Bridging the Geometry Gap from Grades 8 and 9 to Grade 10: Suggestions for Adjusting the Cognitive Demand of Geometry Tasks

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INTRODUCTION

Many geometry tasks in Grade 8 and 9 textbooks in South Africa may be relatively easy for learners who do well in mathematics because the tasks demand little deductive reasoning. By contrast, when learners get to Grade 10, they encounter geometry riders which are much more difficult and require several steps of deductive reasoning. That said, we see much evidence of difficult geometry riders creeping down into Grade 9 via assessment tasks, and this goes back to the Annual National Assessments of 2012-2014. It is well-known that geometry is an area of difficulty for learners and teachers, not just in South Africa but internationally too. The solution is not to push geometric proof tasks down into Grades 8 and 9 in the hope that this will prepare learners to cope with FET geometry. We propose instead that numeric tasks which are typical of Grades 8 and 9 should be strategically extended to prepare learners for the demands of FET geometry.

In a previous article (Ratnayake, Takker & Pournara, 2021) we discussed ways of building up and breaking down complex diagrams to help learners gain access to visual aspects of Euclidean geometry. In this article we focus on tasks dealing with triangles, lines and angles, and we share ideas of how to adjust the cognitive demand of these tasks. We end with a challenge for teachers to try with learners who are already performing well in geometry and who like to grapple with words and diagrams.

We begin with Task A in Figure 1 and we invite the reader to pause and to attempt it.

In ΔPQR, $PQ = QR$, PQ is extended to L and RQ is extended to K. TQS is a straight line and $RP \parallel TS$. Show that QS bisects $P\hat{Q}K$.

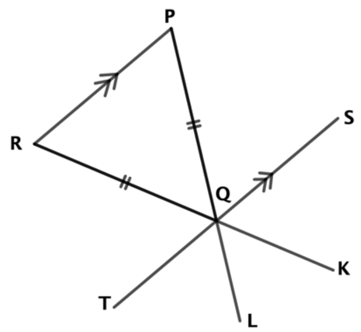


FIGURE 1: Task A.

In order to prove that QS bisects $P\hat{Q}K$ one needs to show that $P\hat{Q}S = S\hat{Q}K$, and you could have done this in various ways. Perhaps you focused on the parallel lines. Perhaps you focused on the exterior angle(s) of the triangle. Either way, as you worked you may have been thinking that this task is too difficult for most Grade 8 and 9 learners. Nonetheless, it deals only with parallel lines and triangles, both of which are part of the Grade 8 and 9 curricula.

Breaking down a complex question into simpler components is a familiar design strategy to make a task more accessible. We will refer to this as *scaffolding* a task. Scaffolding is a typical feature of most formal examination questions in geometry. However, scaffolding may not always be helpful to learners, particularly if they are reasoning in a different way to the person who designed the task. We will show an example of this in the next section.

In addition to scaffolding, we suggest four strategies to adjust the cognitive demand of a geometry task, and we illustrate them through various versions of Task A.

1. **Extending calculation tasks to generalisation:** In calculation tasks, learners work with numeric values, typically to find the sizes of angles. In such tasks it is important to pay attention to which angle sizes we choose to give and what values we use. We also recommend extending numeric tasks to open opportunities to generalise, as we illustrate later on.

2. **Manipulating the diagram:** When diagrams are given in familiar orientation, they are usually easier to work with. However, learners need to get comfortable with various orientations and with different shapes of triangles, for example. We also propose getting learners to extend or adapt the diagram and then to think about the impact on the relationships in the diagram.

3. **Scaffolding proof tasks**: To make a proof task easier, we can provide values of angle sizes[1]. This additional information can be given as numerical values or in the form of algebraic expressions. The latter is typically more difficult for learners. Nevertheless, by providing such information we provide an implicit starting point for learners to begin the task.

4. **Connecting visual and verbal information:** Diagrams are an important part of geometry tasks. Learners need to make sense of information given in the statement (verbally) as well as on the diagram, and they need to transfer any information given only in the statement onto the diagram.

We will now discuss a few examples for each of the above strategies. Although we present them under separate headings, there are clearly overlaps between the four strategies.

EXTENDING CALCULATION TASKS TO GENERALISATION

Consider Task 1 as shown in Figure 2. This is a calculation task based on Task A where the measure (or size) of one angle is given as a numerical value ($52°$). Information is given verbally (in the statement) as well as visually on the diagram. In addition, the task is scaffolded by first asking learners to find the sizes of four angles, some of which can be used to answer part (b). We could have scaffolded the task even more by specifying the order in which to find the sizes of the four angles. For example, finding the size of $S\hat{Q}K$ may be easiest for learners, yet it is the last angle in the list. While part (a) is intended to scaffold towards part (b), a learner who focuses on angles $\hat{R}$ and $S\hat{Q}K$ as equal corresponding angles may not find part (a) helpful.

Part (b) of Task 1 draws learners' attention to the fact that QS bisects $P\hat{Q}K$, although the question is not phrased in this way. Clearly this is an opportunity to introduce the notion of angle bisectors and also to emphasize that when proving that a line bisects an angle we have to show that the adjacent parts of the larger angle are equal, i.e. we have to translate "bisect" into a statement about equal angles.

[1] We appreciate that some may argue that a "proof task" involving specific sizes of angles is not really a proof task. Nevertheless, we include numeric cases within this cluster of tasks because they form part of the trajectory towards formal proof tasks. In such tasks teachers may prefer to say "show that…" rather than "prove that…".

Task 1

In ΔPQR, $\mathrm{PQ} = \mathrm{QR}$, PQ is extended to L and RQ is extended to K. Straight line TQS is drawn parallel to RP.

a) If $\mathrm{R\widehat{P}Q} = 52^{\circ}$ find the sizes of these four angles in any order: $\mathrm{P\widehat{Q}R}$, $\mathrm{P\widehat{Q}S}$, $\mathrm{P\widehat{Q}K}$, and $\mathrm{S\widehat{Q}K}$.

b) What is the relationship between $\mathrm{P\widehat{Q}S}$ and $\mathrm{S\widehat{Q}K}$?

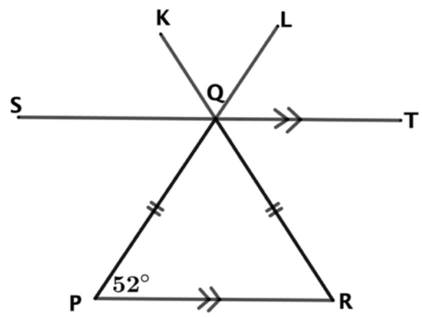


FIGURE 2: Task 1.

There are several ways to extend this task, the simplest of which is to change the angle size and ask learners to check whether the relationship in part (b) still holds. We could invite learners to choose their own angle size and to check again. Eventually we could move to generalising by asking "if the size of $\widehat{\mathrm{P}}$ is x, …" In so doing we help learners to see that the relationship in focus always holds true. Such tasks are not typical in Grade 8 and 9 geometry.

We could make the task slightly more difficult by giving the size of $\mathrm{P\widehat{Q}R}$. This would require learners to calculate the size of $\widehat{\mathrm{P}}$ and/or $\widehat{\mathrm{R}}$. If we choose $\mathrm{P\widehat{Q}R}$ to be an odd number, say $81°$, the sizes of $\widehat{\mathrm{P}}$ and $\widehat{\mathrm{R}}$ include fractional parts. This is important to reinforce that angle sizes are not always whole numbers.

MANIPULATING THE DIAGRAM

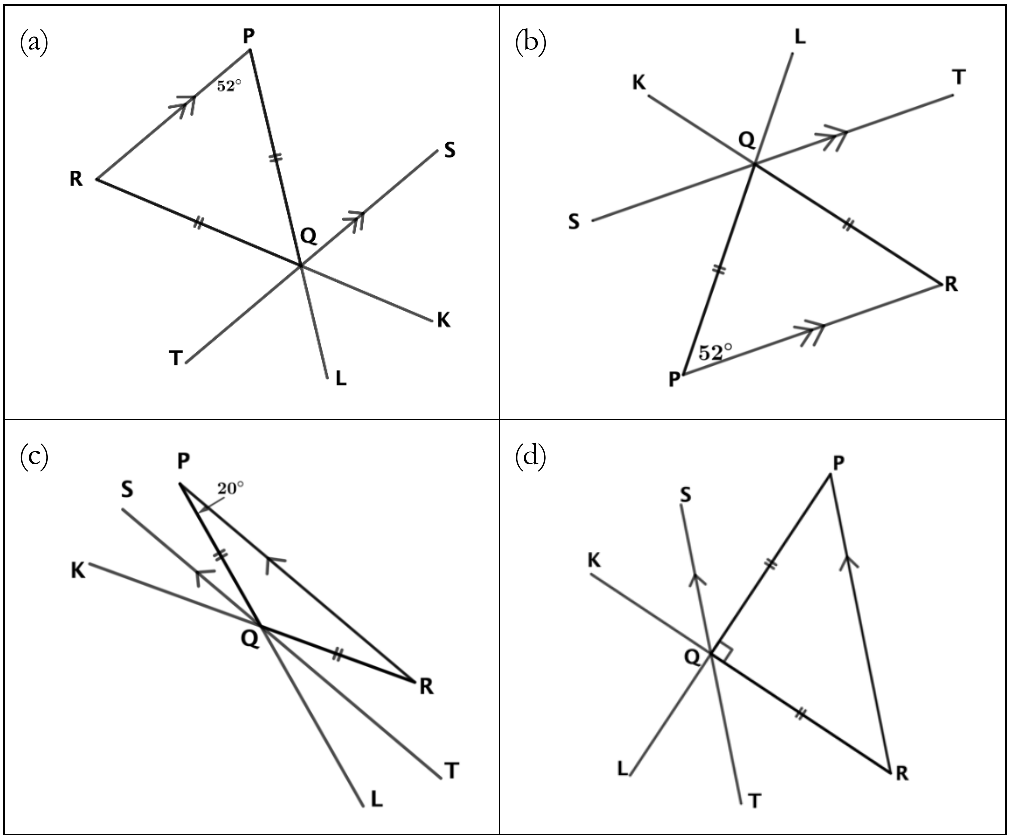


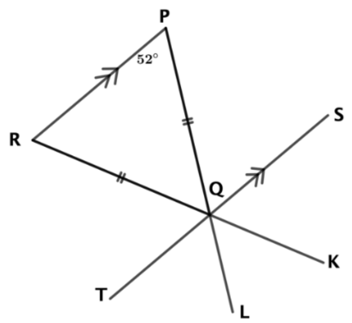
FIGURE 3: Four orientations of a diagram for Task 1.

The diagram in Figure 2 is the most typical diagram for Task 1 because both the base of the triangle and the pair of parallel lines are in horizontal positions. We refer to this as *standard orientation*. Changing this orientation to an atypical or non-standard orientation makes the task visually harder. Figure 3 provides four examples of non-standard orientations of the diagram in Figure 2. Figures 3(a) and 3(b) show two different orientations of the diagram provided in Figure 2. In Figures 3(c) and 3(d) we have also manipulated the type of triangle. While Figure 3(c) has all the same properties as Figure 2, visually it appears more difficult to work with because it is more “squashed up”. In Figure 3(d) we included the special case of a right-angled triangle so that we need not explicitly provide any angle measures. We wonder whether learners will find this easier or more difficult.

Two further extensions involve manipulating the diagram and posing related questions. For example, what if TQS were not drawn parallel to PR? Would the line still bisect the exterior angle, and how would you justify your answer? This requires learners to reason geometrically, but their response will not come in the form of a two-column proof. Rather, they could be invited to write a few sentences and provide supporting diagrams to illustrate their answer. In a similar way, we could ask whether the line drawn parallel to PR will bisect the exterior angle ($\mathrm{P\widehat{Q}K}$ or $\mathrm{R\widehat{Q}L}$) *if the triangle is not isosceles*. In both instances we open opportunities for learners to generate their own versions of the diagram, with their own choice of angles. Learners seldom get the opportunity to produce diagrams in geometry and then to reason from their own diagrams. We believe that this is an important skill to nurture.

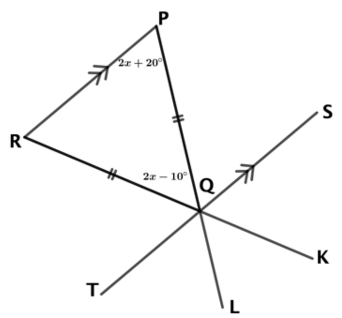
SCAFFOLDING PROOF TASKS

Task 2:
In ΔPQR, $\mathrm{PQ} = \mathrm{QR}$, PQ is extended to L and RQ is extended to K. Straight line TQS is drawn parallel to RP. If $\mathrm{R\widehat{P}Q} = 52^{\circ}$ prove that QS bisects $\mathrm{P\widehat{Q}K}$.



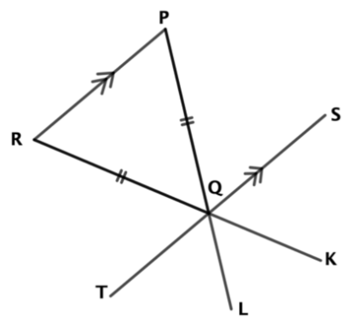
(a)

Task 3:
In ΔPQR, $\mathrm{PQ} = \mathrm{QR}$, PQ is extended to L and RQ is extended to K. Straight line TQS is drawn parallel to RP. $\mathrm{R\widehat{P}Q} = 2x + 20^{\circ}$ and $\mathrm{R\widehat{Q}P} = 2x - 10^{\circ}$.
Show that QS bisects $\mathrm{P\widehat{Q}K}$.



(b)

Task 4:
In ΔPQR, $\mathrm{PQ} = \mathrm{QR}$, PQ is extended to L and RQ is extended to K. TQS is a straight line and $\mathrm{RP} \parallel \mathrm{TS}$.
Show that QS bisects $\mathrm{P\widehat{Q}K}$.



(c)

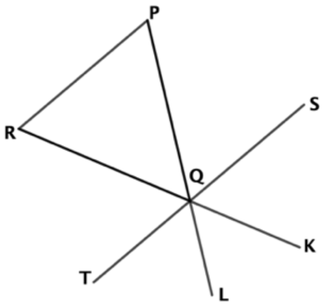
FIGURE 4: Three proof tasks.

In this section we present a few examples of proof tasks and describe ways to adjust the cognitive demand of the tasks. The easiest version of a proof task could be a numeric measure proof task (see Task 2 in Figure 4(a)). In this task the measure of one angle is given and learners are required to prove (or show) that QS bisects $P\widehat{Q}K$. Providing values of angles in algebraic expressions would raise the difficulty level of the task (see Task 3). Notice that we may need to provide sizes of two angles when we use algebraic expressions so that learners can calculate a numeric value. In Task 3, in addition to geometry concepts, the learner needs to apply knowledge of solving linear equations. Task 4 is the most difficult version among the three tasks given in Figure 4. Although Tasks 2 and 3 are both proof tasks and contain no scaffolding, the inclusion of angle-sizes provides an obvious starting point for the learner. By contrast, Task 4 provides no obvious starting point. This makes the task much more difficult for learners. The comparison of these three tasks shows how we can use a form of implicit scaffolding (through angle measures) to provide a starting point for the learner.

CONNECTING VISUAL AND VERBAL INFORMATION

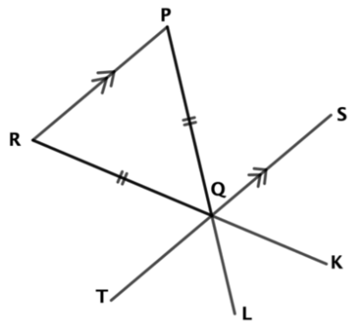
Version 1 – *no specific information on diagram*

In ΔPQR, $PQ = QR$, PQ is extended to L and RQ is extended to K. TQS is a straight line and $RP \parallel TS$. Show that QS bisects $P\widehat{Q}K$.



Version 2 – *most information on diagram. Explicit mention that lines are straight in the verbal statements.*

In the diagram PQL, RQK and TQS are straight lines. Show that QS bisects $P\widehat{Q}K$.



Version 3 – *only equal sides shown on diagram*

In ΔPQR, $PQ = QR$, PQ is extended to L and RQ is extended to K. TQS is a straight line. If $RP \parallel TS$, show that QS bisects $P\widehat{Q}K$.

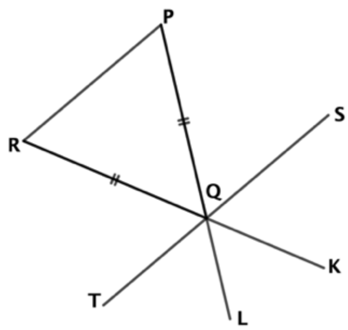


FIGURE 5: Different combinations of information in verbal and visual forms.

In Grades 8 and 9, many geometry tasks involve only a diagram with very little, if any, accompanying text. For example, learners may be given four different diagrams and asked to find the sizes of angles marked x in each diagram. Sometimes this requires them to make assumptions about the diagrams, the most common of which is to assume that all lines are straight. As tasks become more cognitively demanding they usually include a diagram with accompanying text. But do learners read the text? We believe that it is important to pay explicit attention to the text, not only to encourage learners to read it but also to develop their skills in connecting the verbal information with the visual information on the diagram.

Text can be presented in various, albeit fairly obvious, ways, which are illustrated in Figure 5 using slightly different versions of Task 1. For example:

- All information is given in verbal form and nothing specific is shown on the diagram (version 1)
- All information is given in verbal form and repeated on the diagram (as in the original Task 1)
- Most of the information is on the diagram only (version 2)
- Most of the information is in verbal statements with some repeated on the diagram (version 3)

It is important for learners to pay attention to the information given in the verbal statements. For example, the diagram may contain a triangle which *looks* isosceles but may have to be *proved* isosceles. Similarly, at FET level, a quadrilateral may *look* like a parallelogram but one may be required to *prove* it is a parallelogram based on the given information. Perhaps providing some information in verbal statements and the rest of the information on the diagram is a way to encourage learners to pay attention to the verbal and visual aspects of the task, and thus to make sure that they have taken note of all the given information before they attempt the task.

CHALLENGE: PROOF TASKS WITHOUT DIAGRAMS

We seldom encounter geometry riders where a diagram is not provided. However, theorems are often stated in general forms. Thereafter the statement is "translated" to fit a diagram that represents a general case. The skill of translating from verbal descriptions to 2D diagrams is clearly very demanding. We want to challenge teachers to try this with some of their stronger learners. We return to Task A to illustrate some possibilities. Consider the three verbal descriptions below.

TABLE 1: Verbal descriptions for proof tasks.

Version 1 In ΔPQR, $PQ = QR$. PQ is extended to L and RQ is extended to K. Straight line TQS is drawn parallel to RP. *Prove that QS bisects $P\hat{Q}K$.*
Version 2 In ΔPQR, $PQ = QR$. PQ is extended to L and RQ is extended to K. Straight line TQS is drawn parallel to RP. *Prove that TS is an angle bisector of the exterior angles at vertex Q.*
Version 3 Prove that if two equal sides of an isosceles triangle are extended at the common vertex, then the parallel line to the third side, drawn through the common vertex of the two equal sides, bisects the exterior angles formed.

In each case we want learners to make sense of the verbal description and to translate the information into an appropriate diagram. In versions 1 and 2 we name the line segments, angles and triangles. This provides a starting point for learners to draw the triangle, mark the equal sides, etc. We invite the reader to pause and draw a diagram for each version. In doing so, you may realise that learners will likely produce diagrams with different orientations. The orientation will depend on how they position the labels of ΔPQR. This provides a useful point of discussion in class – learners need to check that their "different" diagrams match the verbal description and appreciate that the orientation is irrelevant.

Note also that in version 1 it is not immediately clear where to label S and T on TQS, but their positions do matter. By contrast, in version 2 the positions of S and T do not matter because of the wording of the statement. The difference in the proof instructions is also important. In version 1 the learner needs to consider only $P\widehat{Q}K$. However, in version 2 the learner needs to identify both exterior angles and then to prove that QS bisects both exterior angles ($P\widehat{Q}K$ and $R\widehat{Q}L$).

Version 3 is definitely a more difficult version of Task A because it is stated as a general relationship for the scenario. The wording resembles that of a theorem and it is all contained in a single very complex sentence! The challenge is to break down the sentence into phrases and then to translate each part into a visual representation in a diagram. We recognise that the South African school curriculum does not assess these skills, but they are skills worth developing for those who wish to pursue careers in engineering, architecture, and other design fields, as well as those who will become future mathematics teachers!

Concluding comments

The purpose of this article was to illustrate how the cognitive demand of geometry tasks can be adjusted to bridge the geometry gap from Grades 8 and 9 into Grade 10. In addition to scaffolding, the four strategies of (1) extending calculation tasks to generalisation, (2) manipulating the diagram, (3) scaffolding proof tasks, and (4) connecting visual and verbal information were illustrated to show how tasks can be strategically extended to prepare learners for the demands of FET geometry. We hope teachers will explore these strategies in the classroom as they foster their learners' love for geometry and smooth the transition into the FET phase.

References

Ratnayake, I., Takker, S. & Pournara, C. (2021). Breaking down and building up: Adapting diagrams to develop geometric reasoning. *Learning and Teaching Mathematics, 31*, 10-15.

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