Area of the Largest Rectangle Inscribed in a Right-Angled Triangle

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INTRODUCTION

Given a right-angled triangle ABC, we are interested in finding the area of the largest rectangle $CDFE$ with all four vertices lying on the sides of triangle ABC, as illustrated in Figure 1.

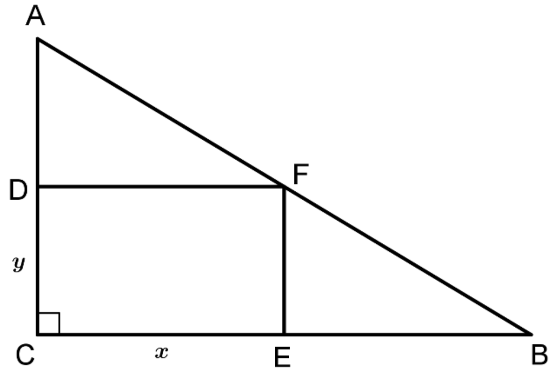

FIGURE 1: Finding the largest rectangle inscribed in triangle ABC.

Let $BC = a$, $AC = b$, $CE = x$ and $CD = y$. Since triangles ADF and ACB are similar, we have:

$$\frac{AD}{AC} = \frac{DF}{CB} \quad \Rightarrow \quad \frac{b-y}{b} = \frac{x}{a}$$

This can be simplified to $ay + bx = ab$.

In this short article we discuss two different approaches to solving this problem, and offer them as alternatives to the trigonometric approach described by Pillay and Bizony (2009). The first approach assumes a basic knowledge of the process of completing the square, while the second approach assumes knowledge of basic differential calculus. For convenience we will use W to denote the area of rectangle $CDFE$.

AN ALGEBRAIC APPROACH

Using the relationship arrived at above, namely $ay + bx = ab$, we have $x = (ab - ay)/b$. We can now express the area of the rectangle in terms of y as follows:

$$W = xy = \frac{(ab-ay)y}{b} = \frac{a}{b}(by - y^2) = \frac{a}{b}\left(\frac{b^2}{4} - \left(\frac{b}{2} - y\right)^2\right) = \frac{ab}{4} - \frac{a}{b}\left(\frac{b}{2} - y\right)^2$$

From this we can see that the maximum area of $W = \frac{ab}{4}$ occurs when $y = \frac{b}{2}$. Since $ay + bx = ab$ it also follows that the maximum area occurs when $x = \frac{a}{2}$. The area of the largest rectangle $CDFE$ is thus half the area of triangle ABC.

A CALCULUS APPROACH

Since $W = xy = \frac{a}{b}(by - y^2)$, we have $\frac{dW}{dy} = \frac{a}{b}(b - 2y)$. Since the second derivative $\frac{d^2W}{dy^2} = \frac{-2a}{b} < 0$ and the first derivative $\frac{dW}{dy} = 0$ occurs when $y = \frac{b}{2}$, we can similarly conclude that the maximum area of W is $\frac{ab}{4}$ and occurs when $y = \frac{b}{2}$ and $x = \frac{a}{2}$.

A RELATED PAPER-FOLDING ACTIVITY

A way of introducing this activity to pupils is through a related paper-folding activity. Provide pupils with right-angled triangles and ask them to explore how to construct a rectangle from the triangle by folding over the vertices of the triangle in such a way that the vertices of the rectangle lie on the sides of the triangle. They should hopefully accomplish this by folding vertices A and B onto C as illustrated in Figure 2. Next ask pupils to explore whether the constructed rectangle is the largest possible rectangle. This then leads into asking pupils to find the area of the largest rectangle, and an introduction to possible solution methods.

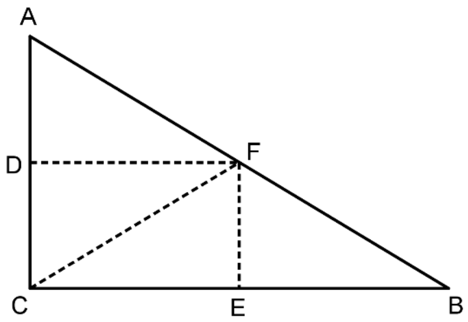

FIGURE 2: Constructing the largest inscribed rectangle by paper-folding.

It is possible during the paper-folding activity that some pupils find a different largest rectangle (Figure 3a). However, it is not difficult to show that the area of this alternative rectangle is also half the area of the triangle. Dropping a perpendicular CH reveals two smaller right-angled triangles, CBH and CAH, each with largest rectangles, $FEJH$ and $GDJH$ respectively, inscribed as before (Figure 3b).

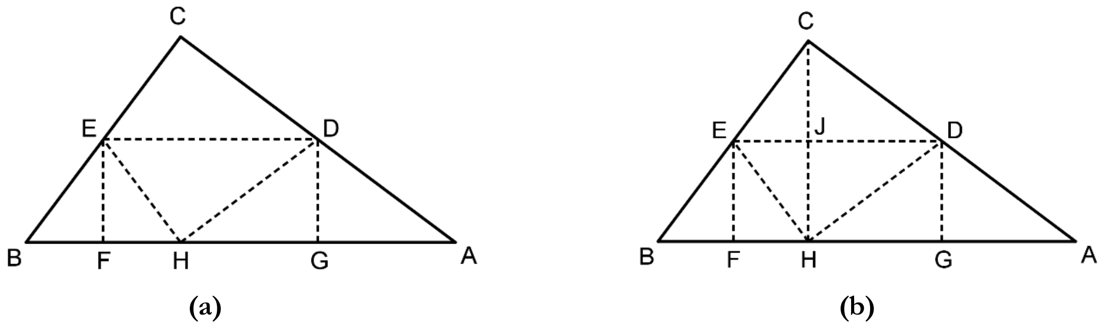

FIGURE 3: Another possible rectangle obtained by paper-folding.

REFERENCES

Pillay, P., & Bizony, M. (2009). Rectangles inscribed in a triangle. *Learning and Teaching Mathematics*, *7*, 13-15.