Using Varied Representations to Build Connections and Conceptual Understanding

Marc North

School of Education, University of Nottingham, UK

marc.north@nottingham.ac.uk

INTRODUCTION

Let your students have a go at the following problem (and have a go yourself too).

> Adding 1 to one of the numbers in the expression (A) 18×37 will give two different expressions:
>
> (B) 19×37 (C) 18×38
>
> Which is bigger: 19×37 or 18×38?

How did your students solve the problem?

- Did they use a calculation? Did different students use different calculations? If you were to show them the different calculations, would they be able to explain how the calculations are linked?
- Would they be able to find another way to solve the problem without needing to use a calculated value?
- Did any of them use a picture? If not, what picture could they use? And, is there more than one way to show this problem visually?
- How might a picture method and a calculation method be linked?

SURPRISING RESPONSES

Having used this task a number of times with groups of both students (in classrooms) and teachers (for professional development), a number of things always surprise me about their responses.

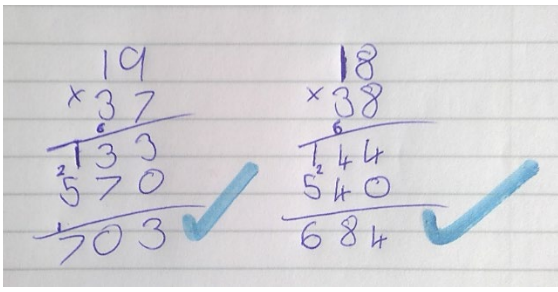
Fixation on calculating accurate answers: When presented with problems like this, most people become preoccupied with the idea that solving this problem involves needing to find a calculated answer of some sort. Linked to this, the accuracy of this calculated answer is then awarded high status (which might be what prompted Person 1 on the next page to put such big ticks behind their answers … and possibly also why all of these work samples stop at the calculated answer and don't actually answer the question of which is larger!).

Prevalence of calculation methods: Building on from this first observation, most students and many teachers use variations of one of the calculation methods shown on the following page to solve the problem. Very few people try to organise and make sense of the problem using a picture representation or a non-calculation approach.

*Confusion about links between methods: S*ome people are surprised that there are multiple calculation methods for solving this problem, that these different methods can be linked to each other, and that the problem can be solved using a picture instead of a calculation.

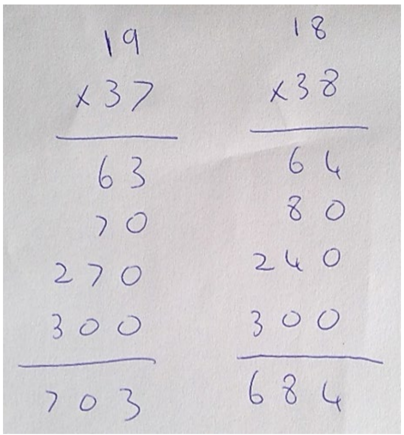
Prevalence of procedural understanding: Most surprising of all, though, is the way in which most people's explanations of their methods and thinking about the problem reflects procedural rather than conceptual understanding.

1



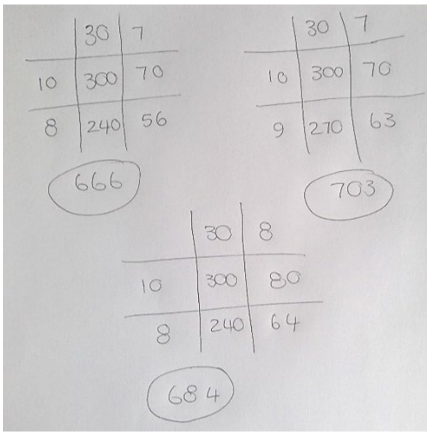
Formal Column Method

2



Expanded Column Method

3



'Grid' Method (Partitioning)

In the following section I explore what thinking conceptually about this problem might involve.

THINKING CONCEPTUALLY ABOUT THE PROBLEM

Procedural understanding and fluency involve knowing mathematical facts and having fluency with mathematical calculations and procedures. By contrast, conceptual (or relational) understanding (Kilpatrick et al., 2001; Skemp, 1978) involves knowing the *why* behind the mathematics – a deep knowledge of the structures that underpin concepts, varied ways of thinking about and representing concepts, relationships between these, why certain methods work, and the situations in which different methods are most effective.

For the multiplication problem given, procedural fluency might enable someone to solve the problem using one or other of the calculation methods shown above. By contrast, conceptual understanding might involve solving the problem by looking more closely at the multiplicative relationships 18 × 37, 19 × 37 and 18 × 38, thinking about what each of these relationships represents and how they are different to each other. For example, 18 × 37 represents a multiplicative relationship of 18 and 37. The underlying mathematical structure of this relationship is multiplication, which stems from the notion of '**groups of**' objects. When thought about in this way, 18 × 37 represents 18 groups with 37 items in each group. The change from 18 × 37 to 19 × 37 involves adding another group (of 37 items); while the change from 18 × 37 to 18 × 38 involves adding one more item to each existing group (i.e. adding an extra 18 items). Using this reasoning it is clear that 19 × 37 is bigger than 18 × 38 (with no calculation needed!).

Thinking conceptually about the problem focuses attention on the underlying mathematical structures that define the relationships in the problem, which makes it possible to understand, organise and solve the problem without needing to resort to a procedural calculation and accurately calculated numerical answers.

Using representations to support conceptual understanding

Picture representations are a useful tool for supporting conceptual understanding for two main reasons. Firstly, you need to have a deep understanding of a problem to organise and show the problem visually. So, supporting students to draw pictures to organise, model and solve problems helps them to develop a deeper understanding of concepts. Secondly, carefully chosen representations allow one to explore mathematical structures and to compare and contrast different ways of working. This shifts the nature of discussions in the classroom away from only finding answers to an explicit focus on connections and relationships – in other words, on conceptual understanding.

Representing the problem visually

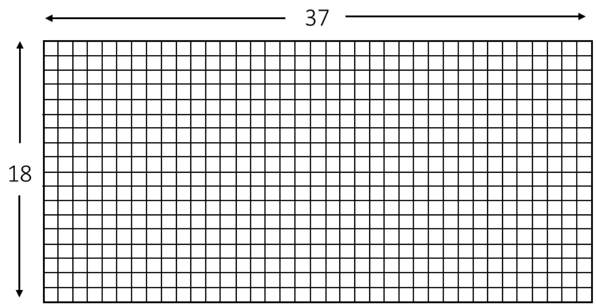
Drawing on this approach, one way to support conceptual thinking about this problem is to ask the question:

"What does the relationship 18 × 37 look like?"

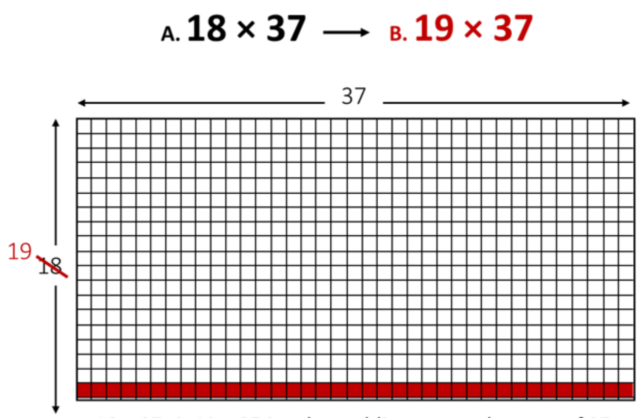
And, by extension:

"How are 19 × 37 and 18 × 38 the same or different to 18 × 37?"

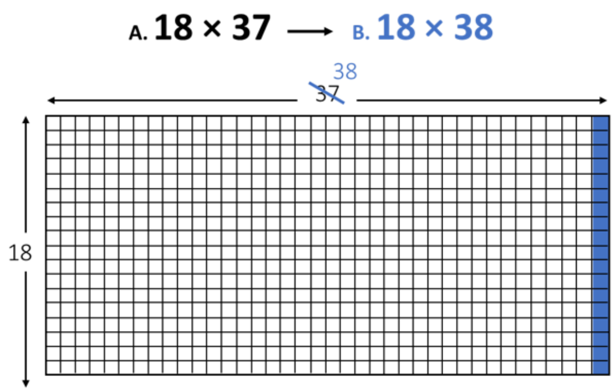
Using the most common grouping model of objects in circles may prove too cumbersome for numbers this big. An alternative is a rectangular area model, with the multiplicand and multiplier in the relationship representing the dimensions (numbers of rows and columns) of the rectangle and the product giving the area of the rectangle (represented by the number of unit squares).



Then, showing the change from 18 × 37 to 19 × 37 and 18 × 38 respectively on this model manifests visually as follows:



18 × 37 → 19 × 37 involves adding on another row of 37



18 × 37 → 18 × 38 involves adding on another column of 18

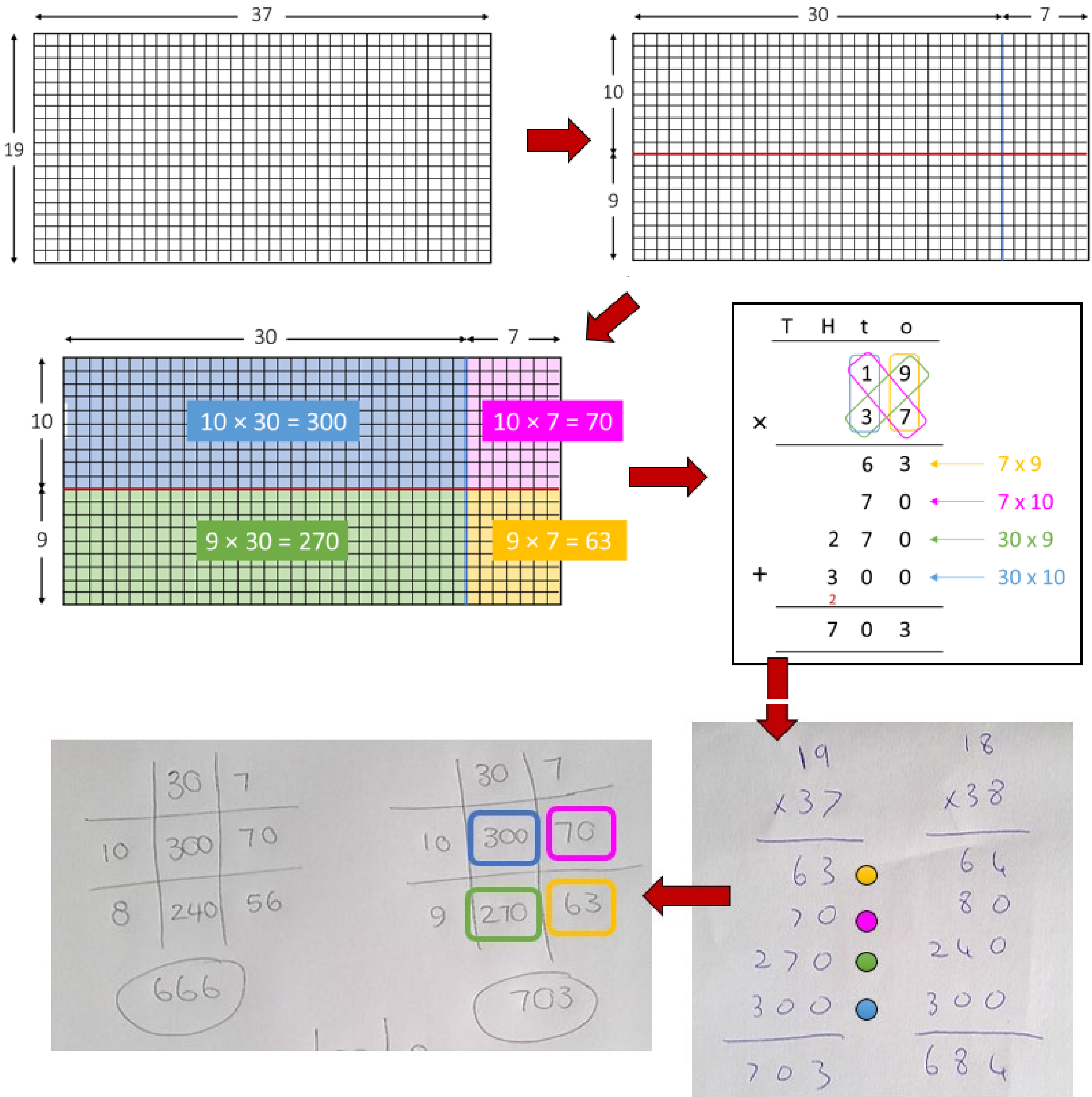
Introducing a pictorial representation for this problem immediately focuses our attention on the nature of the underlying mathematical structures that define these three multiplicative relationships (multiplication as 'groups of'). This makes mapping differences between them easier to conceptualise and track and, so, supports conceptual understanding of these relationships and of the problem. This in turn reduces the need for a procedural calculation; or, if a calculation is still preferred, provides a way to validate a procedural approach.

BUILDING CONNECTIONS BETWEEN METHODS

Conceptual understanding also involves being able to recognise relationships between concepts, between methods, and between different representations (e.g. numeric and pictorial) of the same concept. For the multiplication problem, this involves the links between:

- the area model representation and the calculation methods
- each of the different calculation methods

Starting with the *expanded column* and *'grid'* methods (calculation methods 2 and 3 above), each of these is underpinned by a *partitioning* approach. This partitioning approach can easily be applied and demonstrated visually on the area model, thus building a direct connection to each component of these numerical and pictorial representations.



Building these connections facilitates a deeper conceptual understanding of the origins of each calculated value in the expanded column format. These connections also illuminate why we have to *add* values in a multiplication problem (which is sometimes a source of confusion for children), illustrated via the 'sum of areas' analogy on the rectangular area model. Finally, a leap can be made to an alternative format for the multiplication, showcasing an *expanding brackets* approach to highlight the distributive property of multiplication over addition in partitioning.

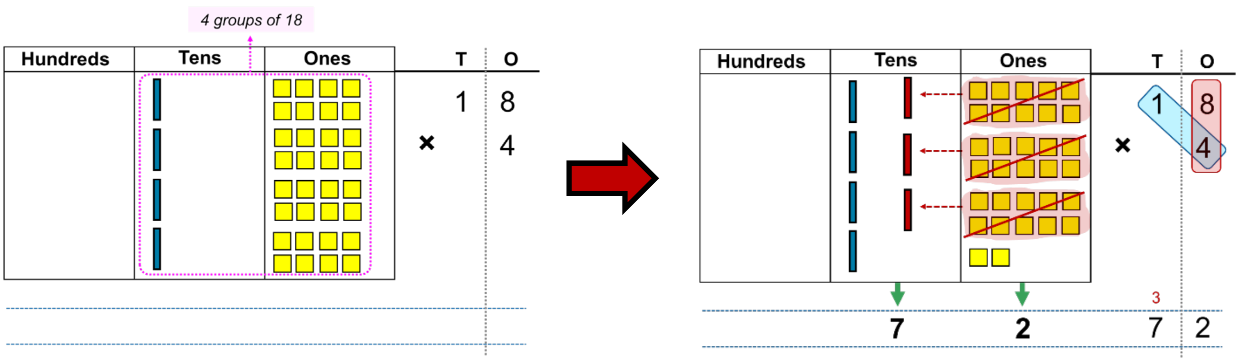
$$19 \times 37 = (10 + 9) \times (30 + 7)$$

$$= (10 \times 30) + (10 \times 7) + (9 \times 30) + (9 \times 7)$$

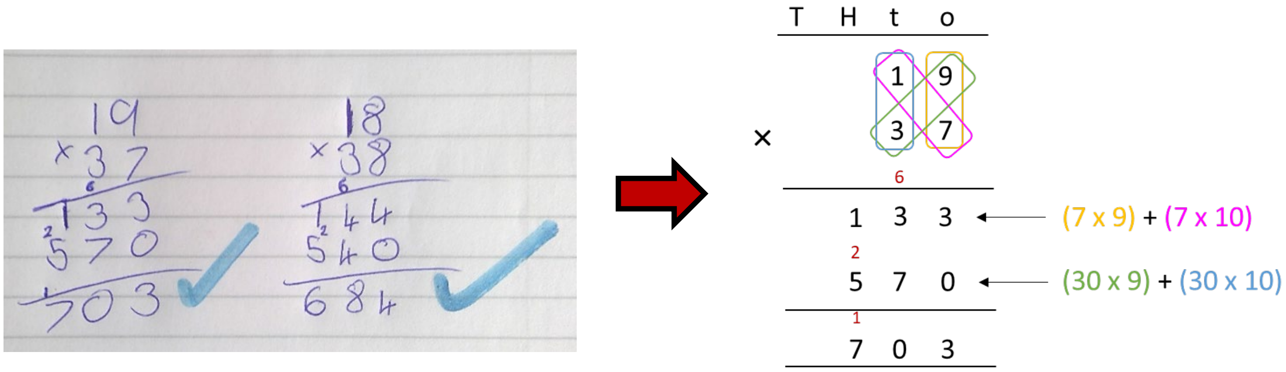
$$= 300 + 70 + 270 + 63$$

$$= 703$$

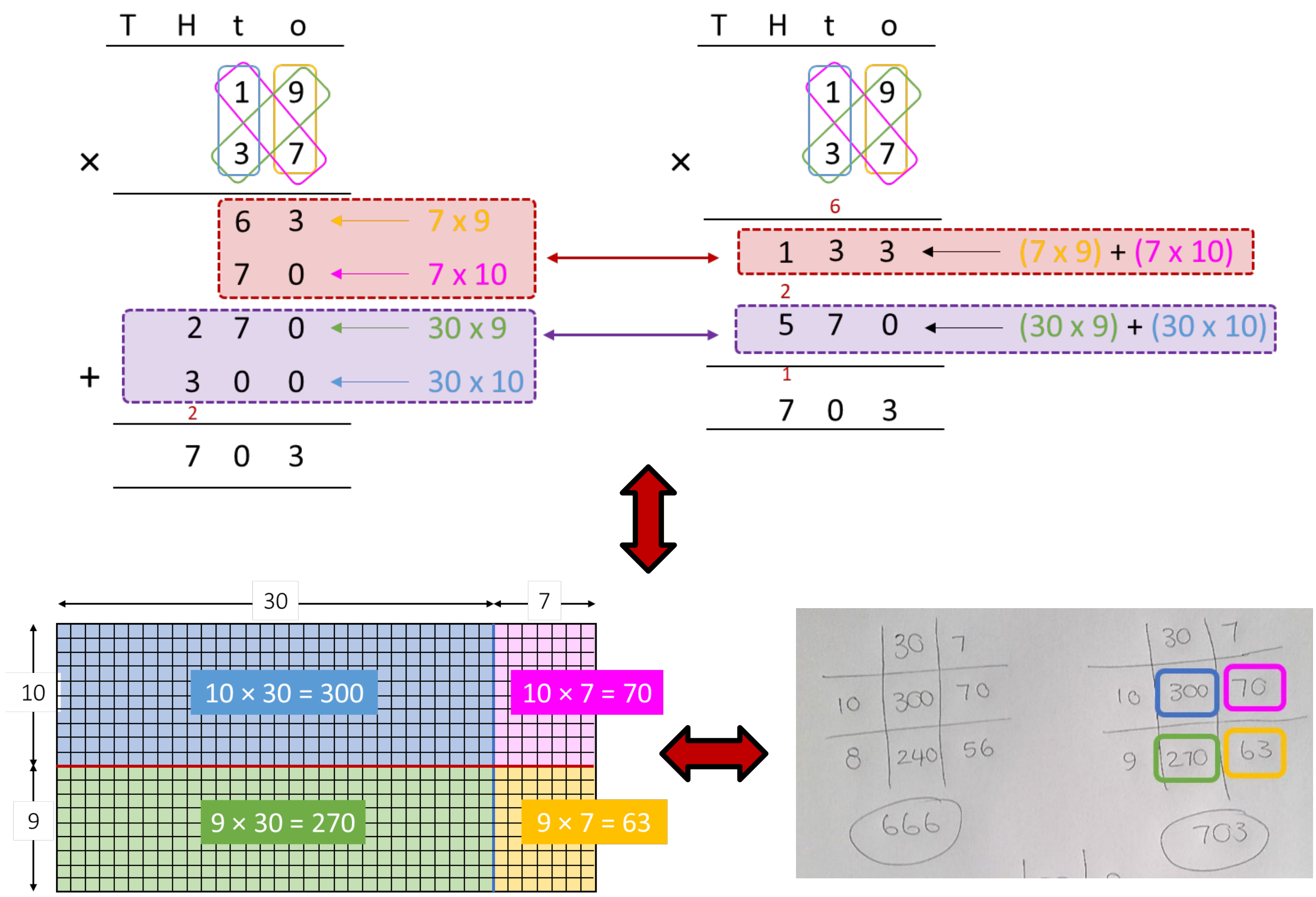
Linking the expanded column method, grid method, and area representation to the formal column method (calculation method 1 above) involves a slight initial diversion to a place value grid – but using a multiplication problem with smaller numbers as the numbers in the original problem are too big to represent effectively on a place value grid. Doing this allows us to explore the evolution of each step in the formal column method, in particular the process of exchanging ones for tens or tens for hundreds.



With this foundational knowledge re-established, we can revert to the multiplicative relationships in the original problem to establish deep understanding of the evolution of each component in the formal column method:



Links can now be made between the expanded column method and this condensed method, and to the grid and area models.



Throughout all of these activities, a further key dimension of supporting conceptual understanding - and one that is sometimes overlooked - is insistence on the use of appropriate technical language that accurately reflects the structure of the mathematics. For example, insisting on "7 ones multiplied by 1 ten, which is 7 multiplied by 10" as opposed to "7 times 1". Similarly, 300 is the product of 30 × 10 and not "3 × 1 with two zeroes added before … why? … Because that's what our teachers told us to do!"

A CHALLENGE!

As explored above, drawing on a carefully chosen picture representation can support deep conceptual understanding by giving us the tools to explore the origins and evolution of each component of a formal procedural method, connect methods, and find links between different representations of the same concept. In addition, the activities shift the nature of classroom discussions away from just finding answers to a dedicated focus exploring relationships and underlying structures.

But, let's also be honest … thinking, understanding and teaching conceptually is hard work! For teachers, doing this requires a deep understanding of mathematics concepts, of varied ways of representing and working with those concepts, and of links between these. For students, it means having to think deeply about mathematical structures and relationships rather than just memorising procedural methods. Despite this, the gains far outweigh the effort. In mathematics lessons, thinking conceptually gives students the tools that they need to check and validate calculated answers, to reset and try a different approach when a preferred procedural method fails, to make sense of unseen problems, and to tackle variations of familiar problems. Beyond the classroom, places of study and employers are desperate for people who can think creatively to organise, model and solve problems, and identify trends and relationships. Thinking conceptually is empowering – and believing this makes teaching for conceptual understanding, no matter how tough, a no-brainer!

So, here's the challenge:

- *How conceptually focused is your teaching?*
- *How conceptually secure are your student's understandings?*
- *How conceptually secure is your own understanding?*
- *How focused are the discussions in your lessons on exploring relationships and structures instead of just finding answers?*
- *How confident are you in using representations to compare methods and explore relationships?*

If your answers to any of these questions are 'less-than-confident', then maybe it's worth rethinking how to prioritise a teaching approach that (re)foregrounds conceptual understanding as the primary goal – and, hopefully, some of the strategies explored in this article will prove helpful here.

REFERENCES

Kilpatrick, J., Swafford, J. and Findell, B. (eds). (2001). *Adding it up: Helping children learn mathematics.* National Research Council.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *The Arithmetic Teacher, 26*(3), 9-15.

ACKNOWLEDGEMENT

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