Learners' Errors with Linear Equations – and How to Solve Them!

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INTRODUCTION

Learners are introduced to algebraic linear equations in Grade 7 through very simple examples that can be solved by inspection, such as $p + 5 = 12$. At high school they progress to questions like $2(m + 3) = 10$, and then to examples where the unknown (or variable) appears on both sides, such as $2x + 1 = x - 7$. Many learners are able to figure out the solution to an equation like $2x - 3 = 9$. Some learners might reason "what times 2, then decreased by 3 gives me 9?" Others might use a process of 'undoing' or 'reversing' by starting with 9, adding 3, and then dividing by 2. Equations like this can be solved without operating on the unknown, but this is not generally true for equations with unknowns on both sides. International research over the past 40 years suggests that the transition from equations with unknowns on one side only to equations with unknowns on both sides is a big jump for learners. The research is unanimous that learners need to be explicitly taught a method for solving equations with the unknown on both sides.

The Wits Maths Connect Secondary (WMCS) project has done a great deal of research and development work on the teaching and learning of algebraic linear equations. We have tested thousands of learners on various items that require them to solve linear equations. All our results point to learners' difficulties with equations when the variable appears on both sides. We have also analysed very thoroughly the Grade 7, 8 and 9 CAPS documentation on equations and it is clear that CAPS does not pay attention to what research has been saying for nearly half a century! For example, the illustrative examples in the curriculum document do not address the important shift from equations with variables on one side to equations with variables on both sides. Furthermore, CAPS pays too much attention to simple examples that can be solved by inspection and hence do not create the need for working with inverses and inverse operations. There is also content listed under equations which is not fundamentally about equations, such as using substitution to populate tables of values – a task more suitable to learning functions.

In this article we identify four issues that underpin learners' difficulties with algebraic linear equations. They are:

(1) not seeing the equal sign as a balance

(2) difficulties in working with negatives and subtraction

(3) difficulties with algebraic simplification

(4) difficulties in working with inverse operations

In this article we expand on each of these issues. Thereafter we provide recommendations for teaching and we include examples of tasks to deal with the abovementioned difficulties which have already proved to be useful for teachers.

Before proceeding, we must be explicit about our position on the teaching of linear equations. We strongly advocate that learners are taught to use inverse operations (additive and multiplicative) to solve equations. This supports the idea that an equation is a statement where two expressions (left side and right side) are equal to each other. By applying inverse operations, we maintain the balance of both sides of the equation. We do not support "short-cut" methods such as transposition, "change sides, change the sign", etc. because these hide the fundamental mathematical ideas on which equation solving is based.

SEEING THE EQUAL SIGN AS A BALANCE

When learners are first introduced to the equal sign, they treat it as signal to do something. This is not surprising because in the early grades they encounter examples such as:

$$4 + 10 = \_\_ \qquad 4 + \_\_ = 10 \qquad \_\_ - 6 = 10 \qquad 10 = 4 + \_\_$$

In each of these examples, learners can reason that adding/subtracting the numbers "gives me" the answer. However, learners also need to view the equal sign as a balance, showing that the left side and the right side are equivalent, e.g. $4 + 10 = \_\_ - 5$. This means they need to see that the result on the left side (in this case 14) "is the same as" the result on the right side. In this example, learners who continue to operate with a do-something view of the equal sign are likely to write 14 in the blank space, thus ignoring the "subtract 5" on the right side. Learners with an equivalence view of the equal sign will treat it as a balance and will recognise that the correct answer is 19. There is lots of evidence, both locally and internationally, that learners in Grades 7 – 9 don't have an equivalence view of the equal sign. This can become an obstacle in making sense of equations.

WORKING WITH NEGATIVES AND SUBTRACTION

One of the big transitions in Senior Phase mathematics is the need to view the minus symbol as both sign and operation. Up to Grade 6, learners treat the minus symbol ($-$) as an operation: subtract or take away. When they are introduced to negative numbers, the symbol takes on an additional meaning because now it also represents a sign, e.g. -2 (negative two). This is confusing for learners. Even more confusing is that the meaning of the symbol can shift from operation to sign (and vice versa) in a single question, as in the example $4 + (-5) = 4 - 5 = -1$. We can read this as "four add negative five" which is then simplified to "four subtract 5" and then the answer is "negative one". The same symbol was treated as negative, then as subtract and then as negative again. It takes time to get used to these new ideas.

When working with equations, we cannot avoid dealing with negatives and subtraction irrespective of whether we speak about "taking to the other side and changing the sign" or working with inverse operations. Consider the following example which a teacher might offer learners:

Solve:	$-5 + 2x = x + 6$	(line 1)
The teacher then writes:	$\mathbf{5} - 5 + 2x = x + 6 \mathbf{+ 5}$	(line 2)

The teacher is collecting variables on the left and constants on the right. So she adds 5 on both sides. On the left, she simply writes 5 and chooses to write it on the left of the expression so that it is easy to compute "5 subtract 5". However, in doing this, the minus symbol (in -5) shifts from representing a sign (negative) in line 1 to an operation (subtract) in line 2. Learners who are comfortable working with integers will ignore these subtle distinctions. But for learners still grappling with different meanings of the minus symbol, this layout on the left side may be a source of difficulty. It might be more helpful to make the changes on the right of each expression (i.e. $-5 + 2x \mathbf{+ 5} = x + 6 \mathbf{+ 5}$) because this would not affect the interpretation of the minus symbol on the left side. It is also worth noting that the operation of adding 5 is now explicit on both sides of the equation whereas this was not the case when 5 was appended to the front of the expression.

DIFFICULTIES WITH ALGEBRAIC SIMPLIFICATION

In the context of linear equations, algebraic simplification is typically limited to: (1) distinguishing like and unlike terms; (2) collecting like terms; and (3) applying the distributive law – and all of this is done with only one variable. Yet, learners still make many errors with such apparently simple algebraic simplification. We conducted a study with over 800 Grade 9 learners from top-performing quintile 5 schools in Gauteng which revealed many basic algebraic errors in learners' attempts to solve linear equations (Pournara, 2020). Most of these errors involved subtraction/negatives and many involved working with *like* terms. This is surprising because we often assume that errors arise from incorrectly combining *unlike* terms such as $x + 5 = 5x$. The following errors were common:

$5x - x = 5$	Learner seemingly focuses on $x - x$ which eliminates the variable and so 5 remains.
$5x - x = 5x$	Learner may be treating x as $0x$, hence $5 - 0 = 5$ and retain x.
$-5x + x = -6x$	Learner likely isolates the leading negative, adds $5x$ and $1x$, then brings back the negative

Algebraic simplification tasks in Grades 8 and 9 typically involve far more complex expressions than those shown above. However, it appears that more attention should be given to these seemingly simple combinations of terms which arise in linear equations. We provide suggestions of such expressions in the last section of the article.

BALANCING AND INVERSE OPERATIONS

An equivalence view of the equal sign is closely linked to the idea of balancing an equation. As previously mentioned, we strongly advocate for the use of inverse operations when manipulating equations to promote the idea of balance. We would therefore expect learners to be familiar with teacher utterances such as "*if you add* 2 *on the left side, then you must add* 2 *on the right side*"; "*if you divide the left side by* –3*, then you must divide the right side by* –3"; "*whatever you do on the left, you do on the right*". However, irrespective of the approach used, learner errors arise in their attempts to manipulate equations. Of course, we can add/subtract any number (or variable) from both sides of an equation and it will remain balanced. We can also multiply/divide both sides by the same value. The point is to make wise choices when adding, subtracting, multiplying or dividing so as to make a sum of zero or a product of one.

Typical learner errors that don't maintain balance include: (1) 'moving a term across the equal sign' without changing sign; (2) subtracting a constant (or a term with a variable) from one side but adding it on the other side; and (3) applying the wrong inverse (e.g. learners might simplify $2x = 6$ by subtracting 2 from both sides rather than dividing by 2, leading to the incorrect solution $x = 4$).

On more careful reflection, we have come to realise that learners' difficulties may stem from "having too many options" in the first step of solving an equation. For example, given $2x - 3 = 5x + 9$, learners could begin by collecting terms with the variable on the left or the right. Therefore they could subtract $2x$ or $5x$ from both sides. Similarly, they must decide whether to collect constants on the left or the right which means they could add 3 to both sides or subtract 9 from both sides. They therefore have four options when they begin to solve the equation. For equations with the unknown on one side only, they have only two options, and these likely reduce to keeping the variable where it is and then isolating it. We return to this issue in the next section.

We have also come to appreciate subtle differences in applying the additive inverse and the multiplicative inverse. When applying the additive inverse, the goal is to eliminate a particular term on one side. By contrast, when applying the multiplicative inverse, the goal is to reduce the coefficient of the variables to 1, but not to eliminate the entire term. In the earlier example ($2x - 3 = 5x + 9$), if we want to eliminate the constant on the right side, we apply the additive inverse of 9 which gives $2x - 3 - 9 = 5x + 9 - 9$, and then simplifies to $2x - 12 = 5x$. We might then apply the additive inverse of $2x$, which is $-2x$ and not just -2, yielding $-12 = 3x$. Given that the goal is to solve for "one x", we must apply the multiplicative inverse of 3, not $3x$, which of course is $\frac{1}{3}$ not $\frac{1}{3x}$. We also see learners subtracting $2x$ from $3x$ in order to get $1x$ on the right. If they subtract correctly on both sides, then they end up with the unknown on both sides again which is a step away from, and not towards, solving for the unknown.

RECOMMENDATIONS FOR TEACHING

In this final section we make five recommendations for teaching and provide examples of tasks to illustrate each recommendation. But first, recommendation zero: avoid spending too much time on algebraic equations with letters on one side only, e.g. $2x = 15$ and even $2 - x = 15$. As we have already explained, learners can solve these without having to operate on the unknown (even if they may have to work a bit harder on the second example listed here).

1. EMPHASISE AN EQUIVALENCE OR BALANCE VIEW OF THE EQUAL SIGN

Fill in the missing number:

a) $6 + 4 = \_\_$
b) $6 + 4 = \_\_ + 7$
c) $6 + \_\_ = 10 + 5$
d) $6 + 4 = \_\_ - 5$
e) $\_\_ + 6 = 6 \times 5$

This task begins with a do-something view of the equal sign in (a) and then immediately shifts to an equivalence view with the subtle introduction of "$+7$" in (b). Learners need to reason "What added to 7 is the same as 6 add 4?" We also introduce subtraction and multiplication on the right side (d and e) to indicate to learners that any operations can be used. Note that the position of the unknown is varied too.

2. GIVE MORE ATTENTION TO ALGEBRAIC EXPRESSIONS THAT LEARNERS ENCOUNTER WHEN SOLVING EQUATIONS

Simplify:

a) $3x - x - 3 =$
b) $3 - x - 3 =$
c) $2x + 6 - x - 6 =$
d) $-x + 5 + x - 3 =$
e) $-2x + 3x + 6 - 6 =$
f) $2(x + 3) - x - 2 =$

This task deals with like and unlike terms involving x and constants. There are several instances of adding and subtracting a constant or term in x to create zero (c, d, e). We also recommend that teachers vary the order of the terms. For example, in (e) the expression begins with $-2x + 3x$. Our research findings show that this is far more difficult for learners to cope with than the equivalent form of $3x - 2x$.

3. Explore different sequences of applying inverses

In the matrix below we focus on a single example, $2x + 4 = x - 5$. We apply different additive inverses and show the new terms which are highlighted by shading them. In A and B, we start with the constants and then move to the variables. In C and D, we start with the terms containing the variable. In A and C, we begin on the left side, subtracting 4 and $2x$ respectively. In B and D, we begin on the right, adding 5 and subtracting x respectively. We deliberately stop after applying the first inverse. This shows the range of simplifications that learners will encounter before applying the next inverse. The matrix also reveals how the partially solved equations differ, depending on what operations have already been performed. For example, in A we get $2x = x - 9$ while in C we get $4 = -x - 5$. While the partially solved equations are equivalent, learners may not yet realise this. However, they should ultimately see that they all the equations reduce to $x = -9$.

We recommend that teachers take time to show all four options to learners. We further recommend that teachers specify which approach to use on particular examples so that learners become comfortable to isolate the variable on the left side or right side, and to work with negative coefficients and constants on either side.

	Start on the left	**Start on the right**
Start with constant term	**A.** $2x + 4 = x - 5$ $2x + 4 - 4 = x - 5 - 4$ $2x = x - 9$	**B.** $2x + 4 = x - 5$ $2x + 4 + 5 = x - 5 + 5$ $2x + 9 = x$
Start with term with variable	**C.** $2x + 4 = x - 5$ $2x - 2x + 4 = x - 2x - 5$ $4 = -x - 5$	**D.** $2x + 4 = x - 5$ $2x - x + 4 = x - x - 5$ $x + 4 = -5$

4. Emphasise different placements of terms when applying inverses

We have already noted the importance of practice in simplifying the kinds of algebraic expressions that learners encounter when solving linear equations. Learners need to get used to working with like and unlike terms in different positions. The matrix below shows different possibilities of placing the "new terms" when applying the additive inverse of terms with variables. Note that we have copied C and D from the matrix above and continued to number E and F. In C and D the new terms are placed immediately after the terms with variables so the like terms are adjacent to each other on both sides of the equal sign. In E and F, we place the new terms on the extreme right of each expression. For the given example, this means that like terms are not adjacent on each side. The same can be done with constants.

	Start on the left	**Start on the right**
Writing terms with variables together	**C.** $2x + 4 = x - 5$ $2x - 2x + 4 = x - 2x - 5$ $4 = -x - 5$	**D.** $2x + 4 = x - 5$ $2x - x + 4 = x - x - 5$ $x + 4 = -5$
Writing 'new term' at end of each expression	**E.** $2x + 4 = x - 5$ $2x + 4 - 2x = x - 5 - 2x$ $4 = -x - 5$	**F.** $2x + 4 = x - 5$ $2x + 4 - x = x - 5 - x$ $x + 4 = -5$

5. DEALING DIRECTLY WITH LEARNER'S ERRORS

We consider learners' errors as opportunities for learning, not failures to be avoided. For this reason we recommend confronting learners with typical errors and then dealing explicitly with the erroneous thinking that leads to the error. Returning to the previous example: $2x + 4 = x - 5$. Consider the following learner response:

Line 1	$2x + 4 = x - 5$
Line 2	$2x + 4 - 4 = x - 5 + 5$
Line 3	$2x = x$
Line 4	$x = 1$

In line 2, the learner has subtracted 4 on the left side but added 5 on the right side. Both moves are partially correct since they eliminate the constant term on each side. However, they disrupt the balance of the equation. Line 3 follows logically based on the errors in line 2. But line 4 does not follow from line 3. In our research we have seen many instances where the last line does not follow from the line above. We suspect that this stems from learners' knowing what the final line should look like (i.e. $x = \cdots$) and so learners force the final line into this familiar form without a logical connection to the previous line.

In dealing with these errors, we recommend that teachers begin by asking learners to check the solution. Learners should then see that they get different results on the left side and the right side:

Left side	Right side
$2x + 4$ Substituting $x = 1$: $2(1) + 4$ $= 6$	$x - 5$ Substituting $x = 1$: $1 - 5$ $= -4$

Since the left side does not yield the same result as the right side, we know that $x = 1$ is not the solution to the equation. This process reinforces the meaning of solution – that the solution is the value which makes the left side equal to the right side, or the value that gives the same result on the left side as the right side. This also promotes an essential practice in solving equations – checking the solution by substitution. It is likely that many learners will have identified the errors in line 2. They can then be encouraged to correct the errors, to obtain a new solution and to check this solution. Note the layout of the two substitutions above. We deliberately write "$= 6$" and "$= -4$" below the line of substitution, rather than alongside. We do so to further emphasise that we are working separately with each expression in the equation and are not setting up another equation. Clearly, there is no mathematical difference in writing the final line next to the line of substitution.

In the WMCS equation materials we provide a range of worksheets with varying foci and different levels of difficulty. We include a large collection of worksheets on numeric equations, some deal only with whole numbers while others require an understanding of integers. Many tasks tackle learners' errors head on, asking learners to identify the errors and to correct them. The extract which follows comes from a worksheet on additive inverses. It shows how we encourage learners to pay attention to what happens to each side of the equation when they apply the additive inverse. The worksheets on equations and several other topics can be freely downloaded from www.witsmathsconnectsecondary.co.za/resources.

Questions

1) Make 5 pairs of additive inverses from the list of terms below. If the additive inverse does not appear in the list, provide it.

   $-4 \quad \frac{1}{4} \quad -x \quad 6x \quad 6 \quad \frac{1}{6} \quad -\frac{1}{2} \quad 4 \quad x \quad 0{,}5$

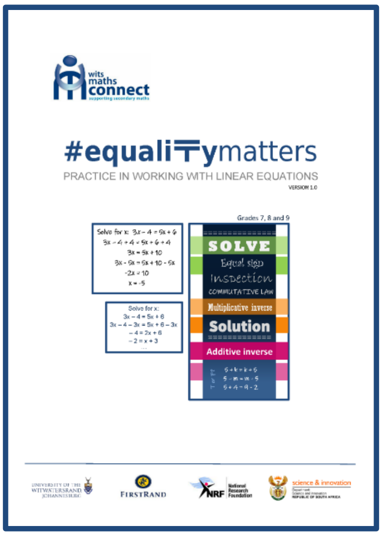
2) In the table below, apply the additive inverse that is indicated. Then write down the new form of the equation after applying the inverse. The first one has been done for you.

Equation	Apply additive inverse of ...	Equation after applying inverse
$x - 4 = 2x$	-4	$x = 2x + 4$
	x	
	$2x$	

3) This equation has k's on both sides of the equal sign: $4k = k + 6$
   a) If you apply the additive inverse of 6 to both sides, will there still be k's on both sides?
   b) Remember that the additive inverse of k is $-k$.
      i) Apply this additive inverse to both sides.
      ii) Are there still k's on both sides?
      iii) What remains on the right side?
   c) Continue to solve the equation and show that the solution is 2.
   d) What type of inverse did you use to continue solving the equation in Q3c?

4) Solve the following equations by applying the additive inverse of the variable.

   a) $3p = p - 2$ b) $p = 3p - 2$ c) $7b = 5b + 4$ d) $20b = 50b - 10$



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