# Exploring Cevians

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### INTRODUCTION

A cevian is a line segment joining a vertex of a triangle to a point on the opposite side (or its extension) as illustrated in Figure 1. Cevians are named after the Italian mathematician Giovanni Ceva (1647-1734) who is probably best known for what is generally referred to as Ceva's theorem – the condition for three general cevians to be concurrent. Medians, altitudes and angle bisectors are special cases of cevians.

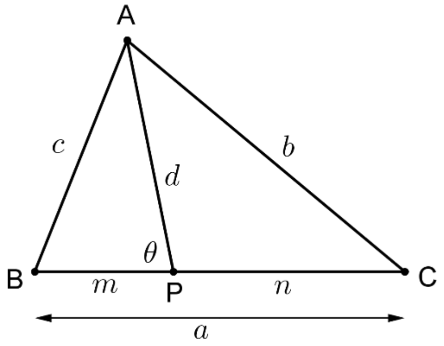



**FIGURE 1:** Triangle $ABC$ with cevian $AP$ of length $d$.

### STEWART'S THEOREM

The length of a cevian can be determined by Stewart's theorem. With reference to Figure 1, the cevian length $d$ is given by the following formula:

$$b^2m + c^2n = a(d^2 + mn)$$

Stewart's theorem can readily be proved using the cosine rule. In triangles $APB$ and $ACP$ we have respectively:

$$c^2 = d^2 + m^2 - 2dm\cos\theta \ \ldots \ (1)$$

$$\begin{aligned} b^2 &= d^2 + n^2 - 2dn\cos(180° - \theta) \\ &= d^2 + n^2 + 2dn\cos\theta \ \ldots \ (2) \end{aligned}$$

Multiplying equation (1) by $n$ and equation (2) by $m$ and then adding allows us to eliminate $\cos\theta$:

$$\begin{aligned} c^2n &= d^2n + m^2n - 2dmn\cos\theta \\ b^2m &= d^2m + n^2m + 2dmn\cos\theta \\ \therefore \ b^2m + c^2n &= d^2m + d^2n + n^2m + m^2n \end{aligned}$$

Factorising the expression on the right-hand side, and replacing $m + n$ with $a$, gets us to the required result:

$$\begin{aligned} b^2m + c^2n &= d^2(m + n) + mn(n + m) \\ &= (m + n)(d^2 + mn) \\ &= a(d^2 + mn) \end{aligned}$$

While Stewart's theorem will be unfamiliar to most high school pupils, both it and its proof are readily accessible. The proof is in fact rather pleasing as it incorporates many aspects of high school mathematics – the cosine rule, trigonometric reductions, elimination, and factorisation of a four-term expression involving grouping. Stewart's theorem can also be proved using Pythagoras's theorem directly by dropping a perpendicular from vertex $A$ to base $BC$ and writing the distances $b$, $c$ and $d$ in terms of this altitude. It is left to the interested reader to complete this proof.

### VAN AUBEL'S THEOREM FOR TRIANGLES

Henri Van Aubel (1830-1906) taught pre-university mathematics at the *Koninklijk Atheneum Antwerpen* in Belgium. Given triangle $ABC$ with three cevians intersecting at a common point $P$, as illustrated in Figure 2, then Van Aubel's theorem for triangles states that:

$$\frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}$$

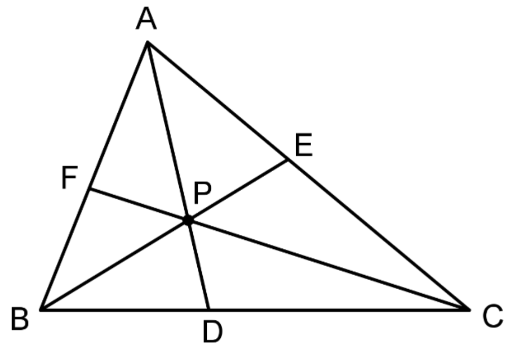



**FIGURE 2:** Triangle $ABC$ with three cevians concurrent at point $P$.

The proof of this interesting result is also very accessible to the high school pupil. There are two basic ideas we will make use of in the proof:

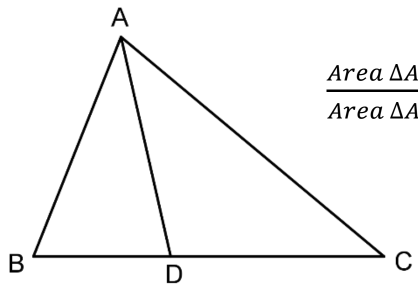
- If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} = \frac{a-c}{b-d}$

  This can readily be understood as follows:

  If $\frac{a}{b} = \frac{c}{d}$ then $c = ka$ and $d = kb$ where $k$ is some scalar constant. Thus:

  $\frac{a+c}{b+d} = \frac{a+ka}{b+kb} = \frac{a(1+k)}{b(1+k)} = \frac{a}{b}$ and $\frac{a-c}{b-d} = \frac{a-ka}{b-kb} = \frac{a(1-k)}{b(1-k)} = \frac{a}{b}$

- The areas of triangles with equal altitudes (perpendicular heights) are in the same proportion as the lengths of their bases.




$$\frac{Area\ \Delta ABD}{Area\ \Delta ADC} = \frac{\frac{1}{2} \times BD \times \perp h}{\frac{1}{2} \times DC \times \perp h} = \frac{BD}{DC}$$

**FIGURE 3:** Triangles $ABD$ and $ADC$ with areas in the ratio $BD{:}DC$.

We can now prove Van Aubel's theorem for triangles as follows:

$$\left.\begin{array}{l}\dfrac{Area\ \Delta ACF}{Area\ \Delta BCF}=\dfrac{AF}{BF}\\[2ex]\dfrac{Area\ \Delta APF}{Area\ \Delta BPF}=\dfrac{AF}{BF}\end{array}\right\}\quad \therefore\quad \frac{AF}{FB}=\frac{Area\ \Delta ACF-Area\ \Delta APF}{Area\ \Delta BCF-Area\ \Delta BPF}=\frac{Area\ \Delta APC}{Area\ \Delta BPC}\quad ...(1)$$

$$\left.\begin{array}{l}\dfrac{Area\ \Delta ABE}{Area\ \Delta CBE}=\dfrac{AE}{EC}\\[2ex]\dfrac{Area\ \Delta APE}{Area\ \Delta CPE}=\dfrac{AE}{EC}\end{array}\right\}\quad \therefore\quad \frac{AE}{EC}=\frac{Area\ \Delta ABE-Area\ \Delta APE}{Area\ \Delta CBE-Area\ \Delta CPE}=\frac{Area\ \Delta ABP}{Area\ \Delta CBP}\quad ...(2)$$

Adding equations (1) and (2) gives:

$$\frac{AF}{FB}+\frac{AE}{EC}=\frac{Area\ \Delta APC}{Area\ \Delta BPC}+\frac{Area\ \Delta ABP}{Area\ \Delta CBP}=\frac{Area\ \Delta APC+Area\ \Delta ABP}{Area\ \Delta BPC}$$

But: $$\frac{AP}{PD}=\frac{Area\ \Delta ABP}{Area\ \Delta DBP}=\frac{Area\ \Delta APC}{Area\ \Delta DPC}$$

$$\therefore\quad \frac{AP}{PD}=\frac{Area\ \Delta ABP+Area\ \Delta APC}{Area\ \Delta DBP+Area\ \Delta DPC}=\frac{Area\ \Delta ABP+Area\ \Delta APC}{Area\ \Delta BPC}$$

$$\therefore\quad \frac{AP}{PD}=\frac{AF}{FB}+\frac{AE}{EC}$$

## CONCLUDING COMMENTS

While Stewart's theorem and Van Aubel's theorem for triangles will both be unfamiliar to most high school pupils (other than those who have received special Olympiad training), both theorems are readily accessible. More importantly, their *proofs* rely only on concepts and techniques covered within the school Mathematics syllabus. Exploring such proofs in the classroom (with appropriate scaffolding and guidance) can expose pupils to interesting theorems and results beyond the confines of the school syllabus, and at the same time show how basic concepts and techniques learnt at school find application beyond the walls of the classroom.