Exploring Algebra with 'Magic Squares'

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INTRODUCTION

A magic square is a square array of numbers in which the sum of the numbers in each row, column and diagonal is the same. A traditional magic square of dimensions n-by-n would contain the positive inters $1, 2, 3, \ldots, n^2$. So, for example, a 3-by-3 magic square would contain the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9. An example of such a magic square is shown alongside. Note that each row, column and diagonal adds to 15, and this value is known as the 'magic constant' or 'magic sum' for the square.

2	7	6
9	5	1
4	3	8

In this article I use the idea of a 3-by-3 magic square and show how it can be adapted to create activities for exploring and engaging with basic algebra. These activities would be suitable for pupils in Grades 8 and 9.

NON-TRADITIONAL MAGIC SQUARES

Although traditional magic squares use only positive integers, the idea can be extended to include negative numbers as well. Getting pupils to complete magic squares using integers is a great warm-up activity for mental arithmetic. Once pupils have determined the magic constant for the square they can then fill in the empty cells to complete the grid. A few examples are shown below.

9		12
	8	
4		

⇩

9	3	12
11	8	5
4	13	7

		3
		7
1		-4

⇩

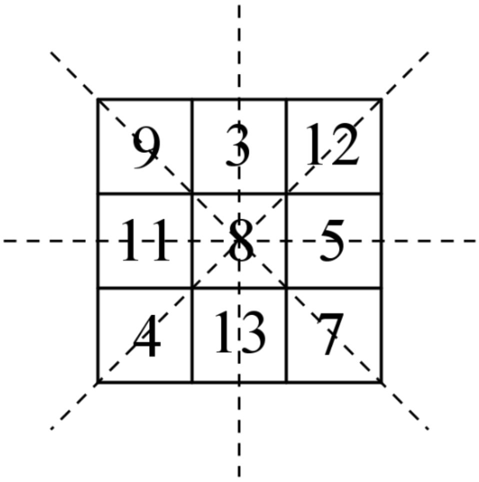
8	-5	3
-3	2	7
1	9	-4

2	-21	4
	11	

⇩

2	-21	4
-3	-5	-7
-14	11	-12

For the three 'magic squares' illustrated, the magic constants are 24, 6 and -15 respectively. Note that this magic constant is three times the middle number of the respective magic square. Because of this, the middle number is always the average of the three numbers in any row, column and diagonal containing the middle number. In addition, the two numbers at either end of each of these rows, columns and diagonals will be equally spaced either side of the middle number. Pupils can use these properties to create their own magic squares. Once they have decided on what number they want to place in the middle square, thus fixing the magic constant for the square, they can then systematically complete the grid using the above properties.



ALGEBRAIC MAGIC SQUARES

We can build on this idea to create algebraic 'magic squares'. Consider the algebraic magic square shown below. Since we know that the sum of each row, column and diagonal should be the same as three times the middle number, pupils can practise their basic algebra skills by cross-checking that this is indeed the case.

$x+5$	$x-2$	$x-3$
$x-8$	x	$x+8$
$x+3$	$x+2$	$x-5$

Notice the symmetry in the middle row and column as well as the two diagonals. Drawing this to pupils' attention can lead to meaningful discussions around additive inverses (e.g. the $+2$ and -2 in the middle column), and the relationship between algebraic expressions (e.g. $x+3$ being one more than $x+2$, and $x+5$ being seven more than $x-2$).

Once pupils have confirmed that the grid is an algebraic magic square, they can then replace x with any chosen value to create their own numeric magic square. For example, substituting $x=5$ and $x=-2$ respectively creates the following 'magic squares'. Pupils can then practise their mental arithmetic to cross-check that all rows, columns and diagonals sum to the magic constant.

10	3	2
-3	5	13
8	7	0

3	-4	-5
-10	-2	6
1	0	-7

Rather than representing the number in the middle cell simply with x, one can represent this value using any algebraic expression. The only requirement is that each row, column and diagonal should sum to three times this middle expression. A few examples are shown below in increasing order of complexity. As with the simpler example on the previous page, pupils can practise their algebraic skills by cross-checking that all rows, columns and diagonals sum to the same expression. They can thereafter substitute a value of their own choosing for x to create a numeric 'magic square', and then cross-check that it works.

x	$x+1$	$x+5$
$x+7$	$x+2$	$x-3$
$x-1$	$x+3$	$x+4$

$4x-6$	$x+1$	$x+2$
$-x+7$	$2x-1$	$5x-9$
$3x-4$	$3x-3$	4

$4-x$	$3-x$	$5-4x$
$5-5x$	$4-2x$	$3+x$
3	$5-3x$	$4-3x$

A very valuable activity is to provide pupils with algebraic magic squares such as the ones above, but to leave a number of cells empty. Pupils are then required to think carefully about what algebraic expressions should go into each block to ensure the grid remains a 'magic square' – i.e. that each row, column and diagonal sums to the same expression. A few examples are shown below.

$x+1$	$x+3$	
	$x-2$	
	$x-7$	

		$3x-20$
$4x-11$	$x-6$	$4x+5$

$x+6$	$2x+7$	$2-3x$
		$4-x$

Taking the middle grid as an example, we can sum the three algebraic expressions in the bottom row to obtain the 'magic number' $9x-12$. Since the two known cells in the right-most column sum to $7x-15$, the top right cell must be $2x+3$, i.e. the expression that needs to be added to $7x-15$ to give $9x-12$. Carrying on in similar vein one can then complete the grid. One could even take the idea one step further and incorporate more than one variable, as illustrated in the example alongside. As a final challenge one could ask pupils to create their own algebraic 'magic square' from a completely empty grid.

	$3x+w$	
	$2x+3w$	
$x+6w$	$x+5w$	

CONCLUDING COMMENTS

This article shows how the idea of a 3-by-3 magic square can be adapted to create activities for exploring and engaging with basic algebra. These activities include comparing different algebraic expressions, substitution, grouping of like terms, as well as adding and subtracting algebraic expressions. The activities described could be used either as introductory exercises for basic algebra engagement, or as more explorative exercises once basic algebraic competency has been established.