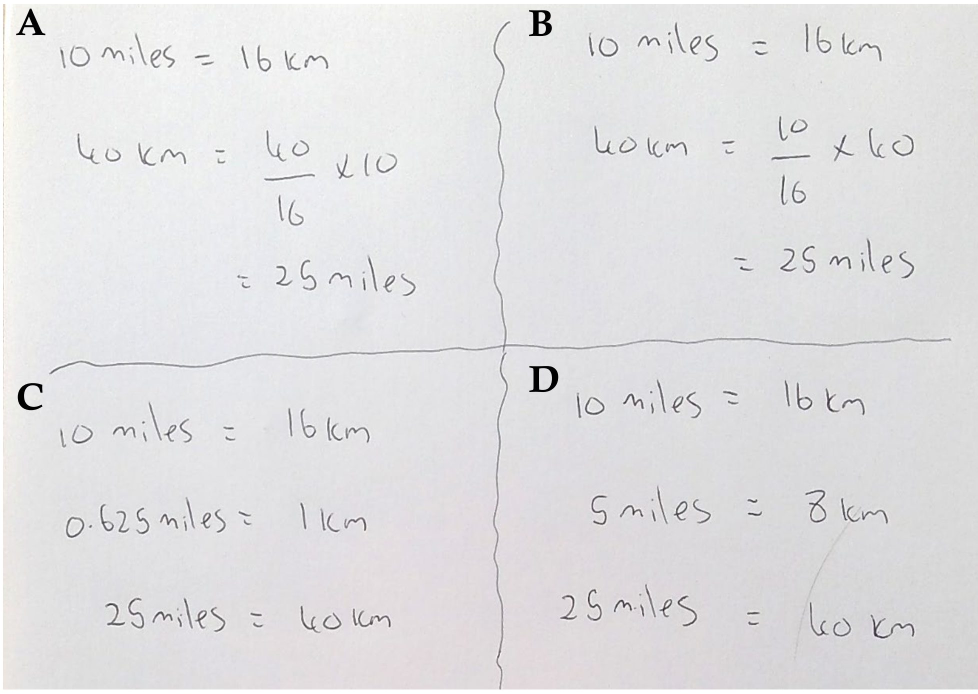


FIGURE 4: Different methods for solving the conversion rate problem.

We could also represent the problem using a 'ratio table' (Figure 5), which provides a two-way table-like format for structuring the known and unknown variables and values in the miles-kilometre relationship.

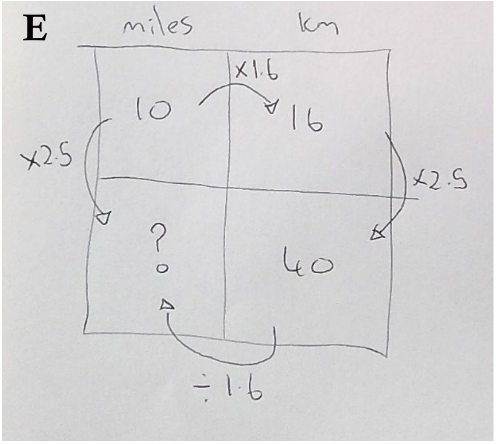


FIGURE 5: A ratio table method for solving the conversion rate problem.

Although each method is equally valid and effective in providing a solution, it can be difficult to see how each method is linked and to understand why each method leads to the same outcome. Many students, particularly lower attaining students, may struggle to recognise that these are simply different ways of working with the same relationship of miles to kilometers rather than completely different methods. The difference in each method relates to how we work with this relationship and the ways in which we choose to adapt the original relationship to work out an unknown value. And, while these methods are effective for finding an answer, they don't reveal much about the nature of the relationship between miles and kilometers. The method that uses the multiplication '×1,6' in the ratio table is the first method to give direct insight into how miles and kilometers are related proportionally.

Another potential challenge is that none of the methods bear a close resemblance to the problem scenario of distances on a road. This scenario involves a *conversion of units* on two measurement systems that have different scales – similar to what you might find on a ruler. As such, to support the *progressive formalisation of models* approach, one possibility would be to choose a ruler-type representation so that there is a natural transition from the problem to the model and method used to solve the problem. A bar model immediately springs to mind as a potential representation, but the challenge here is that a bar model is an area model showing the relationship of part(s) to a whole, whereas this scenario involves two linear measurement systems. So, a *double number-line (DNL) model* proves more useful here (Figure 6) (and also for any scenario involving a multiplicative relationship between two values).

FIGURE 6: A 'model' of the conversion rate problem.

A key feature in the diagram above is that the model chosen to represent and explore this problem (the DNL) – the *model of* this scenario – maps seamlessly to the structure of the problem represented in the realistic context. This continuity between structure of context and structure of representation is important as it helps students understand why a particular representation has been chosen to model the problem.

Extracting the model from the context and presenting it in a more general form makes it easier to work with (Figure 7). At any point, however, we could reintroduce or super-impose the context to re-ground and support understanding (for example, if a different or more complex problem is encountered).

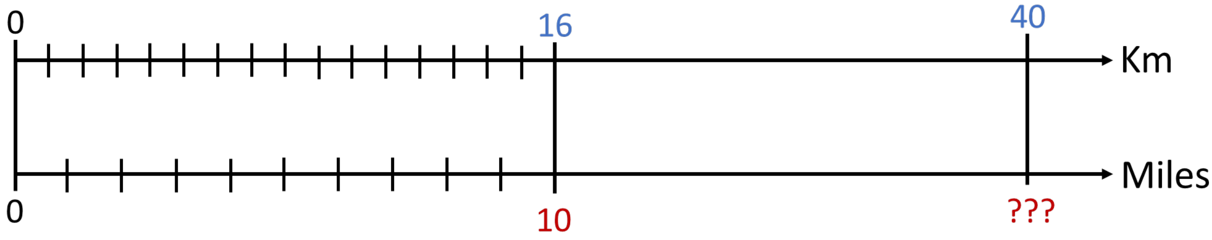


FIGURE 7: A double number line (DNL) of the conversion rate problem.

A DNL is a model comprising two number-lines that have the zero-values aligned but with different variables with different scales on each number line. Although DNLs like the one in Figure 7 are useful because you can count along the line markers to calculate and check answers, they are sometimes difficult and time-consuming to draw because of the need for accuracy with the scales on each line. 'Empty' DNLs such as the one in Figure 8 are easier and more efficient to draw, but don't provide the same check and count facility.

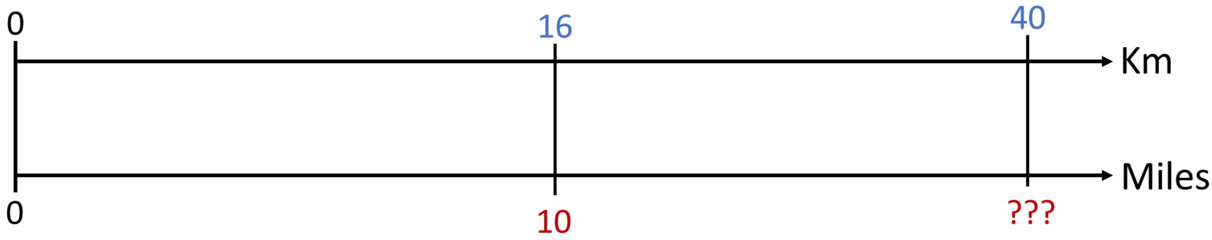


FIGURE 8: An 'empty' double number line of the conversion rate problem.

A key benefit of the DNL model is that it provides a *single* visual resource for:

- exploring and comparing different methods – and, so, for solving the problem (in different ways);
- and, for investigating the nature of the relationship between the variables involved (miles and kilometers in this instance).

This same model can also be used across a range of problems and content topics that involve a comparison of two variables (for example, exchange rate conversions; scaling problems involving enlargement and reduction; ratio and rate problems) – which is what makes it an effective *model for* investigating mathematical structures.

In terms of comparing different methods, consider the double number-line in Figure 9 (the dotted arrow lines indicate a calculation process and the numbers in circles indicate the order of these calculations).

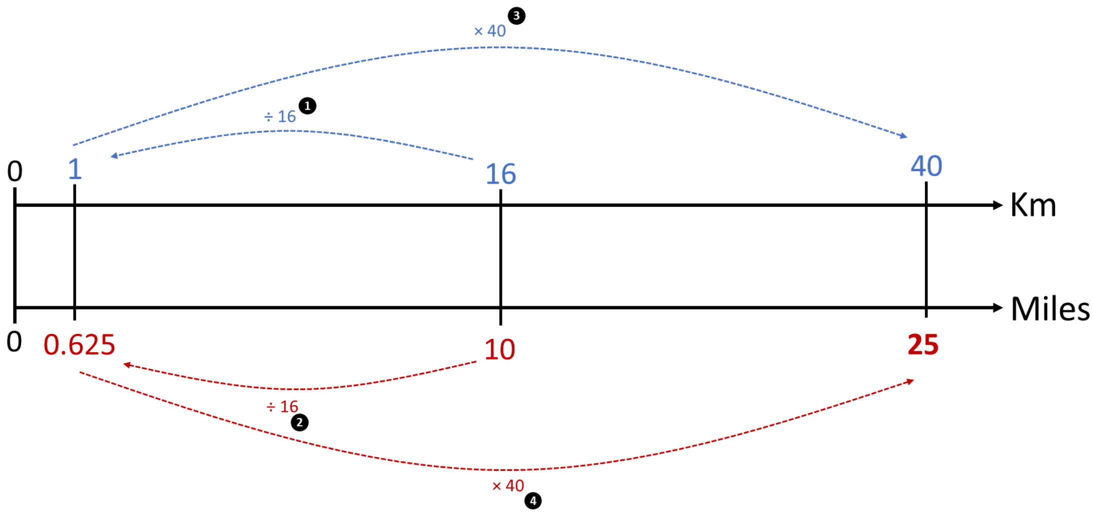
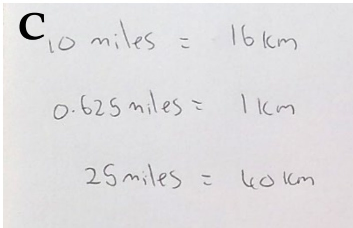
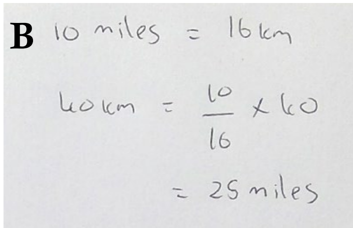


FIGURE 9: A 'unit' method shown on the DNL.





The operations on this DNL provide a visual map of the different stages involved in the numerical calculation **(C)** shown above in Figure 4 that involves what is commonly referred to as the 'unit' method. In other words, scaling one value down to a single unit, then scaling this value up to the required value, and repeating this series of calculations for the other variable in the relationship.

This is also the same method that is used in **(B)** - which is hard to notice when only looking at the calculations, but easier to spot on the DNL.

Now consider the DNL in Figure 10.

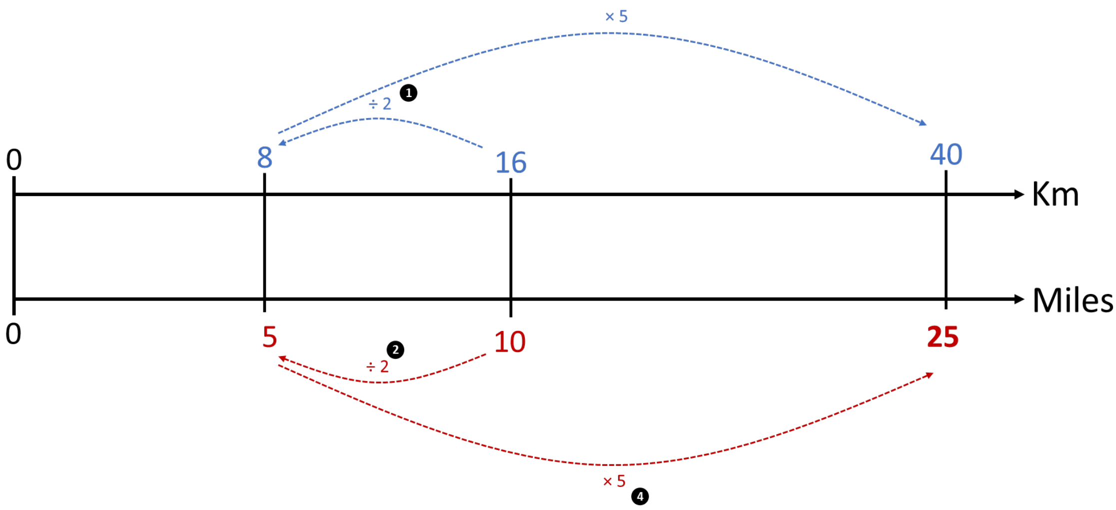
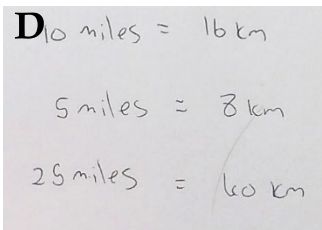


FIGURE 10: A different scaling method shown on the DNL.



This DNL provides a visual map of the steps in the numerical calculation shown in **(D)** above. This method also involves a scaling approach, but this time halving first and then scaling up by a factor of 5.

By showing both approaches (unit and scaling) on the same model, it becomes easier to have a conversation about how the methods are the same and different, about which method might be more or less efficient, and also about why seemingly different methods both lead to the same result.

Now consider the DNL in Figure 11.

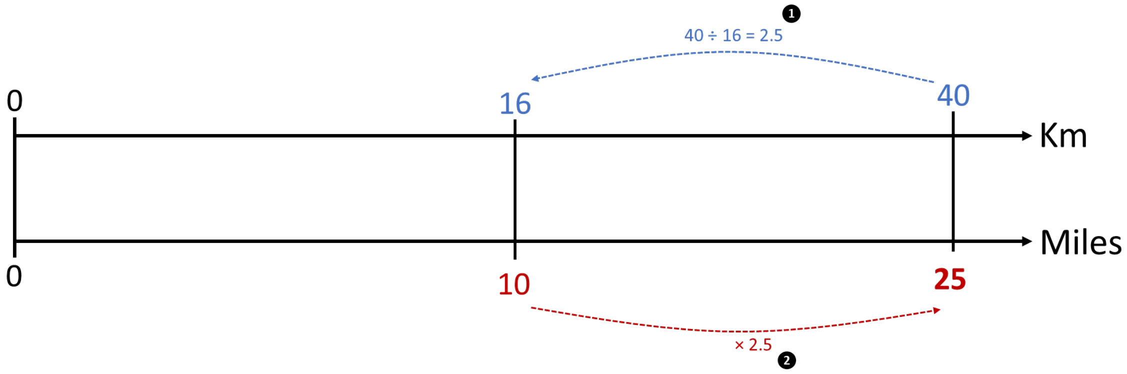
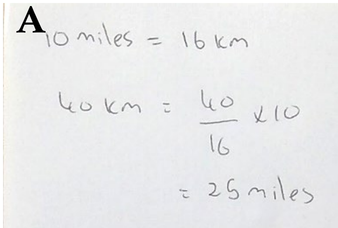
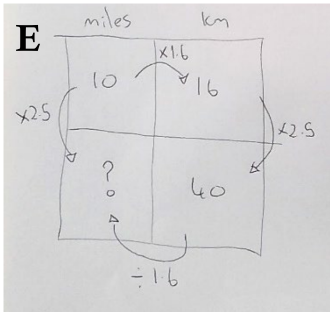


FIGURE 11: A third scaling method shown on the DNL.

This DNL provides a visual map of the steps in the numerical calculation shown in **(A)** above and also in the vertical calculation steps in the ratio table in **(E)** (Figure 5) (how easily might students spot that these are the same calculations?). This method also involves a scaling approach that identifies the direct multiplier from 16 km to 40 km (which is determined by calculating 40 ÷ 16) and applying this scale factor (of 2,5) to work out the unknown corresponding miles value.





Now consider the DNL in Figure 12.

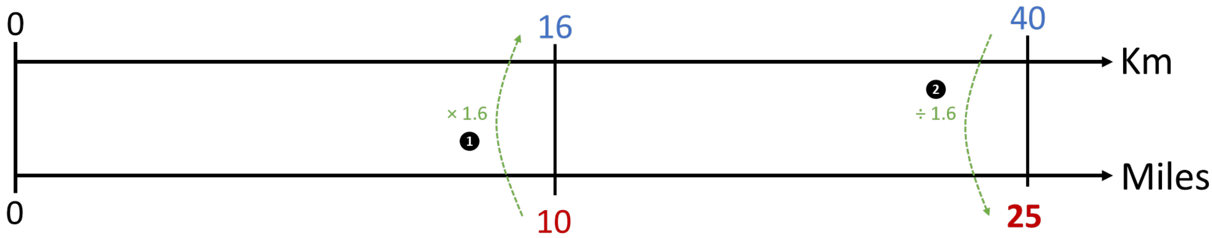


FIGURE 12: A different method that uses the 'functional relationship' shown on the DNL.

The method used on this DNL is unique and subtly different to all other methods. All methods shown on the other DNLs use a scaling approach that involves scaling within each measurement separately (working 'along' the number lines). ***In comparison, this is the only method that works directly with the 'functional' relationship between the miles and kilometer variables (working 'between' the number-lines) to identify that each kilometer value is 1,6 times bigger than its corresponding mile equivalent.*** This method corresponds to the horizontal calculation on the ratio table in **(E)**.

A key observation here is that *it is the functional relationship that defines how these two variables are related* – the one variable (kilometres) is 1,6 times larger than the other variable (miles). And, this relationship gives the most *efficient* way to work with these two variables. So, although working with scalar relationships help us to find the answers to problems involving miles and kilometers, they do not represent the formal relationship between the variables and are commonly not the most efficient way of working.

Now let's consider what happens if we make a slight adjustment to the DNL by rotating one of the number-lines 90° to the left – the result is shown in Figure 13 (*Note that I've had to resize the two axes (number-lines) from the original DNL width so that the picture doesn't take up the whole page!*):

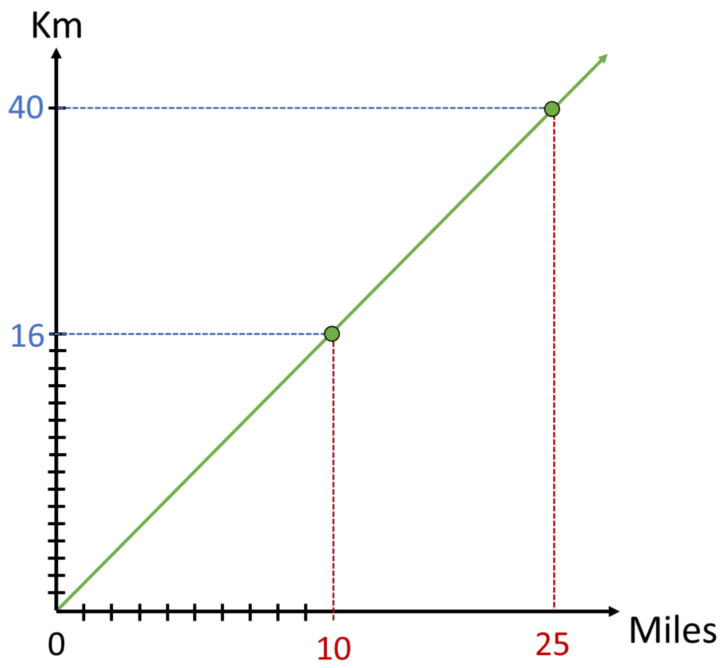


FIGURE 13: A plot of the conversion rate relationship on a set of axes.

The result is a set of axes, with each axis containing a different scale and representing a different variable. The 'functional' relationship between the two variables (miles and km) – the 'between the lines' relationship – is represented by dots (●) (or 'points') on the space between the axes. Each point represents a relationship between a miles value and a corresponding km value such that the km value is always 1,6 times larger than the miles value (or the miles value is 1,6 times smaller than the km value) – 10 miles and 16 km; 40 km and 25 miles; and so on). The collection of all of the dots that share this relationship of $km = 1{,}6 \times miles$ is a **straight line that starts at the origin (0;0)** (since 0 miles is 0 km – shown on the DNL by both number lines starting at 0).

Crucially, it is the ***functional relationship*** (the relationship between the lines on the DNL) that defines how the variables miles and km are related, and it is this relationship that we can represent graphically. Working with the scalar relationships, which commonly represents how many students operate on this type of relationship, does not always give deep insight into the actual nature of the relationship between different variables. This might explain why some students find graph work so difficult and also why some struggle to see the connection between numerical methods and graphical representations of these.

Being deliberate about using models of and models for – and being explicit about this

The discussion above has illustrated how a carefully and deliberately chosen representation that bears close resemblance to a contextual situation can be used to both solve a mathematical problem *(a model of)* and to describe and investigate ways of working with a mathematical relationship *(a model for)*. Although we commonly use representations to help us to understand and then solve problems, the representations provide an equally powerful tool for investigating relationships and for exploring, comparing, contrasting, and linking different methods and ways of working with those relationships.

By being deliberate with the representations that we use and by actively using those representations to both solve problems and explore relationships and compare methods, we support students to see the connectedness of different methods and also the connectedness of different representations (for example, the DNL and a line-graph). And, by being explicit about the decisions that we make, we help our students to decode our thought process. This supports students' independent learning behaviours and enables them to reconstruct appropriate problem-solving strategies independently. In short, the tools that we use are most effective if they are the right tools for the job, if they are used appropriately, and if we communicate clearly to others how to use them.

Acknowledgement

Thank you to Jimmy Diggs from Northgate Primary school in Nottingham and Lorna Glenn from Pear Tree Community Junior School in Derby City (United Kingdom) for sharing photos of how they use different representations in their mathematics lessons.

References

Bruner, J. S. (1966). *Toward a theory of instruction* (Vol. 59). Harvard University Press.

Hough, S., Solomon, Y. J., Dickinson, P., & Gough, S. (2017). *Investigating the impact of a Realistic Mathematics Education approach on achievement and attitudes in Post-16 GCSE resit classes*. Manchester Metropolitan University.

North, M. (2018). Which Picture and Why? Being Deliberate and Explicit when Using Representations. *Learning and Teaching Mathematics*, 24, 3-8.

North, M. (2020). 'Our Brains Really Don't Like Abstract Ideas!' Using Representations Deliberately and Carefully to Help Students Access Abstract Mathematics. *Learning and Teaching Mathematics*, 28, 3-7.

Van den Heuvel-Panhuizen, M. and Drijvers, P. (2014). Realistic Mathematics Education. In Lerman, S. (ed.), *Encyclopedia of Mathematics Education*. Springer: Netherlands. https://www.icrme.net/uploads/1/0/9/8/109819470/rme_encyclopaediamathed.pdf

Willingham, D. T. (2010). *Why don't students like school?: A cognitive scientist answers questions about how the mind works and what it means for the classroom*. John Wiley & Sons.