I'm an Asymptote – Cross Me if You Can!

Alan Christison & Duncan Samson[1]

[1]St Andrew's College & The Diocesan School for Girls, Makhanda (Grahamstown)

alanchristison04@gmail.com d.samson@sacschool.com

INTRODUCTION

Students in South African schools generally first encounter the concept of an asymptote when they are introduced to the hyperbola and exponential functions in Grade 10. A typical description of an asymptote in such contexts is generally given as "a line towards which a curve approaches, but never touches". While this is true in the context of such hyperbola and exponential functions encountered at school, it is an inadequate description of an asymptote *in general*, as students who are exposed to the Further Studies curriculum soon discover. A more general description of an asymptote would be something as follows:

> An *asymptote* of a curve is a line such that the distance between the curve and the line approaches zero as one or both of the x or y coordinates approaches infinity.

Given this more general description, asymptotes can be vertical, horizontal or oblique, and a curve can intersect an asymptote any number of times before the region where the distance between the curve and the asymptote approaches zero.

In this article we will begin by considering vertical, horizontal and oblique (or slant) asymptotes of rational functions of the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials that share no common divisors other than 1.

VERTICAL ASYMPTOTES

Vertical asymptotes occur at values for which the denominator of a rational function equals zero. By way of example, the function $f(x) = \frac{x-1}{x+2}$ has a vertical asymptote when $x + 2 = 0$, i.e. at $x = -2$. In order to investigate the behaviour of the graph on either side of the asymptote we can, for example, determine $f(-2{,}1)$ and $f(-1{,}9)$. More formally, we discover that $\lim_{x \to -2^-} f(x) \to +\infty$ and $\lim_{x \to -2^+} f(x) \to -\infty$. The graph of $f(x)$ gets infinitely closer to the line $x = -2$ as $y \to \pm\infty$. Similarly, the graph of $g(x) = \frac{x^2+3}{x^2-5x-6}$ has vertical asymptotes when $x^2 - 5x - 6 = 0$, i.e. at $x = -1$ and $x = 6$. Note that since division by zero is undefined, the function itself is undefined when the denominator is zero. This implies that a graph will *never cross* its vertical asymptote(s).

HORIZONTAL ASYMPTOTES

Horizontal asymptotes occur in rational functions when the degree of the numerator is either the same or less than the degree of the denominator. If the degree of the numerator is less than the degree of the denominator then the horizontal asymptote will be the x-axis, i.e. the line $y = 0$. In the case of the degree of the numerator being the same as the denominator then the equation of the horizontal asymptote is simply the ratio of the leading coefficients of the numerator and denominator. By way of example, $f(x) = \frac{6x-5}{2x+4}$ will have horizontal asymptote $y = \frac{6}{2} = 3$. We can make sense of this by carrying out a process of long division and writing $f(x)$ in the form $f(x) = 3 - \frac{17}{2x+4}$. When $x \to \pm\infty$, $\frac{17}{2x+4} \to 0$, thus $f(x) \to 3$.

While a graph will never cross its vertical asymptote(s), it *is* possible for a graph to cross its horizontal asymptote(s). Let us now investigate under what conditions such crossings may occur. Consider the graph of $f(x) = \frac{2x^3+x-1}{x^3+1}$ as shown in Figure 1.

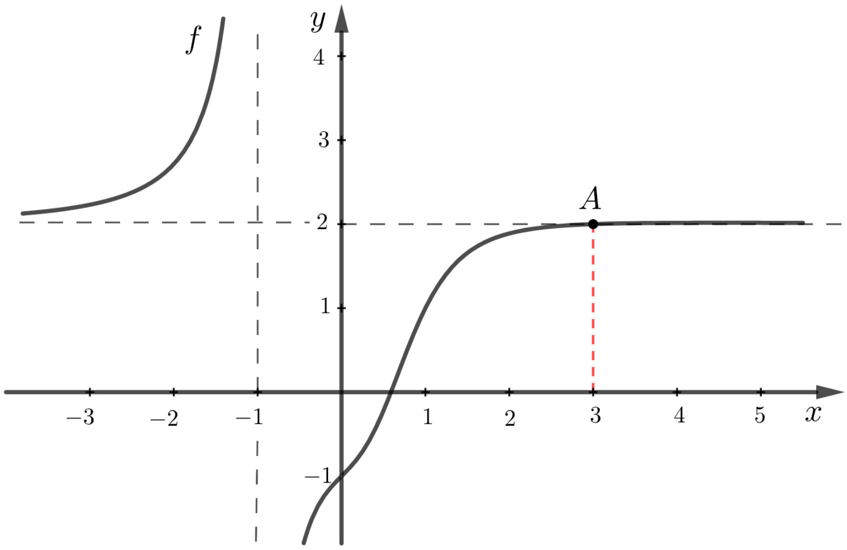

FIGURE 1: The graph of $f(x) = \frac{2x^3+x-1}{x^3+1}$

A *vertical* asymptote occurs when $x^3 + 1 = 0$, i.e. at $x = -1$. Since the degree of the numerator and denominator of the rational function $f(x)$ are the same, a *horizontal* asymptote occurs at $y = 2$, i.e. at the ratio of the leading coefficients of the numerator and denominator. Note that the curve crosses the horizontal asymptote at the point $A(3; 2)$. We can make sense of this by carrying out a process of long division and writing $f(x)$ in the form $f(x) = 2 + \frac{x-3}{x^3+1}$. From this it is clear that as $x \to \pm\infty$, $\frac{x-3}{x^3+1} \to 0$ and $f(x) \to 2$. Thus $y = 2$ is a horizontal asymptote. However, note that $f(x) = 2$ when $\frac{x-3}{x^3+1} = 0$, i.e. at $x = 3$. The graph of $f(x)$ thus intersects (crosses) the asymptote $y = 2$ at the point $(3; 2)$.

Let us now consider the function $f(x) = \frac{5x-7}{x-2}$ which can be written in the form $f(x) = 5 + \frac{3}{x-2}$. We can immediately see that the graph will have a horizontal asymptote at $y = 5$. However, $f(x)$ will only equal 5 when $\frac{3}{x-2}$ equals zero, and since this is not possible we can conclude that the graph will not cross the horizontal asymptote.

By contrast, the function $f(x) = \frac{2x^4+x^2-2}{x^4+1}$, which can be written in the form $f(x) = 2 + \frac{x^2-4}{x^4+1}$, will cross the horizontal asymptote $y = 2$ at two points, $A(-2; 2)$ and $B(2; 2)$, since $\frac{x^2-4}{x^4+1}$ will equal zero at $x = \pm 2$. We also see that since the denominator $x^4 + 1$ cannot equal zero, the graph won't have any vertical asymptotes. The graph of $f(x) = \frac{2x^4+x^2-2}{x^4+1}$ is shown in Figure 2.

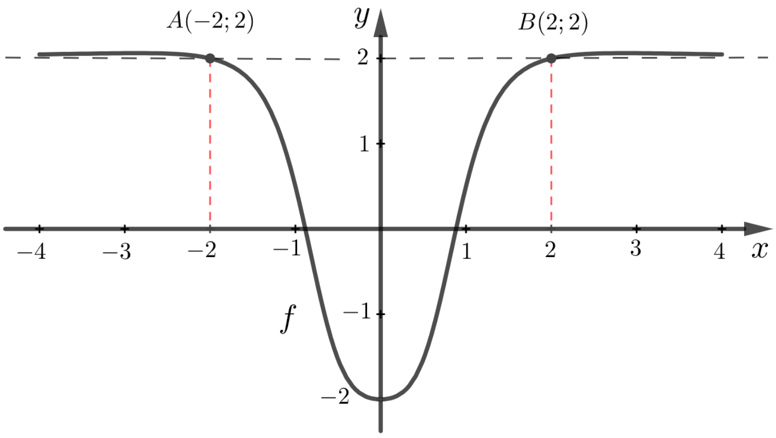

FIGURE 2: The graph of $f(x) = \frac{2x^4+x^2-2}{x^4+1} = 2 + \frac{x^2-4}{x^4+1}$

As previously noted, when the degree of the numerator is less than the degree of the denominator then the horizontal asymptote will be the x-axis, i.e. the line $y = 0$. By way of example, consider the graph of the function $f(x) = \frac{x+1}{x^2+2}$. Let us explore what happens as x tends to infinity:

$$\lim_{x\to\pm\infty} \frac{x+1}{x^2+2} = \lim_{x\to\pm\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{2}{x^2}} = \frac{0+0}{1+0} = 0$$

The graph will thus have a horizontal asymptote at $y = 0$, i.e. the x-axis. Note that $f(x) = \frac{x+1}{x^2+2} = 0$ when $x = -1$. The graph thus crosses the horizontal asymptote at the point $(-1;0)$.

OBLIQUE (SLANT) ASYMPTOTES

An oblique asymptote occurs in a rational function when the degree of the numerator is exactly 1 greater than the degree of the denominator. As with the horizontal asymptote, it is possible for the curve of a function to cross an oblique asymptote. Consider the function $f(x) = \frac{2x^3-3x^2-x}{3x^2-3}$ which is shown in Figure 3 below.

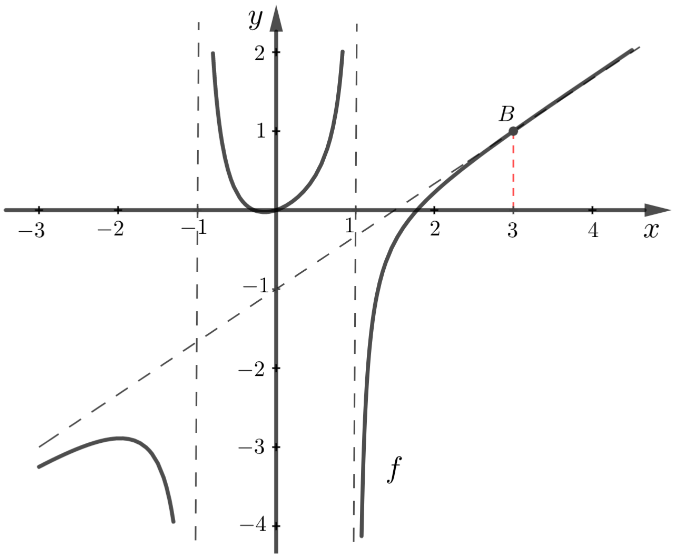

FIGURE 3: The graph of $f(x) = \frac{2x^3-3x^2-x}{3x^2-3} = \frac{2}{3}x - 1 + \frac{x-3}{3x^2-3}$

Using long division we can write $f(x)$ in the form $f(x) = \frac{2}{3}x - 1 + \frac{x-3}{3x^2-3}$. Since the denominator is zero when $x = \pm 1$, we have two vertical asymptotes, $x = 1$ and $x = -1$. The oblique asymptote has equation $y = \frac{2}{3}x - 1$. This can be understood as follows:

$$\lim_{x\to\pm\infty}\left[f(x) - \left(\frac{2}{3}x - 1\right)\right] = \lim_{x\to\pm\infty}\left[\frac{x-3}{3x^2-3}\right] = 0$$

In other words, as x tends to infinity, the vertical distance between $f(x)$ and the line $y = \frac{2}{3}x - 1$ tends to zero, from which we can conclude that $y = \frac{2}{3}x - 1$ is an oblique asymptote. Carrying out long division and writing the equation of $f(x)$ in the form $f(x) = \frac{2}{3}x - 1 + \frac{x-3}{3x^2-3}$ also exposes the fact that the graph of $f(x)$ will cross the oblique asymptote. This will occur at the point where $x = 3$ since the fraction will collapse to zero and we are left with $f(x) = \frac{2}{3}x - 1$. The graph of $f(x)$ thus crosses the oblique asymptote at the point $B(3; 1)$. Note that in general for the case $f(x) = ax + b + \frac{k}{Q(x)}$ where $a \neq 0$ and k is a non-zero constant, the curve of f will *not* cross the oblique asymptote $y = ax + b$. In the case where the numerator of the fractional part (i.e. the remainder after long division) leads to additional zeros, the curve of the function will cross the oblique asymptote more than once.

RATIONAL FUNCTIONS WITH COMMON FACTORS

Care needs to be taken before assuming that the number of zeros in the numerator of the remainder will always lead to points of intersection of the curve and the horizontal/oblique asymptote. By way of example, consider the function $f(x) = \frac{2x^2+3x-27}{x^2+x-12} = 2 + \frac{x-3}{x^2+x-12}$. In this particular case the graph of f and the horizontal asymptote $y = 2$ do *not* intersect at $x = 3$. We can make sense of this by factorising and simplifying the original function:

$$f(x) = \frac{(2x+9)(x-3)}{(x+4)(x-3)} = \frac{2x+9}{x+4} = 2 + \frac{1}{x+4}$$

We can now see that the graph of $f(x)$ has a horizontal asymptote at $y = 2$, a vertical asymptote at $x = -4$, and a removable discontinuity at $x = 3$. Importantly, there is no vertical asymptote at $x = 3$, and since $\frac{1}{x+4}$ can never equal zero, the graph of $f(x)$ will *not* cross the horizontal asymptote.

CURVILINEAR ASYMPTOTES

Although curves may also be asymptotic to another curve, this is not a common inclusion within the definition of an asymptote. However, since it is a natural extension of what we have discussed so far in relation to rational functions, it is worth exploring such a scenario. Consider, for example, the following function:

$$f(x) = \frac{x^3 - 3x^2 + 2x - 5}{x} = x^2 - 3x + 2 - \frac{5}{x}$$

Since $\lim_{x\to\pm\infty}[f(x) - (x^2 - 3x + 2)] = \lim_{x\to\pm\infty}\left[\frac{5}{x}\right] = 0$, it follows that the vertical distance between $f(x)$ and the parabola $y = x^2 - 3x + 2$ tends to zero as x tends to infinity, and hence that the parabola is a curvilinear asymptote. This is illustrated in Figure 4.

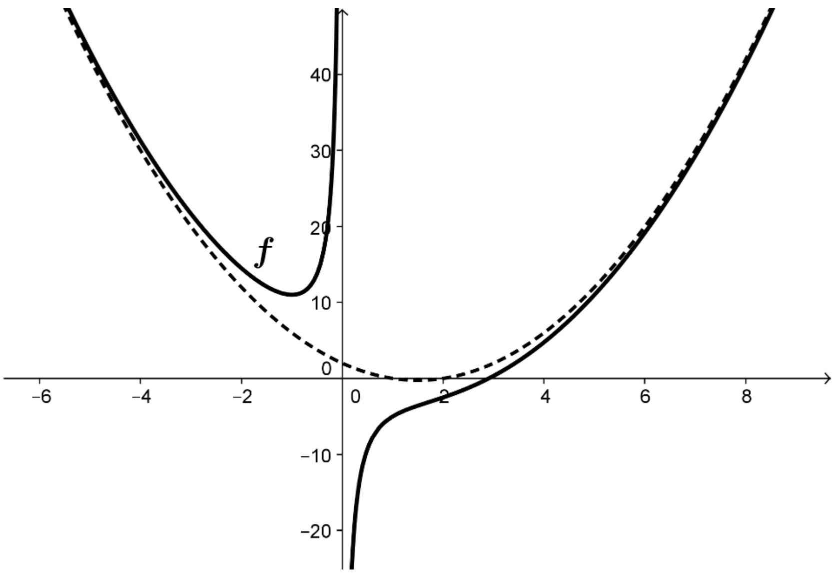

FIGURE 4: The graph of $f(x) = \frac{x^3-3x^2+2x-5}{x} = x^2 - 3x + 2 - \frac{5}{x}$

RATIONAL FUNCTIONS

We can now summarise things as follows. For rational functions f of the form

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$$

where $P(x)$ and $Q(x)$ are polynomials that share no common factors other than 1, then:

- Vertical asymptotes occur at $x = k_i$ for all constants k_i such that $Q(k_i) = 0$
- If $n > m$ then there are no horizontal asymptotes
- If $n = m$ then there is a horizontal asymptote at $y = \frac{a_n}{b_m}$
- If $n < m$ then there is a horizontal asymptote at $y = 0$, i.e. the x-axis
- If $n = m + 1$ then there is an oblique (slant) asymptote $y = \frac{a_n}{b_m}x + c$
- The graph of f will never cross its vertical asymptotes
- The graph of f may or may not cross its horizontal/oblique asymptotes

NON-RATIONAL FUNCTIONS

As a final extension, let us consider an example of a non-rational function. Since limits are an important aspect of asymptotes, non-rational functions need to be carefully examined. Consider the following function:

$$g(x) = \frac{x.e^x}{e^x + 1} + 1 = x + 1 - \frac{x}{e^x + 1}$$

The graph of $g(x)$ is shown in Figure 5. The function has a horizontal asymptote at $y = 1$, as well as an oblique asymptote $y = x + 1$. The curve of $g(x)$ intersects *both* asymptotes at $x = 0$.

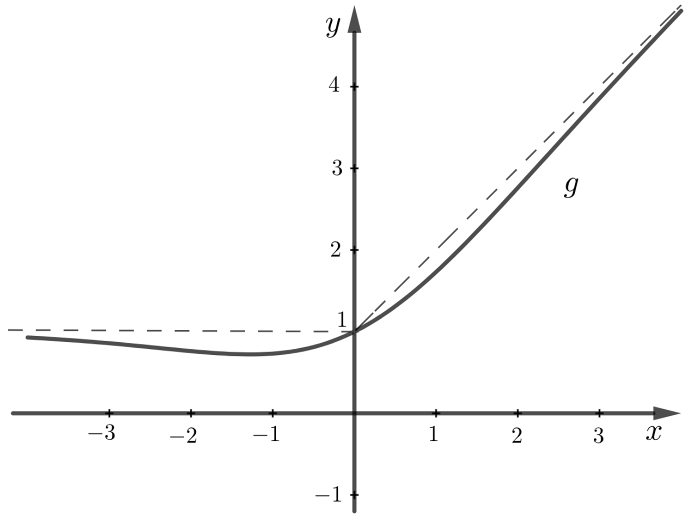

FIGURE 5: The graph of $g(x) = \frac{x.e^x}{e^x+1} + 1 = x + 1 - \frac{x}{e^x+1}$

To understand these asymptotes we need to consider the limits of $g(x)$ as $x \to \infty$ and as $x \to -\infty$. First we write $g(x)$ in the indeterminate form $g(x) = \frac{x.e^x+e^x+1}{e^x+1}$ and then make use of L'Hospital's rule.

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} \frac{x.e^x + 2.e^x}{e^x} = \lim_{x\to+\infty} x + 2 \to \infty$$

$$\lim_{x\to-\infty} g(x) = \lim_{x\to+\infty} g(-x) = \lim_{x\to+\infty} \frac{-x + 1 + e^x}{1 + e^x} = \lim_{x\to+\infty} \frac{-1 + e^x}{e^x} = \lim_{x\to+\infty} \frac{e^x}{e^x} = 1$$

We thus see that the graph of f will approach the oblique asymptote $y = x + 1$ as x approaches positive infinity, and will approach the horizontal asymptote $y = 1$ as x approaches negative infinity. The domain of the oblique asymptote is $[0; \infty)$ while the domain of the horizontal asymptote is $(-\infty; 0]$. These are examples of one-sided asymptotes.

CONCLUDING COMMENTS

This article illustrates that *only under certain circumstances* is an asymptote a line that the graph of a function "approaches but never crosses/touches". The graphs encountered in the Core Mathematics curriculum in South Africa, i.e. the hyperbola, exponential and logarithmic functions, match these circumstances, and for this reason there is often a misconception that the graph of a function may *never* cross an asymptote. Students taking Further Studies Mathematics at school, or students encountering higher mathematics at tertiary level, are often somewhat surprised to see that asymptotes may indeed be crossed, and need to reconceptualise their understanding of what an asymptote is. It is perhaps prudent that students be informed at the outset of a more complete definition of an asymptote, and then of a "subset definition" as will apply to all applicable functions being considered at the time. It is hoped that this article may assist in understanding under which circumstances such a "subset definition" will occur.

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