An Interesting Geometric Conservation Property – A Multiple Solution Task

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INTRODUCTION

Euclidean geometry is one of the most beautiful branches of mathematics, and a large number of conservation properties can be found in it. By 'conservation properties' we refer to specific geometrical properties of a given scenario that remain unchanged when other changes are effected. By way of example, the area of a triangle remains constant (i.e. is conserved) when one of its vertices is dragged along a straight line parallel to the side formed by the other two vertices. In this article we present and prove an interesting conservation property and explore it using a number of different approaches.

THE TASK

Figure 1 shows trapezium $ABCD$ whose base angles sum to 90° (i.e. $\hat{A} + \hat{D} = 90°$). $AD = m$ and $BC = n$. If M and N are the midpoints of AD and BC respectively, prove that $MN = \frac{m-n}{2}$.

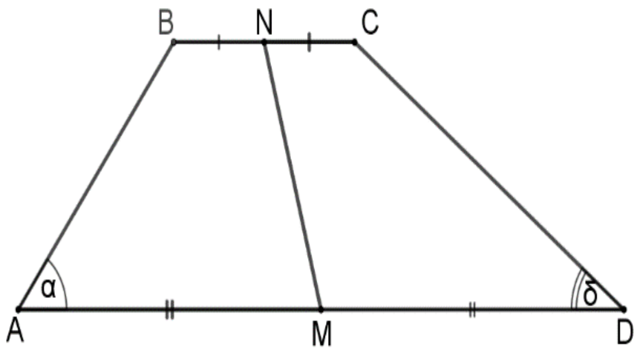

FIGURE 1

To investigate that the property $MN = \frac{m-n}{2}$ is indeed conserved for trapeziums whose base angles sum to 90°, the following applet may be used: https://www.geogebra.org/m/kha8dys4

PROOF 1 – USING ANALYTICAL GEOMETRY

Without any loss of generality, let us place the trapezium in the Cartesian plane with vertex A at the origin. If we set the coordinates of point B as $(b; c)$ then the coordinates of C, D, M and N can all be written in terms of b, c, m and n. We now extend lines AB and DC to their point of intersection, E. Note that since the base angles of the trapezium sum to 90°, it follows that $A\hat{E}D = 90°$, i.e. $AE \perp DE$. All of this is illustrated in Figure 2.

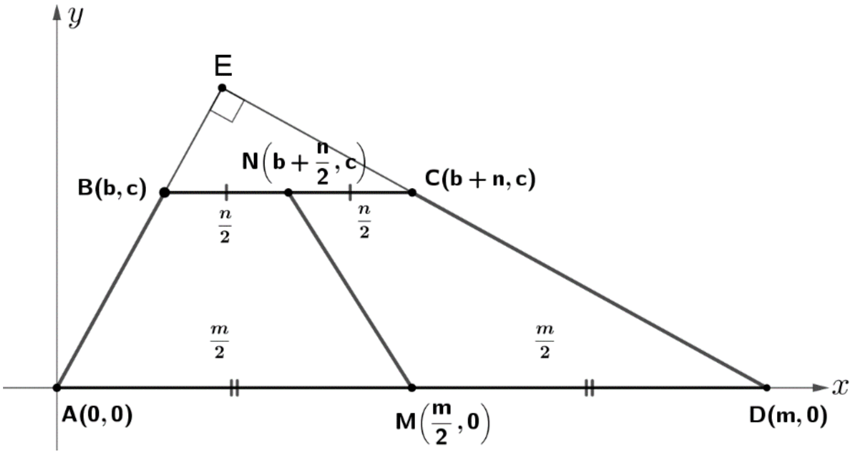

FIGURE 2

Using the distance formula we have:

$$MN=\sqrt{\left(b+\frac{n}{2}-\frac{m}{2}\right)^2+(c-0)^2}=\sqrt{\left(\frac{2b+n-m}{2}\right)^2+c^2}$$

Since $AB\perp CD$ it follows that $m_{AB}\times m_{CD}=-1$, thus:

$$\frac{c}{b}\times\frac{c}{b+n-m}=-1\ \Rightarrow\ c^2=b(m-b-n)$$

We thus have:

$$MN=\sqrt{\left(\frac{2b+n-m}{2}\right)^2+b(m-b-n)}=\sqrt{\frac{m^2-2mn+n^2}{4}}=\sqrt{\frac{(m-n)^2}{4}}=\frac{m-n}{2}$$

PROOF 2 – USING THE MEDIAN OF A RIGHT-ANGLED TRIANGLE

The next proof makes use of the fact that the median of a right-angled triangle is equal to half the length of the hypotenuse. This useful result can be understood as follows (Figure 3). Given right-angled triangle LMN with $\widehat{M}=90°$, draw median MO with O on the hypotenuse LN. From the converse of Thales' theorem it follows that O must be the centre of the circumscribed circle passing through L, M and N. Thus LO, MO and NO are all radii of the circumscribed circle, from which it follows that $MO=\frac{1}{2}LN$.

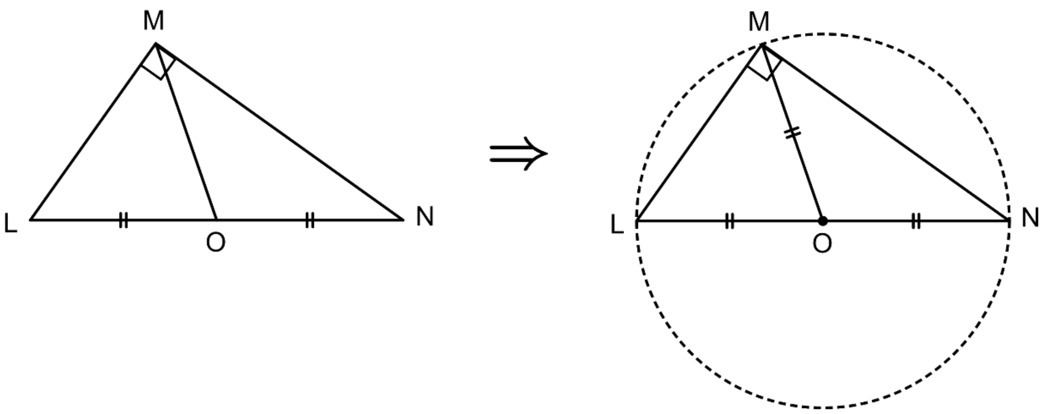

FIGURE 3

We can now use this result as follows. From N, draw NE parallel to BA with E on base AD. Similarly, draw NF parallel to CD with F on base AD. This results in the formation of two parallelograms, $BAEN$ and $CDFN$ (Figure 4).

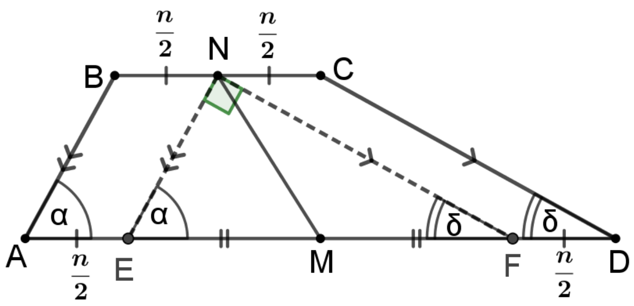

FIGURE 4

Since $\alpha + \delta = 90°$, it follows that $E\hat{N}F = 90°$. And since $EF = AD - AE - FD = m - n$, and the median of a right-angled triangle is equal to half the length of the hypotenuse, we have $MN = \frac{m-n}{2}$ as before.

PROOF 3 – USING PROPORTIONALITY

From B, draw BG parallel to EM with G on base AD to form parallelogram $BGMN$. Again from B, draw BF parallel to ED with F also on base AD to form parallelogram $BFDC$ (Figure 5). We can now make use of proportionality to prove the required property. Since $BF \parallel ED$ we have $\frac{AB}{AE} = \frac{AF}{AD}$. Similarly, since $BG \parallel EM$ we have $\frac{AB}{AE} = \frac{AG}{AM}$. From this it follows that $\frac{AF}{AD} = \frac{AG}{AM}$. But since $AD = 2AM$ it follows that $AG = \frac{1}{2}AF$, i.e. G is the midpoint of AF. Since $A\hat{B}F = 90°$, we can once again using the property that the median of a right-angled triangle is equal to half the length of the hypotenuse, i.e. $BG = \frac{1}{2}AF$. But $AF = m - n$ ($FD = BC = n$), and $BG = MN$ (parallelogram $BGMN$), thus $MN = \frac{m-n}{2}$ as before.

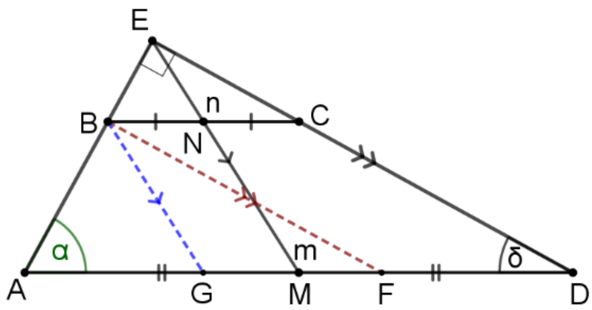

FIGURE 5

PROOF 4 – USING STEINER'S THEOREM

Steiner's theorem for a trapezium, named after Jakob Steiner (1796-1863), can be stated as follows: For every trapezium $ABCD$, the midpoints of the two parallel sides, the point of intersection of the diagonals, and the point of meeting of the continuation of the two non-parallel sides all lie on the same line, i.e. are collinear (Figure 6). For a proof of Steiner's theorem for a trapezium, see for example Stupel and Ben-Chaim (2013). Steiner's theorem can be explored using the following applet: https://www.geogebra.org/m/ypbuf2nh

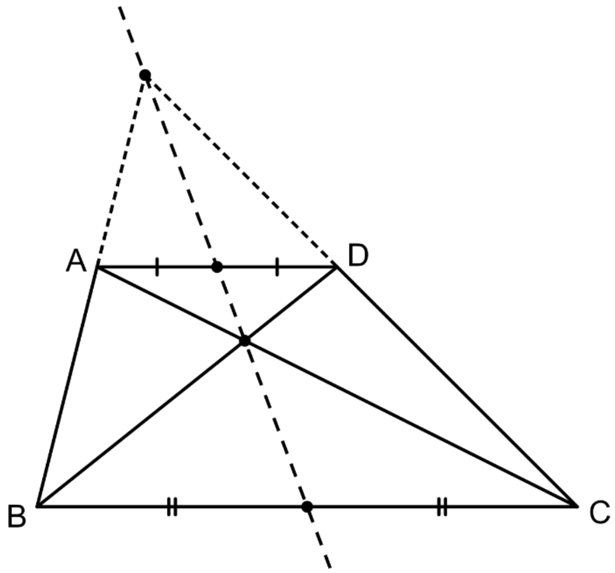

FIGURE 6

We can now use Steiner's theorem for trapeziums to prove that $MN = \frac{m-n}{2}$.

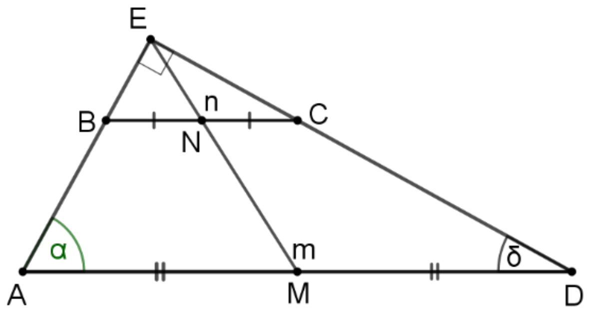

FIGURE 7

With reference to Figure 7, points E, N and M are collinear (Steiner's theorem). Once again using the fact that the median of a right-angled triangle is half the length of the hypotenuse, in ΔBEC we have $EN = \frac{n}{2}$, and in ΔAED we have $EM = \frac{m}{2}$. We thus have $MN = EM - EN = \frac{m-n}{2}$ as before.

CONCLUDING COMMENTS

In this article we have explored an interesting conservation property, proving it using a number of different approaches. There are certainly other ways one could go about establishing this result – for example by using vector algebra. Multiple solution tasks such as this one provide a rich setting for mathematical exploration and engagement. As an alternative approach, students could be presented with the scenario along with a particular proof presented purely in the form of symbolic expressions – i.e. without any text. Pupils would then need to carefully work through the various symbolic statements and try to make sense of the proof by adding in appropriate text. This is a wonderful way to focus attention and to encourage critical engagement.

REFERENCES

Stupel, M., & Ben-Chaim, D (2013). A fascinating application of Steiner's theorem for trapezium: geometric constructions using straightedge alone. *Australian Senior Mathematics Journal*, 27(2), pp. 6-24.