# Reflecting on Reflection


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### INTRODUCTION

In this article we explore the process of reflecting a point across a straight line in the Cartesian plane. We begin with the standard reflections across the two coordinate axes, i.e. the $x$-axis and the $y$-axis, and then extend this to reflections across other vertical and horizontal lines. We then move on to the standard reflections across the $y = x$ and $y = -x$ lines. This leads to a consideration of lines of reflection of the form $y = x + c$ and $y = -x + c$. Finally we explore the general case of reflecting a point across any line of the form $y = mx + c$.

The purpose of this article is to illustrate how a simple concept (such as the reflection of a point across a straight line) can lead, through well-considered incremental steps, to a wealth of mathematical exploration. This type of guided investigation has the potential to lead to rich classroom discussions, deep mathematical thinking, conceptual enrichment, and consideration of the important process of generalisation.

### REFLECTING ACROSS THE COORDINATE AXES

Pupils can readily relate to the concept of reflection. Symmetry in the world around us is often a result of some or other form of reflection. Pupils are also familiar with the idea of a mirror image – i.e. an image that appears to be the same distance 'behind' the mirror plane as the object being mirrored is in front of it. By considering a few simple examples, pupils are usually able to quickly establish a general rule for reflecting points across the $x$-axis and the $y$-axis:

- When reflecting across the $y$-axis the *$y$-coordinate stays the same* while the *$x$-coordinate changes sign*.
- When reflecting across the $x$-axis the *$x$-coordinate stays the same* while the *$y$-coordinate changes sign*.

Expressing these verbal statements symbolically we have the following general results for the reflection of a general point $(a; b)$:

- When reflecting across the $y$-axis: $(a; b) \rightarrow (-a; b)$
- When reflecting across the $x$-axis: $(a; b) \rightarrow (a; -b)$

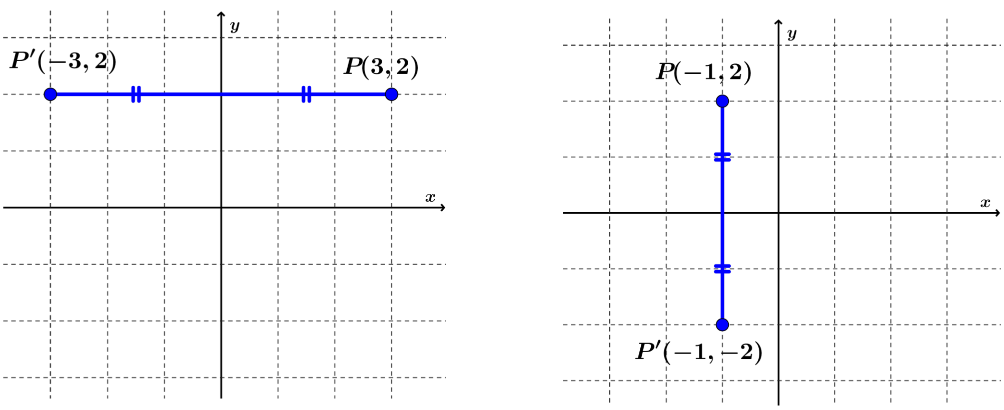


**FIGURE 1:** Reflecting across the coordinate axes.

### REFLECTING ACROSS OTHER VERTICAL AND HORIZONTAL LINES

When reflecting across the $y$-axis, for example, the perpendicular distance from the object $P$ to the $y$-axis is the same as the distance from the $y$-axis to the image $P'$. We can use this fundamental concept of reflection to reflect a point across *any* vertical or horizontal line, not only the coordinate axes. Consider the point $P(5;4)$ that we wish to reflect across the vertical line $x = 2$. The perpendicular distance from $P$ to the line $x = 2$ is 3 units. The image $P'$ is thus 3 units on the other side of the line $x = 2$ at the same 'height' (i.e. with the same $y$-coordinate).

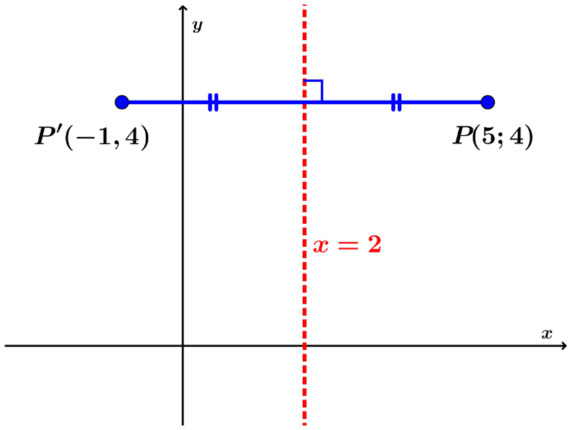


**FIGURE 2:** Reflecting $P(5;4)$ across the line $x = 2$.

Let us now consider this scenario more generally by reflecting the point $P(a;b)$ across the line $x = k$. There are two similar yet slightly different ways we could determine the image $P'$.

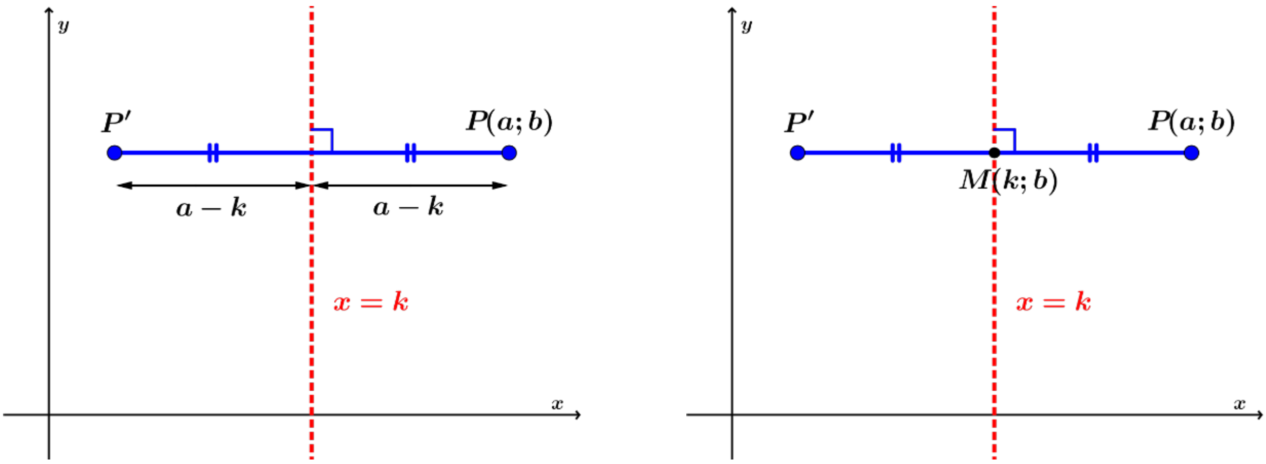


**FIGURE 3:** Reflecting $P(a;b)$ across the line $x = k$.

Let us represent the coordinates of the image by $P'(x;y)$. Firstly, since the reflection is perpendicular to a vertical line, the image will be on the same horizontal line as the original point. The $y$-coordinate of $P'$ is thus $b$. The perpendicular distance from $P(a;b)$ to the vertical line $x = k$ is $a - k$ units. The horizontal distance between $P$ and $P'$ is thus $2(a - k)$. We thus have $x = a - 2(a - k) = 2k - a$. The coordinates of the image are thus $P'(2k - a; b)$. Alternatively, since the point midway between $P(a;b)$ and $P'(x;y)$ is $M(k;b)$, we have $\frac{a+x}{2} = k$, from which $x = 2k - a$ as before. In general, the image of a point $P(a;b)$ reflected across a vertical line $x = k$ is $P'(2k - a; b)$. It is left to the reader to confirm that the image of a point $P(a;b)$ reflected across a *horizontal* line $y = k$ is $P'(a; 2k - b)$.

### REFLECTING ACROSS THE LINE $y = x$

By considering a few simple examples, pupils are usually able to quickly establish a general rule for reflecting points across the oblique line $y = x$, namely that the $x$-coordinate of the original point becomes the $y$-coordinate of the image, and that the $y$-coordinate of the original point becomes the $x$-coordinate of the image. Expressed symbolically, we have the following general rule for the reflection of a point $(a; b)$ across the line $y = x$: $(a; b) \rightarrow (b; a)$.

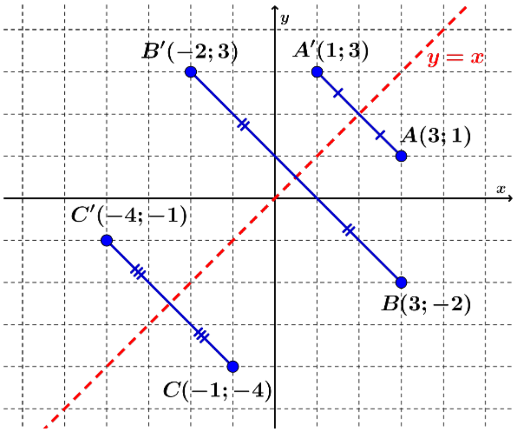


**FIGURE 4:** Reflecting across the line $y = x$.

The line $y = x$ has a gradient of 1 and a $y$-intercept of zero, i.e. the line passes through the origin. What about reflecting a point across a line with a gradient of 1 which doesn't pass through the origin? Consider the point $P(7; 1)$ that we wish to reflect across the oblique line $y = x - 2$. Using a Cartesian plane with a grid makes this process very simple. All we need to do is move from point $P$ perpendicularly to the line of reflection, and then continue the same distance on the other side to get the image $P'(3; 5)$.

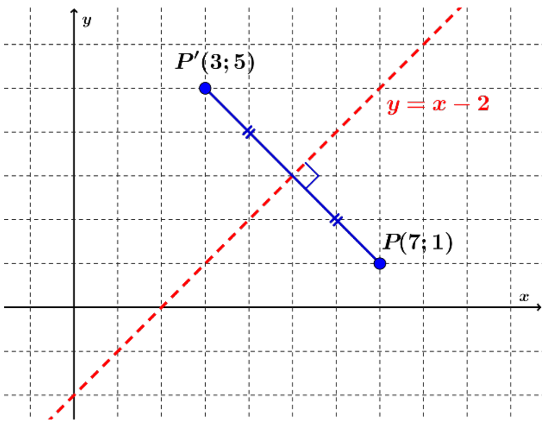


**FIGURE 5:** Reflecting $P(7; 1)$ across the line $y = x - 2$.

How might one achieve this without an accurate drawing? One approach is to perform a translation that reduces the problem to a simpler configuration with which we are more familiar. If we translate the line of reflection as well as the point $P$ vertically upwards by 2 units, then the problem is reduced to simply reflecting the point $(7; 3)$ about the line $y = x$. Since this is a standard result that we already know, we have the image point $(3; 7)$. All we now need to do is reverse the original translation. Thus, moving $(3; 7)$ vertically downwards by 2 units gives $P'(3; 5)$.

Let us now consider this scenario more generally by reflecting the point $P(a;b)$ across the line $y=x+c$. We begin by transforming the line of reflection to $y=x$ by translating it vertically by $-c$ units. We likewise translate point $P$ vertically by $-c$ units to give $(a;b-c)$. We then reflect this point about the line $y=x$ to get $(b-c;a)$, and finally reverse the original transformation by translating the point $c$ units vertically to get $P'(b-c;a+c)$.

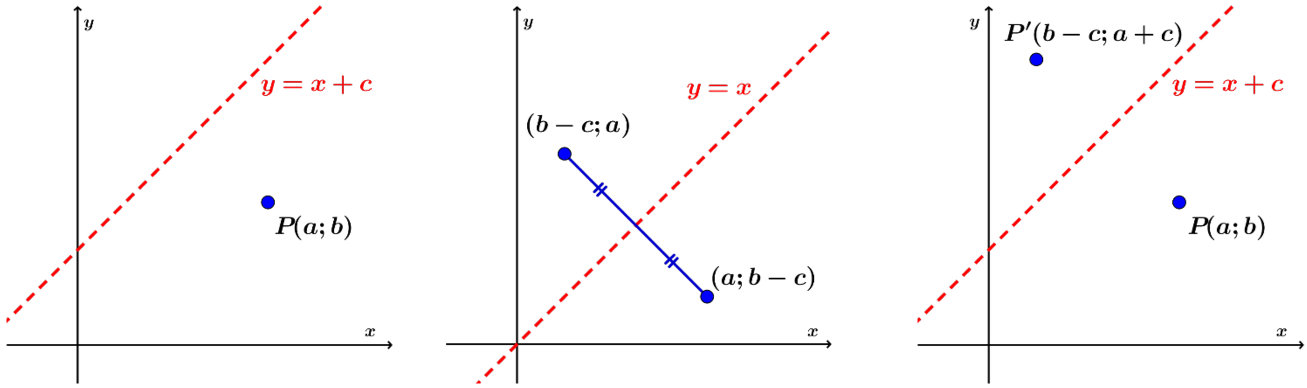


**FIGURE 6:** Reflecting $P(a;b)$ across the line $y=x+c$.

A similar approach can be used for reflecting a point across oblique lines with a gradient of $-1$. The general rule for the reflection of a point $(a;b)$ across the line $y=-x$ is $(a;b)\to(-b;-a)$. It is left to the reader to confirm that, more generally, the reflection of a point $P(a;b)$ across the line $y=-x+c$ gives the image point $P'(c-b;c-a)$.

### REFLECTING ACROSS THE LINE $y = mx + c$

Let us now take it one step further by considering reflections across oblique straight lines having any gradient. As a starting point let's consider a specific case, namely the reflection of the point $P(4;5)$ across the line $y=2x+3$.

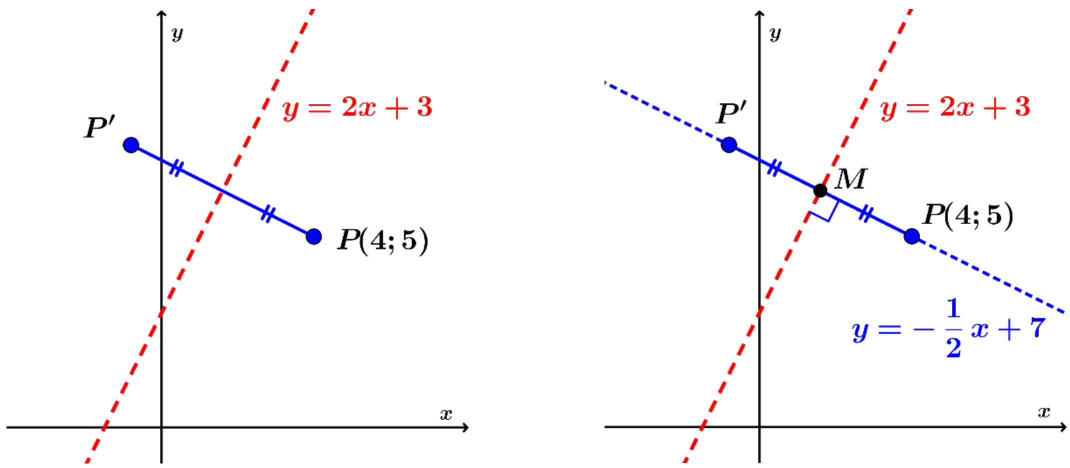


**FIGURE 7:** Reflecting $P(4;5)$ across the line $y=2x+3$.

How might we go about determining the coordinates of the image $P'$? Thinking through a strategy conceptually we could arrive at the following three steps:

- Determine the equation of the line perpendicular to $y=2x+3$ passing through $P(4;5)$.
- Determine the coordinates of $M$, the point of intersection of these two perpendicular lines.
- Determine the coordinates of $P'$ such that $M$ is the midpoint of $PP'$.

Since the line of reflection, $y = 2x + 3$, has a gradient of $2$, the line perpendicular to it must have a gradient of $-\frac{1}{2}$. Substituting the point $(4; 5)$ into $y = -\frac{1}{2}x + c$ gives the equation of the perpendicular line as $y = -\frac{1}{2}x + 7$. Next we determine the point of intersection of these two lines to get $M(1{,}6; 6{,}2)$. Finally we determine the coordinates of the image $P'(x; y)$ such that $\frac{x+4}{2} = 1{,}6$ and $\frac{y+5}{2} = 6{,}2$. This gives us the image point as $P'(-0{,}8; 7{,}4)$.

Let us now consider this more generally by reflecting the point $P(a; b)$ across the line $y = mx + c$. The line perpendicular to $y = mx + c$ has gradient $-\frac{1}{m}$. Substituting the point $(a; b)$ into $y = -\frac{1}{m}x + k$ gives the equation of the perpendicular line as $y = -\frac{1}{m}x + \left(b + \frac{a}{m}\right)$. Next we determine the point of intersection of $y = mx + c$ and $y = -\frac{1}{m}x + \left(b + \frac{a}{m}\right)$ to obtain $M\left(\frac{bm + a - cm}{m^2 + 1}; \frac{bm^2 + am + c}{m^2 + 1}\right)$. Finally we determine the coordinates of the image $P'(x; y)$ such that $\frac{x+a}{2} = \frac{bm + a - cm}{m^2 + 1}$ and $\frac{y+b}{2} = \frac{bm^2 + am + c}{m^2 + 1}$. This gives us the image point as:

$$P'\left(2\left(\frac{bm + a - cm}{m^2 + 1} - \frac{a}{2}\right); 2\left(\frac{bm^2 + am + c}{m^2 + 1} - \frac{b}{2}\right)\right)$$

This can be rearranged to give:

$$P'\left(\frac{a(1 - m^2) + 2m(b - c)}{m^2 + 1}; \frac{2am + b(m^2 - 1) + 2c}{m^2 + 1}\right)$$

Testing this general result by reflecting the point $P(4; 5)$ across the line $y = 2x + 3$ gives us the image point $P'(-0{,}8; 7{,}4)$ as before:

$$P'\left(\frac{4(1 - 2^2) + 2(2)(5 - 3)}{2^2 + 1}; \frac{2(4)(2) + 5(2^2 - 1) + 2(3)}{2^2 + 1}\right) \rightarrow P'(-0{,}8; 7{,}4)$$

## CONCLUDING COMMENTS

The purpose of this article was to illustrate how a simple concept (such as the reflection of a point across a straight line) can lead, through well-considered incremental steps, to a wealth of mathematical exploration. We began with the simple scenario of reflecting a point across the coordinate axes. This was then extended to reflections across any vertical or horizontal line. We then moved on to oblique lines by considering reflections across the lines $y = x$ and $y = -x$. This was then extended, through the use of translations, to lines of the form $y = x + c$ and $y = -x + c$. Finally we considered the more general scenario of reflecting a point $P(a; b)$ across the line $y = mx + c$.

This type of guided investigation has the potential to lead to rich classroom discussions, deep mathematical thinking, conceptual enrichment, and consideration of the important process of generalisation. It could perhaps be structured into a Grade 12 portfolio investigation or project. Although these types of investigations are often time-consuming, the depth of mathematical engagement that accompanies them is almost always worth it.