Making the "Structure" Visible: Using Numeric Equations to Introduce Algebraic Equations

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INTRODUCTION

Learner difficulties in algebra have been a concern internationally, particularly since algebra is known to be a "gatekeeper" to learners' achievement in school mathematics. Although algebra is formally introduced in Grade 6, its foundations can be laid much earlier in primary school. By appropriately modifying typical arithmetic tasks we can initiate learners into thinking algebraically by making explicit the connection between arithmetic and algebraic reasoning. The transition from arithmetic to algebra can be supported by changing the focus of arithmetic tasks, for example by moving from calculations to identifying relationships between numbers, by stating properties of numbers in general terms, and by making local generalisations. Through the process of decomposing and recomposing numbers, expressions and equations, learners' attention can be drawn to the structure of mathematical statements.

In this article I discuss examples of tasks which use numeric equations as a context for introducing algebraic thinking. The focus of the suggested tasks is different to how they are typically used in the primary school. Rather, they make use of three big ideas that have the potential to support learners in making the transition from arithmetic to algebra, namely (a) focusing on relationships between numbers in an equation, (b) creating equivalent expressions using decomposition and re-representation, and (c) forming and testing local generalisations.

RELATIONSHIPS BETWEEN NUMBERS

In how many different ways can we make 10? This is an example of a simple arithmetic task typically used in the early stages of mathematical learning. Such tasks have traces of algebraic thinking since learners start engaging with different ways of decomposing 10. At very early stages, learners can recognise the changing relationship between two addends if they are arranged systematically, for example 1 and 9, 2 and 8, 3 and 7, and so on[2]. Learners can be encouraged to articulate the changing relationship between pairs of numbers, for example adding 1 to the first addend and taking away 1 from the second addend. The same task can be used to discuss the commutative property for addition of whole numbers, for example $2 + 8 = 8 + 2$. When learners are challenged to think of different kinds of numbers that can be used to make 10 (see Task 1 in Figure 1), they become aware of the infinite possibilities. These possibilities include making 10 using rational numbers, e.g. $\frac{17}{3} + 4\frac{1}{3}$, decimal representations, e.g. $3{,}75 + 6{,}25$, negative numbers, e.g. $(-2) + 12$, roots and powers, e.g. $3\sqrt{4} + 2^2$, and combinations of these. Broadening the example space for such simple equations by foregrounding learners' knowledge of different kinds of numbers is a basic introduction to thinking algebraically.

[2] Similar patterns can be explored with decomposing 10 using other arithmetic operations on two whole numbers.

Task 1: Make 10 in different ways using the following conditions: 1. by adding two whole numbers 2. by adding two rational numbers, integers, squares, etc. or a combination of these 3. using any number and operation	*Task 2:* What is the same and what is different in these two equations? 1. $\_\_\_ + \_\_\_ = 10$ 2. $\_\_\_ + 3 = 10$

FIGURE 1: Tasks 1 and 2

If we rewrite the same task of making 10 using numbers and symbols it would be $\_\_ + \_\_ = 10$. However, if the task was modified to $\_\_ + 3 = 10$, the unknown can only take on one value. In Task 2, learners are asked to interrogate the two equations, with one and two unknowns respectively, with respect to what remains the same and what has changed. The purpose of the task is to understand that when there is one unknown in an equation, the list of possibilities is constrained.

Task 1 involves decomposing a number in different ways. This prepares learners to see equivalent representations of the same number. This is a key idea in solving equations where every step is an equivalent representation of the previous step. Task 2 draws learners' attention to the conditions under which a variable can take either one or different values. In the equation with two unknowns, the variables can take different values. Learners can be made aware that if we fix the value of one unknown, the other unknown becomes a dependent variable.

FORMING EQUIVALENT EQUATIONS

When solving algebraic equations, we systematically write a series of equations which are equivalent forms of one another. For example, $a + 5 = 7 + b$ can be re-represented[3] as $a - 2 = b$ or in a decompressed form as $a + 5 = 5 + 2 + b$ to make the connection between the expressions on either side of the equal sign more visible. Recognising and forming equivalent equations is an important part of the equation solving process. Task 3 (see Figure 2) draws learners' attention to the relationship between perceptually different but structurally similar numeric equations by foregrounding the idea of balancing.

Consider the given numeric equation $4 + 3 + 6 = \square + 5$ to be the original equation. Here, learners are introduced to three different ways of creating equivalent equations (see Task 3, Q1). These are:

1. operating using the same quantity on both sides of the equal sign, for example adding 1 on both sides or subtracting 1 from both sides of the original equation (parts a and b),
2. operating with the number and its additive inverse on the same side of the equation, for example adding and subtracting 1 on the same side of the equation (part c), and
3. operating with the same quantity on both sides of the equation but changing its position in the expressions on either side, thus helping learners to use the associative property for addition of whole numbers (part d).

[3] In LTM issue 30 we discussed how decompression and re-representation are key ideas in the teaching of exponents (Takker, Masondo & Pournara, 2021). Both of these ideas are useful in equation solving as well.

While solving Task 3, learners can be asked to highlight those parts of the equations which are the same as the original equation. Their attention can then be drawn to the parts which are *not* the same. Distinguishing between parts of the equation which are the same and those which are different will make the structural similarity explicit for learners.

Learners can be asked to predict the value of □ for each of the four equivalent equations and then find the value to check their prediction. The validation of the structural similarity of the four equations comes from the same solution for each equation.

If solved as an arithmetic task, the focus of Task 3 would be on calculating the unknown. But algebraically, the focus is on using the relation between the expressions on either side of the equal sign, and the ideas of equivalence or balancing. While solving the equation helps in validating the relation, shifting learners' attention to generating equivalent expressions or equations is helpful.

Task 3: Given: $4+3+6=\square+5$ 1. Look for what is the same and different in this set of four equations. a) $4+3+6+1=\square+5+1$ b) $4+3+6-1=\square+5-1$ c) $4+3+6+1-1=\square+5$ d) $4+3+7+6=7+\square+5$ 2. Predict whether the value of the □ will be the same or different for each equation. Give a reason for your answer. 3. Solve the four equations. 4. Was your prediction in Q2 correct?	*Task 4:* Given: $10+13+4=11+\square$ 1. Look for what is the same and different in this set of four equations. a) $10+13+4-4=11+\square$ b) $10+13+4-5=11+\square$ c) $10+13+4-7=11+\square$ d) $10+13+4-10=11+\square$ 2. Predict whether the value of the □ will be the same or different for each equation. Give a reason for your answer. 3. Solve the four equations. 4. Was your prediction in Q2 correct?

FIGURE 2: Tasks 3 and 4

In Task 4 (shown in Figure 2), the original equation $10+13+4=11+\square$ has addends on both sides of the equal sign, as in Task 3. In the first question, the original equation is changed by subtracting a number from the left side. Learners are expected to evaluate the impact of the change on the left side of the equation on the value of the unknown on the right side of the equation. Again, highlighting parts of the equations which are the same might help learners to see how changing the left side influences the value of the □ on the other side. Like Task 3, learners are asked to predict whether the solution will be the same for each equation (Q2) and then check their prediction (Q3 and Q4).

In both Tasks 3 and 4, the expressions on either side of the equal sign are manipulated either by the same or a different quantity in relation to the original equation. The key idea is to maintain balance between both sides of the equation. By focusing on what is the same and what is different across equations, learners' attention is drawn to the structure of mathematical statements and its effect on the value of the unknown. Learners could be encouraged to form conjectures about the relation between equivalent (Task 3) and non-equivalent (Task 4) equations and then check their predictions by finding the solution to the equations.

EXAMINING GENERALITY OF MATHEMATICAL STATEMENTS

Teachers often use algebraic notation to express general relations about numbers and their properties. For example, the algebraic identity $a(b-c)=ab-ac$, is typically introduced by asking learners to replace *letters* with numeric values. Such exercises have the potential to obfuscate the distinction between the processes of verification and proof. The process of verification is useful as it helps learners to use specific numbers to evaluate a statement or to use the equivalence between the left and right side of an equation. However, learners need to be challenged by asking them "how do we know that the identity will work for *all* whole numbers or integers or fractions?" The move from using numeric equations (used in Tasks 1 to 4) to purely algebraic generalisations can be supported through generalisations involving both numbers and symbols.

Task 5 (in Figure 3) might seem similar to Task 1, but we are now using different symbols for unknowns. In Task 5, learners are expected to use the given equation, "two numbers add to give 13", to infer other mathematical statements by manipulating the expressions on either or both sides of equal sign. In other words, we are now encouraging learners to use the two key ideas that were discussed earlier – to focus on the relationship between numbers and to manipulate expressions to form equivalent representations.

The focus of this task is not on finding the solution, but rather on using a given relation to examine other relational statements. For example, what happens to the value of the unknown if we subtract 5 from the left side? The complexity is increased by subtracting and adding two different numbers, 20 and 25, in the next question. In this case, learners can be encouraged to discuss the resulting impact on the right side of the equation. Gradually, learners are supported in the generalisation of the relation for any number, n. The focus of the task is on maintaining balance between the left and right side of the equation by manipulating either side.

Task 5: Given that $\square + \Delta = 13$,	*Task 6:* If we know that $7 + \square = 0$, then which of the following statements are true? Give reasons.
a) What is $\square + \Delta + 1 =$ _______?	a) The value of $\square$ is 7.
b) Is the below statement true or false? $\square + \Delta - 5 = 8$	b) $\square$ can be any whole number.
c) Is this statement balanced? $\square + \Delta - 20 + 25 = 13$	c) The value of $\square$ is 0.
d) Is this statement balanced? $\square + \Delta + n = 13 + n$	d) The value of $\square$ is -7.
	e) $\square$ and 7 are additive inverse of each other.
	f) $\square$ is the additive identity for addition.
	g) 7 can be replaced by any whole number.
	h) 7 can be replaced by any rational number.
	i) 7 can be replaced by any integer.

FIGURE 3: Tasks 5 and 6

In Task 6 (shown in Figure 3), learners encounter the property of the additive identity for whole numbers by relating it to the additive inverse, a key idea in solving algebraic equations. Here learners examine the case for 7 using correct and incorrect statements, for example by examining whether the value of the $\square$ is 0. Gradually they are introduced to the general property, using the words "additive inverse" and "identity". Further, learners are encouraged to extend the generalisation for numbers other than 7. For example, if 7 is replaced by any whole number, integer, or rational number, would the identity still hold? Such tasks encourage learners to examine the conditions under which a generalisation holds true and to expand the scope by checking it for other types of numbers.

CONCLUDING COMMENTS

In this article I discussed how arithmetic tasks with their modified focus can be used to introduce algebraic thinking. The three big ideas which connect the structure of numeric equations and algebraic equations are:

a) relationships between expressions on either side of the equal sign,
b) manipulating the expressions to create equivalent forms while maintaining balance between the left and right side of an equation, and
c) formulating and examining generalisations while expressing them using general words and phrases.

The tasks[4] used in this article are designed by selecting an original equation and then modifying some parts of the equation while keeping others as the same. This design feature allows representing an equation in different ways: a decompressed version and a re-representation using different operations, symbols and notation. The use of equivalent and non-equivalent representations also helps with drawing learners' attention to the mathematical structure of the equation. It is hoped that the use of such tasks can help smoothen the transition from arithmetic to algebra by creating a hybrid space where the hidden connections between these topics can be made more explicit.

REFERENCES

Takker, S., Masondo, W., & Pournara, C. (2021). Decompression and re-representation: Key ideas in teaching exponents. *Learning and Teaching Mathematics, 30*, pp. 16-22.

ACKNOWLEDGEMENT

This research and development work was supported financially by the First Rand Foundation, the Department of Science and Innovation and the National Research Foundation. The author thanks Prof. Craig Pournara and Dr Duncan Samson for their suggestions on a draft version of this article.

[4] Several tasks used in this article are taken from worksheets on linear equations which, along with worksheets on other topics, can be freely downloaded from: www.witsmathsconnectsecondary.co.za/resources