FACTORISING 3RD DEGREE POLYNOMIALS: AN ALTERNATIVE APPROACH TO FINDING THE QUADRATIC QUOTIENT

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INTRODUCTION

Cubic functions, which are polynomials of degree three of the form $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d are real constants with $a \neq 0$, form an integral part of the South African School Mathematics curriculum in terms of curve sketching, graph interpretation and contextualized maxima and minima problems. The ability to solve cubic equations efficiently is therefore an essential skill in many applications in school mathematics. There are a number of different methods which can be used to factorize cubic expressions and/or solve cubic equations. The most commonly used method found in contemporary textbooks and classrooms is the use of *factor theorem* to find the *linear factor* and the *division algorithm* and/or *synthetic division* to find the *quadratic quotient.*

The use of division algorithm and/or synthetic division methods to find the quadratic quotient given the linear factor of the cubic expression often confuse a lot of students, as illustrated in the example below.

FINDING THE QUADRATIC QUOTIENT

Example 1: Given that $f(x) = 12x^3 + 16x^2 - 5x - 3$ and that $2x - 1$ is a factor of $f(x)$, factorize $f(x)$ completely.

Solution:

Method 1: The division algorithm

$$
\begin{array}{r|l}
 & \mathbf{6x^2 + 11x + 3} \\
\hline
2x - 1 & 12x^3 + 16x^2 - 5x - 3 \\
(-) & \underline{12x^3 - 6x^2} \\
 & \quad 22x^2 - 5x - 3 \\
(-) & \quad \underline{22x^2 - 11x} \\
 & \qquad 6x - 3 \\
(-) & \qquad \underline{6x - 3} \\
 & \qquad\quad 0
\end{array}
$$

The *division algorithm* provides a lot of detail that appear cumbersome to handle. So a short cut to the *division algorithm* is the *synthetic division* which is used to find the coefficients of the terms and constant of the quadratic quotient *when the divisor is a*

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linear polynomial with a leading coefficient of one. With this method of synthetic division, students normally present difficulties when the leading coefficient (of the linear factor of the cubic expression) is different from one. Like in Example 1, the linear factor $2x-1$ has 2 as the leading coefficient. Therefore, we write $2x-1=2\left(x-\frac{1}{2}\right)$. So now we can carry out the process of *synthetic division*.

Method 2: Synthetic division

$$\begin{array}{r|rrrr} & \mathbf{12} & \mathbf{22} & \mathbf{6} & \mathbf{0} \\ \hline \frac{1}{2} & 12 & 16 & -5 & -3 \end{array}$$

Process of synthetic division in this case is as follows:

$$\boxed{\mathbf{12}} \div 2 = 6 \rightarrow \boxed{6+16=\mathbf{22}} \rightarrow \boxed{22 \div 2 = 11} \rightarrow \boxed{11-5=\mathbf{6}} \rightarrow \boxed{6 \div 2 - 3 = \mathbf{0}}$$

The coefficient of the quadratic term of the quotient coincides with that of the cubic term of the cubic expression.

We note that the coefficients of the terms and constant of the quadratic quotient in the result are double those found using the division algorithm, which students usually fail to notice. Hence students would factorise the resultant quadratic quotient in the synthetic division and <u>wrongly</u> conclude that

$12x^3+16x^2-5x-3=(\mathbf{2x-1})(\mathbf{4x+6})(\mathbf{3x+1})$, where as

$(\mathbf{2x-1})(\mathbf{4x+6})(\mathbf{3x+1})=24x^3+32x^2-10x-6$…….. (3)

The correct factorization of $f(x)$ <u>based on the result of the synthetic division</u> should be:

$$\begin{aligned}\left(x-\tfrac{1}{2}\right)(\mathbf{12}x^2+\mathbf{22}x+\mathbf{6}) &= 2\left(x-\tfrac{1}{2}\right)(6x^2+11x+3) \\ &= (2x-1)(\mathbf{6x^2}+\mathbf{11x}+\mathbf{3}) \\ &= (2x-1)(2x+3)(3x+1) \\ &= 12x^3+16x^2-5x-3 \text{……..(4)}\end{aligned}$$

We also note that the coefficients of the cubic expression (3) are double those of cubic expression (4), although if the cubic expressions (3) and (4) are converted into cubic equations, the solution set would be the same. Therefore, the purpose of this article is to present an alternative approach to finding the quadratic quotient of the cubic polynomial with a known linear factor regardless of whether the leading coefficient of the linear factor is one or different from one.

FORMULA FOR FINDING QUADRATIC QUOTIENT OF A 3RD DEGREE POLYNOMIAL

For any cubic expression of the form $\boldsymbol{ax^3 + bx^2 + cx + d}$, with a linear polynomial factor $\boldsymbol{mx + n}$, where a, b, c, d, m and n are real constants <u>and</u> a, m and n are different from zero, the quadratic quotient can be expressed in the form:

$$\frac{a}{m}x^2 + \left(\frac{cn-dm}{n^2}\right)x + \frac{d}{n} \quad \underline{\text{or}} \quad \frac{a}{m}x^2 + \left(\frac{bm-an}{m^2}\right)x + \frac{d}{n}$$

PROOF OF THE FORMULA FOR FINDING QUADRATIC QUOTIENT

Suppose that we have the cubic expression of the form: $ax^3 + bx^2 + cx + d$ ... (1)

where a, b, c and d are real constants with $a \neq 0$, that we wish to factorise into the form $(mx + n)(px^2 + qx + r)$ .... (2)

where p, q and r are real constants to be determined and $mx + n$ is a factor of (1) with m and n different from zero. Multiplying out the factorised form (2) gives:

$pmx^3 + (mq + np)x^2 + (mr + nq)x + nr$, which, on comparison with term by term of (1), gives:

$a = \boldsymbol{p}m,\ b = m\boldsymbol{q} + n\boldsymbol{p},\ c = m\boldsymbol{r} + n\boldsymbol{q}$, and $d = n\boldsymbol{r}$.

From these relationships we solve for $\boldsymbol{p}, \boldsymbol{q}$, and $\boldsymbol{r}$:

$$\boldsymbol{p} = \frac{a}{m};\ \boldsymbol{r} = \frac{d}{n};\ \boldsymbol{q} = \frac{bm-an}{m^2} = \frac{cn-dm}{n^2}$$

From these results we substitute $\boldsymbol{p}, \boldsymbol{q}$, and $\boldsymbol{r}$ into (2) to give:

$$ax^3 + bx^2 + cx + d = (\boldsymbol{mx + n})\left[\frac{a}{m}\boldsymbol{x}^2 + \left(\frac{bm-an}{m^2}\right)\boldsymbol{x} + \frac{d}{n}\right] = (\boldsymbol{mx + n})\left[\frac{a}{m}\boldsymbol{x}^2 + \left(\frac{cn-dm}{n^2}\right)\boldsymbol{x} + \frac{d}{n}\right]$$

Now, in **example 1** above, we find the quadratic quotient of $f(x)$ using the formula:

Step 1: Identify the coefficients and constants of the cubic polynomial and linear factor: $a = 12,\ c = -5,\ d = -3,\ m = 2, n = -1$

Step 2: Substitute the values into $\frac{a}{m}x^2 + \left(\frac{cn-dm}{n^2}\right)x + \frac{d}{n}$ and simplify:

$$\frac{12}{2}x^2 + \frac{(-5)(-1)-(-3)(2)}{(-1)^2}x + \frac{(-3)}{-1} = \boldsymbol{6x^2 + 11x + 3.}$$

CONCLUDING REMARKS

The formula for finding quadratic quotient of a cubic expression can be used provided the linear polynomial factor of the cubic expression exists. I have used this formula in question with the students and they appreciated it. Students said working with a formula in mathematics is simple because one needs to identify the values of the variables in the formula, substitute and simplify, citing the use of quadratic formula to solve any type of quadratic equation as an example. I believe that the use of the formula provides an interesting alternative to the existing methods of finding quadratic quotient when the linear polynomial factor is found through the use of the *factor theorem*.

REFERENCES

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