Constructive evaluation of definitions in a Sketchpad context

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The ICT revolution has had a significant impact on curriculum development and delivery at both school and tertiary level in South Africa. This is particularly evident in the incorporation of Information and Communication Technologies (ICT) into the teaching and learning of mathematics. For example, as part of LO3 in the South African National Curriculum Statement – Mathematics (Grades 10-12), learners will “use construction and measurement or dynamic geometry software (like Geometer’s Sketchpad), for exploration and conjecture” and “…engage with new tools in a range of applications, and … become more proficient in processes leading to proof” (DoE, 2003).

Dynamic geometry software (DGS), like Geometer’s Sketchpad, can serve as an interactive context for making generalizations about geometric objects and thus lead to proof-generating situations, wherein justification and reasoning is encouraged. Moreover, it can play the role of mediator in the transition between argumentation and proof through its “dragging function” (de Villiers & Hanna, 2008). When elements of a drawing are moved, this feature allows the construction to respond dynamically to the altered conditions (Goldenberg & Cuoco, 1998) by maintaining the invariant. Moreover, while dragging, students often switch back and forth from figures to concepts and from abductive to deductive modalities, which helps them progress from the empirical to the theoretical level (de Villiers & Hanna, 2008).

Fortunately, the National Curriculum Statement Grades 10-12 (Schools) lays a foundation for the use of technology such as dynamic geometry software for exploration and investigation. In particular, Grade 10 learners ought to explore necessary and sufficient conditions for the various quadrilaterals and investigate ways of defining various polygons. Furthermore, according to the new FET curriculum, Grade 10 learners ought to realise that definitions are not absolute but fixed on the basis of principles that will result in conciseness and efficiency. Definitions, axioms and preceding theorems thus serve as starting points for deductive arguments in the expansion of the axiomatic system. For the quadrilaterals, decisions as to which system of definitions to use can depend on either partition or inclusiveness approach, although the latter is favoured in the interests of efficiency. Learners need to be aware of these factors in determining definitions.

However, research by Linchevsky, Vinner and Karsenty (1992), among others, on definitions in mathematics have indicated that many pre-service teachers do not understand that definitions in geometry have to be economical (contain no superfluous information) and that they are arbitrary (in the sense that several alternative definitions may exist). It is plausible to conjecture that this is probably due to their past school experiences where definitions were largely supplied to them directly. It would appear essential that in order to increase future teachers’ understanding of geometric definitions, and of the concepts to which they relate, it is necessary to engage them at some stage in the process of defining geometric concepts.

Furthermore, pre-service teachers’ own experiences in learning mathematics in an ICT environment greatly influence the way they will teach mathematics to their future students.

“The process of teaching mathematics and learning mathematics is iterative: the way pre-service mathematics teachers are taught influences their understanding of and beliefs about mathematics; their understanding of and beliefs about mathematics influence the way they teach; and the way they teach influences their students’ understanding and beliefs about mathematics.” (Billstein, 1993).

This means that mathematics courses for prospective mathematics teachers should be designed to help them learn how to learn mathematics in an ICT environment of good teaching practices, which can in turn help them to learn how to teach mathematics in an ICT environment. Invariably, this means that pre-
service mathematics teachers should be engaged in worthwhile mathematical tasks within an ICT environment that supports exploration, problem solving, reasoning and communication. Mathematics teacher educators who teach pre-service mathematics teachers can have a significant impact on mathematics classroom practice through the instructional strategies they model.

Hence, the research reported here concentrated mainly on student teachers’ (i.e. prospective mathematics teachers’) understanding of the nature of definitions and the development of their ability to evaluate and formulate definitions in a Sketchpad context (see Govender, 2002; Govender & de Villiers, 2002). Use of Sketchpad was made to expose the students to a process of defining, as a creative activity in which students can be fully involved, and not view definitions as an imposed body of knowledge immune to any change or development. Specifically, the following research questions were addressed:

- What prior understanding of the nature of definitions do student teachers have prior to being engaged in a process of formulating definitions for themselves?
- To what extent do the student teachers’ understandings of the nature of definitions change while involved in a process of evaluating definitions by means of construction, measurement and dragging within a Sketchpad context?
- How competent are the student teachers in evaluating other definitions after being engaged in the preceding process?

Several useful implications have evolved from this work and may be able to influence both the teaching and learning of geometry in school. Initially, student teachers were given the following tasks in order to determine their prior understanding of the nature of definitions:

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<th>Task 1</th>
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<td>How would you describe what a rhombus is, over the phone, to someone who is not yet acquainted with it?</td>
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<th>Task 2</th>
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<td>Which of the following descriptions do you think you would be able to use? Circle these descriptions.</td>
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<tr>
<td>a. A rhombus is any quadrilateral with opposite sides parallel.</td>
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<tr>
<td>b. A rhombus is any quadrilateral with perpendicular diagonals.</td>
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<tr>
<td>c. A rhombus is any quadrilateral with two perpendicular axes of symmetry (each through a pair of opposite angles).</td>
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<tr>
<td>d. A rhombus is any quadrilateral with perpendicular, bisecting diagonals.</td>
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<tr>
<td>e. A rhombus is any quadrilateral with two pairs of adjacent sides equal.</td>
</tr>
<tr>
<td>f. A rhombus is any quadrilateral with all sides equal.</td>
</tr>
<tr>
<td>g. A rhombus is any quadrilateral with one pair of adjacent sides equal, and opposite sides parallel.</td>
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The majority of the student teachers appeared to have an intuitive understanding of the arbitrary nature of definitions; several student teachers understood a definition (description) of a given figure to be a list of properties of that given figure, which in fact is an uneconomical way to describe a figure (thus suggesting Van Hiele level 2 understanding) and some student teachers incompletely defined (described) figures by viewing necessary conditions as sufficient conditions.

Secondly, student teachers were engaged in constructively evaluating different definitions for a rhombus through the following tasks (Task 3 & Task 4):
Although initially the student teachers required some guidance in constructing the required givens (prerequisite points) for each script, they quickly became independent as they progressed through the 7 scripts. Working through each script provided good learning opportunities for the student teachers to check whether the conditions for each script were sufficient to produce a rhombus. Due to space limitations, we shall here only discuss the script Rhombus 7. An example of an on-screen sketch produced by this script is shown in Figure 1 (though appearance may vary depending on the relative positions of the pre-requisite points A and B).

When the construction was finished, the researcher firstly questioned the student teachers about the displayed lengths and gradients on the screen. The student teachers showed understanding of the displayed measurements. Upon asking the students whether the script constructed a correct rhombus, all responded that it was correct, apparently judging purely from a visual perspective. Only upon encouragement did the students check out their claim by measuring the sides to see if they were really all equal.

Furthermore, all 18 student teachers matched the script Rhombus 7 correctly to description g. The student teachers were then requested to drag the figure on Sketchpad and observe whether it always remained a rhombus. After the student teachers had dragged the figure around numerous times they were confident that this figure always remained a rhombus. The student teachers also indicated that the given information was sufficient as well as necessary.

17 out of 18 students listed all four (i.e. c, d, f and g) correct descriptions as the ones that best describe the rhombus in Task 2. Only one student did not choose all four, but managed to identify three out of the four correct descriptions, namely c, d and f.

The student teachers’ responses to Task 4, in comparison to their earlier responses to Task 2, clearly suggest the following as a result of being involved in the process:

- They appear to have developed a deeper understanding of the arbitrary nature of definitions.
- They showed improved ability to select correct alternative definitions for a rhombus.

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• The scripts and the use of dynamic dragging provided the student teachers with the opportunity to check whether the conditions in the given definitions were sufficient for the accurate construction of a rhombus.
• As a consequence, the student teachers exhibited a better understanding of when conditions are: necessary and sufficient; necessary but not sufficient; sufficient but not necessary (and therefore also some ability to distinguish between economical and uneconomical definitions).

Concluding Remarks
The use of construction and measurement to evaluate the correctness of geometric statements (conjectures) before proofs are carried out is of course common practice among mathematical researchers. As a teaching approach it is also not new. For example, a similar approach was used effectively in the USEME teaching experiment during 1977/78 (see Human & Nel, 1989). Similarly, Smith (1940) reported marked improvement in pupils' understanding of "if-then" statements by letting them first make constructions to evaluate geometric statements. In his research he found that it enabled pupils to learn to clearly distinguish between the "given condition(s)" and the "conclusion(s)", and laid the conceptual groundwork for an improved understanding of the eventual deductive proof.

However, this study is markedly different in that it took place within the context of dynamic geometry, where a geometric configuration can be continuously dragged into different shapes to check for invariance. Ideally, students should test geometric statements by making their own constructions within Sketchpad. However, since this requires a rather high level of technical knowledge of the software, it was decided to provide them with ready-made scripts that they could play through step by step and observe as the figure was gradually constructed. As the scripts are dependent on the arbitrary construction and positioning of the "given points", they sometimes produce crossed quadrilaterals, which was a little confusing to some students. Accordingly, in the revised version of this activity (see De Villiers, 2003) use will instead be made of the "Hide/Show" button facility of Sketchpad to produce figures step by step, ensuring that they all initially appear to be a rhombus. Only upon further dragging would students then be able to ascertain whether it always remains a rhombus, and therefore whether the conditions are really sufficient.

It should also be noted that since the dynamic geometry software provided conviction to all the student teachers, the role of the eventual deductive proofs (i.e. to prove the sufficiency of the definitions) was conceptualized as that of systematization rather than that of verification.

Although it was not a main focus of this study, the issue of hierarchical vs. partition definitions for a rhombus arose quite a few times while interviewing (or in discussion with) the student teachers. However, the dynamic nature of the rhombi constructed in Sketchpad seemed to make the acceptance of the hierarchical classification of a square as a special rhombus far easier than in a traditional non-dynamic environment, as the student teachers could easily drag the constructed rhombus until it became a square. This is, however, a matter for further research.

References:
Bilstein, R. (1993). Improving K-S Pre-service Mathematics Education in Department of Mathematics. Proceedings of the National Science Foundation Workshop on the Role of Faculty from the Scientific Disciplines in the Undergraduate Education of Future Science and Mathematics Teachers, 146-149.

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