GROUP THEORY – A RELEVANT TOPIC FOR PROSPECTIVE TEACHERS?

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Group theory may not be considered to be a core topic in a teacher education programme, in particular not in a programme for prospective primary school teachers. The reason for such a view could be that group theory is an abstract mathematical topic and considered to be far away from topics that these teachers will actually teach. In this paper I will present an example from a course in a master programme for prospective teachers where group theory forms an important part. I will briefly describe the programme and how group theory is integrated into the programme. Using examples and referring to theory I will present arguments why I think group theory is a relevant topic here. The relevance is seen both in the value of the topic itself and how it connects to other central mathematical topics. Also important is the value I see that group theory has in giving a perspective on the epistemological nature of mathematics as a science.

INTRODUCTION: THE GROUP CONCEPT

The concept of a *group* is normally introduced in university programmes in mathematics in the second year of study, after the students have been through basic courses in calculus and linear algebra, and perhaps some number theory and geometry. The concept is considered to be rather abstract, and typically, courses and textbooks presenting group theory often contain the words *abstract algebra* (e.g. Fraleigh 1976). The group concept developed in the 19th century. Victor J. Katz, in his book on the history of mathematics, writes that "[a]lgebra in 1800 meant the solving of equations. By 1900, the term encompassed the study of various mathematical structures, that is, the sets of elements with well-defined operations, satisfying certain specified axioms" (Katz 1998: 650). This development includes the development of the group concept, as well as other algebraic concepts such as *ring* and *field.* In the early decades of the 19th century problems connected to solving algebraic equations by radicals were very much in focus, and in particular the work of Abel and Galois on the impossibility of solving the quintic equation represents milestones in the history of mathematics. Both Abel and Galois worked with permutations of the roots of the equations and Galois even used the notion "group of permutations" (Katz 1998: 667). Galois died in 1832 and in 1846 his manuscripts were published in *Journal des mathématiques.* Following this publication several French mathematicians lectured on Galois' results and worked further on them. In 1870 Camille Jordan (1838-1922) published his work *Traité des substitutions et des equations algébriques* in which he defined "a *group* to be a system of permutations of a finite set with the condition that the product (composition) of any two such

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permutations belongs to the system" (Katz 1998: 669). Other central persons in the development of the group concept were Arthur Cayley (1821-1895) in England and Leopold Kronecker in Germany. Cayley used the word *group* and he also gave credit to Galois for his idea of a group of permutations. Kronecker, on the other hand, did not give a name to the system he defined, and he also did not interpret it in terms of permutation groups arising from Galois theory (Katz 1998: 672).

In modern terminology a *group* is defined as a set G together with a binary operation (multiplication) which is closed in G (i.e. for x and y in G, the composition xy is also in G) and satisfying the following properties:

1. The multiplication is associative, that is to say $(xy)z = x(yz)$ for any three (not necessarily distinct) elements from G.
2. There is an element e in G, called an identity element, such that $xe = ex = x$ for every x in G.
3. Each element x of G has a (so-called) inverse x^{-1} which belongs to the set G and satisfies $x^{-1} x = x x^{-1} = e$.
   (Armstrong 1988: 6).

The properties above can be taken as axioms based on which one can derive properties for sets and binary operations satisfying the axioms, without considering more specifically what the elements in the set or the operations actually are. There are numerous examples of familiar sets and binary operations satisfying the properties listed above, such as the integers with the operation addition and the rational numbers, with zero removed, and the operation multiplication.

The axioms above are formulated with as few requirements as possible. Note for example that in all the examples we can think of there is only one identity element. However, this is not a requirement in the axioms but it follows as a consequence. Assume that in a group there are two identity elements e and e'. From 2 it follows, taking $x = e'$, that $e' e = e e' = e'$ since e is an identity element. But also $e e' = e' e = e$ since e' is an identity element. Hence, $e = e'$. Similarly it can be shown that for each x the inverse element x^{-1} is unique, although this is also not required in the axioms.

A master programme in mathematics education

For a number of years I have been teaching on a two-year master programme in mathematics education with the aim of educating teachers for students age 6-16. The students have completed at least three years of a teacher education programme before entering the master programme. Some has a complete teacher education beforehand and takes the master on top of that. The first course that the students encounter is called *Learning and Teaching of Mathematics.* This course fills the whole first semester of the programme and comprises a number of topics. The intention is that the course (as well as the whole master programme) shall contribute to developing

the students' competence both in mathematics and mathematics education. Below is a list of topics included in the course Learning and Teaching of Mathematics.

- Theories for learning and teaching.
- Knowledge and understanding
- Mathematical knowledge for teachers
- The nature of mathematics as a science
- Perspectives on and approaches to algebra and geometry
- Definitions, proof and argumentation
- Group theory

The course is large and the list above shows topics that may seem of a very different nature. However, much work has been done to create links between the topics and in this paper I will in particular focus on how the topic of group theory is linked to the other topics. By pointing to such links I will also make a case for why group theory can be seen to be a relevant topic even for primary school mathematics teachers.

THE NATURE OF MATHEMATICS

The French mathematician René Thom said that "[a]ll mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (Thom 1973). If one subscribes to this statement, it means that as a teacher one cannot avoid having a view on the nature of mathematics but one may not be conscious of one's view. I believe that it is of value for a teacher to have more than tacit knowledge about the ontological and epistemological basis of mathematics. Taking this stance I believe that in teacher education one should be concerned with questions like

- What characterises the objects that one works with in mathematics?
- In what ways do they exist and how can one come to know about them?
- What characterises knowledge of mathematical objects?
- How can one ascertain if something is true in mathematics?

These are philosophical questions which can be linked to theories of learning, and to theories about knowledge and understanding. Making this link, with the subject (here mathematics) in the centre, will potentially provide the students with opportunities to better understand what these questions really mean.

Perhaps the most characteristic property of mathematics, distinguishing it from other sciences, is its axiomatic structure. Ideally, a mathematical theory should be based on some basic concepts and a number of axioms, and by logical deduction one should be able to obtain the valid truths (the theorems) in the system. The most well known axiomatic system is probably Euclid's theory for plane geometry. Euclid's idea was that from 23 definitions, five common notions and five axioms, or postulates, all truths about plane geometry should be deduced (Heath 1956).

Euclid's structure has been part of basic mathematical knowledge for centuries and Euclidian geometry is, for many reasons, important for teachers to know about. However, as an example of a "pure" axiomatic system Euclid's system has some disadvantages. This is due to the figural aspect of geometry. All geometrical concepts have conceptual properties, built on definitions, axioms and logically derived properties but they can also be represented as a visual picture. They have spatial properties and at the same time general conceptual properties. This dual nature is what Fischbein (1993) refers to as *figural properties*. The figural aspect of geometry can be a support for the reasoning because one may get help from a figure to see connections but it can also be an obstacle in the sense that one avoids the logical reasoning because the result "can be seen from the figure". Hence, students see no need to do a proof, the desired result is "obviously true", and also it is hard to distinguish what is known from what is not known, because the visual image shows both the properties that are logically established and those that remain to be established. It also adds to the confusion that even in proofs that are taken to be mathematically valid one can find arguments that directly refer to a figure (see Fischbein 1993: 139-140).

Group theory is different in this respect because in group theory the objects in question are defined only through the axioms. The objects cannot be visualised and so there is no source of knowledge about them, except what is contained in the axioms, and any further claims about their properties must be justified by referring to the axioms, or, as the development proceeds, to theorems that are already proved.

According to Duval all mathematical objects are abstract ideas and "[t]he only way to have access to them and deal with them is using signs and semiotic representations" (2006: 107). Nevertheless, some mathematical objects are so familiar (e.g. numbers, geometrical shapes) that students have difficulties separating the objects from their semiotic representations. Introducing a general structure such as a group, where the elements could be different mathematical objects (numbers, matrices, functions) the learners are forced to think in an abstract manner.

THE STRUCTURAL ASPECT OF ALGEBRA

James Kaput has given a content analysis of algebra and identified five interrelated strands around which algebraic reasoning can be organised (see e.g. Kaput 1999; Kaput & Blanton 2001).

1. Algebra as generalising and formalising patterns and constraints.
2. Algebra as syntactically guided manipulation of formalisms.
3. Algebra as the study of structures and systems abstracted from computations and relations.
4. Algebra as the study of functions, relations, and joint variations.
5. Algebra as a cluster of modelling and phenomena-controlling languages. (Kaput & Blanton 2001: 346-347)

Acknowledging that algebra is a fundamental part of school mathematics the course Learning and Teaching of Mathematics has algebra as one of its central topics. School algebra is often considered to be about 'solving equations' and 'calculating with letters'. Taken a little further, algebra can be seen as generalised arithmetic and as a way of establishing generalisations about particular properties of numbers (e.g. "the sum of the first *n* odd numbers is the *n*th square number"). This would then correspond to item 1 in Kaput and Blanton's (2001) list. Work with numerical patterns, e.g. number sequences given by figurate patterns, has become widespread both in school and in teacher education. This emphasises the functional aspect of algebra (find an algebraic expression for the *n*th figure in a pattern), which is quite close to and can be developed into the modelling aspect (items 4 and 5 in Kaput and Blanton's list). Obviously, in all this, it is necessary to manipulate formalisms (item 2 in the list). Examples of item 3, the study of structures and systems, are not easily accessible in school algebra but, referring to the historical development of algebra outlined in the Introduction, this aspect of algebra is necessary to give a sense of what algebra as a scientific mathematical discipline entails.

The structural aspect of algebra has been elaborated on by Kirshner (2001) who expresses concern about algebra being reduced to a matter of memorising and applying rules. He presents two approaches to algebra; a structural approach which "builds meaning internally from the connections generated within a syntactically constructed system" and a referential approach with the property that it imports "meaning to algebraic symbol systems from external domains" (Kirshner 2001: 84-85). Acknowledging that the structural approach is fundamental in mathematics (algebra) as a science it can be argued that prospective teachers should experience this approach and in my view, group theory is a very well suited topic to gain such experience.

An alternative to group theory to illuminate the structural aspect of algebra could have been to work with linear algebra and the axiomatic structure of a vector space. In our master programme we have chosen group theory because this topic has very close connections to elementary geometry (symmetry) and also to basic properties of numbers.

DIHEDRAL GROUPS AND PERMUTATIONS

Working with groups, the connection to geometry through the concept of symmetry can be made central and this is certainly the case in the course I am describing. This is also reflected in the choice of textbook, *Groups and Symmetry* (Armstrong 1988). Much emphasis is placed on studying the symmetry groups of regular polygons, the so-called *dihedral groups.* The simplest group of this type is the symmetry group of the regular triangle, denoted D_3. It is seen that this group consists of six elements, three reflections and three rotations. Working with compositions of the elements of this group, the students get a first experience with a non-commutative operation (non-Abelian group). This experience is well suited to make sense of the fact that

commutativity is not taken as part of the group axioms. Looking at what happens with a regular triangle when reflecting and rotating it, it becomes clear that all these operations can be expresses as permutations of the three corners. It is also not hard to see that indeed all possible permutations of the corners can be obtained by some reflection or rotation. Comparing the group tables for the symmetry group of the triangle (D_3) and the group of permutations of three elements (S_3) one will see that the two tables have the same structure, i.e. the two groups are *isomorphic.*

This is a good starting point for some genuine investigations, presenting a good opportunity to let the students engage in *inquiry based learning.* A number of recent projects in mathematics education have been based on *inquiry* as a fundamental concept (see e.g. Jaworski, Fuglestad, Bjuland, Breiteig, Goodchild, & Grevholm 2007). Inquiry is here seen in the sense of Wells who characterises inquiry as "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (Wells 1999: 122). Inquiry can be seen on many levels; in the mathematical tasks for pupils' learning in the classroom, in the developmental process of planning for the classroom, and in the research process of systematically exploring this developmental process (Jaworski 2007: 15). I will here focus on *inquiry-based tasks* for the student teachers' learning.

In order that the teachers should be able to design inquiry-based tasks for their pupils, I take it to be of great value that the teachers, as students or in in-service training, should have had some experience of working in an inquiry-based way themselves. Of course there is no lack of problems that can be given to students with which they can engage in genuine inquiry. Such problems can e.g. be taken from elementary geometry and as an example of this I could refer to Bjuland (2004), who has shown how Viviani's theorem (Weisstein n.d.) can be a source for investigations among student teachers. However, the point I am making here is that by using group theory as a basis for inquiry, it is not just a matter of taking an isolated problem from an otherwise familiar context (e.g. plane geometry) but it is a matter of developing an unknown theory, with new concepts, new notation, and obtaining results that are new in the sense that the students did not know about them beforehand. In fact, they could not even formulate them before having engaged with developing the necessary concepts and notation. In my work with student teachers this is as close I have ever experienced students to engage in activity that could be said to be close to mathematical research.

As an example I will return to the relationship between the dihedral group D_n and the group of permutations S_n (the symmetric group of degree n). Having established that D_3 is isomorphic to S_3 a natural question will be to ask what is it like in the case $n = 4$? Obviously all the elements of D_4 (the symmetry group of the square) can be identified as elements of the symmetric group S_4, and also it is easily seen that there are elements in S_4 that cannot be found in D_4, e.g. it is not possible to have two neighbouring points in the square swap places, so e.g. the permutation (1 2) does not have a corresponding element in D_4. The result of this consideration is that D_4 must

be a proper subgroup of S_4. A natural question to ask at this point is whether one can find a geometric figure whose symmetry group is isomorphic to S_4. The first obvious candidate, the square, has failed, so it is not clear whether such a figure actually exists, or if it does, what figure it can be. Having seen that a regular plane figure with four corners does not have sufficient freedom to perform all the permutations of S_4, a possibility could be to look for a three dimensional figure with four corners. Introducing the third dimension will give more freedom, so there is a hope that this may give results. Therefore the tetrahedron could be thought of as a possible candidate.

I will denote the symmetry group of the tetrahedron by T. Now asking the question whether T is isomorphic to S_4 is for the students an example of authentic inquiry (Lindfors 1999), a question to which they do not already know the answer. They know that the order of S_4 is 24, so attempting to solve the problem by comparing the group tables of the two groups is a tedious task. Therefore the situation calls for the use of mathematical theory. Obviously, doing reflections and rotations of the tetrahedron entails permuting the four corners, so T has to be a subgroup of S_4 just as D_4 also is a subgroup of S_4. The students know Lagrange's theorem (Armstrong 1988: 57), so they know that the order of T must be a factor of 24. Investigating the rotations of the tetrahedron the students will find that there are rotational axes through each of the four corners and the midpoint of the opposite face allowing rotations of 120 and 240 degrees. There are also rotational axes through the midpoints of each of three pairs of opposite edges, allowing rotations of 180 degrees. Counting the identity this gives 12 rotations altogether. But, there has to be symmetries not accounted for by the rotations (due to planes of reflection), so the order of T must be strictly greater than 12. And hence, by Lagrange's theorem, it must be exactly 24, so T is isomorphic to S_4. This example shows the power of abstract (general) theory applied to a problem from geometry. Using the language of Kirshner (2001) one can say that the example shows an encounter between the structural and the referential approach to algebra.

Being able to work efficiently with elements of the dihedral groups requires an efficient notation. I have used the notation given in the textbook (Armstrong, 1988) but emphasised that embedded in this notation there are many conventional choices. In the notation I have used in describing D_4, r denotes a 90 degree counter clockwise rotation and s denotes a reflection around a fixed perpendicular bisector of two sides of the square. Furthermore, rs means the composition by first doing s and then r. By making other choices one would get the same results although they at the surface would look different. This I take to be a valuable experience for prospective teachers. The notation presented here is efficient in that it can easily be used in computations. I will show one example. Given the two elements rs and r^2s of D_4, the task is to compute the composition $(r^2s)(rs)$. A fundamental property in the dihedral group D_n is that $sr = r^{n-1}s$, which can be seen geometrically. Using this, the associative property, and that $r^4 = s^2 = e$, one can write

$$(r^2s)(rs) = r^2(sr)s = r^2(r^3s)s = r^5s^2 = r.$$

Proceeding in a similar way the whole group table for D_4 can be filled out without too much difficulty.

In light of the fact that notation is largely conventional it is of interest to investigate what notation students will choose if left to themselves, and to what extent this notation is efficient. Mona Nosrati (2013) has made a study of students in Norwegian upper secondary school who has been given as a task to investigate and describe the symmetries of the square. She found that different students developed quite different notation, using different semiotic resources. The notation that was developed consisted of different signs, e.g. arrows, and numbers indicating angle of rotation, and also instructions in natural language. It was found that based on the choice of notation for the basic operations the students experienced different challenges when describing compositions of these operations. Some notation was not at all useful for describing compositions. In Duval's terms the difference experienced here could be seen as a difference between discursive and non-discursive registers (Duval 2006: 110). In my case I have from the very beginning of the work with group theory introduced a discursive register which is efficient for computations but, as Nosrati has shown, students will come up with other choices when left to themselves. This could be another area of inquiry where group theory could show its strength: Discuss what could be seen as the most efficient symbolic representation of the phenomena that are observed.

MATHEMATICAL KNOWLEDGE FOR TEACHERS

Following the seminal work of Shulman (1987) about pedagogical content knowledge, the question about what should constitute a sound knowledge base for teachers of mathematics has been discussed by many authors. Building directly on Shulman's work, Deborah Ball and a number of co-authors have proposed a characterisation of mathematical knowledge for teachers, MKT for short (e.g. Ball, Thames & Phelps 2008). The question is discussed from a slightly different perspective by Rowland, Huckstep and Thwaites (2005). Using the model in Ball et al. (2008, p. 403) where MKT is divided into subject matter knowledge and pedagogical content knowledge, it seems clear that knowledge of group theory must count as subject matter knowledge. But how can this knowledge be of use e.g. in teaching mathematics in primary school? An example of this is presented below.

The rose with six petals shown in Figure 1 is probably well known to all children who have experimented with a compass. I have used this rose as a starting point for discussing symmetry with children at the age of 8-9 (Rønning 2011). The task given to the children was to colour the rose in such a way that the picture would be symmetric. The numbers on the petals were not there; they are introduced here for easier reference. It could easily be imagined to extend the task to include working with more than one axis of symmetry, although I did not do this in my work with the children. What follows is therefore not based on actual experience but it could easily

be imagined taking place. Assume that a pupil, Pupil A, has placed an axis of symmetry (s) through the tips of the petals 1 and 4 of the rose in Figure 1, and another axis (m) halfway between the petals 3 and 4, and 1 and 6. Another pupil, Pupil B, may also have used s but instead of m used an axis (n) through the tips of petals 3 and 6. Starting to colour the petals, Pupil A will experience that all petals will have to have the same colour, whereas Pupil B is able to produce a rose where the petals 1, 3, and 5 have one colour and the petals 2, 4, and 6 have a different colour. In what ways can knowledge of group theory inform the teacher about what is happening here, and what kind of response to give to these two pupils? Is just one colour the only possibility for Pupil A? Does it make sense that the teacher could suggest to Pupil A to start colouring in a different way in order to try to get a rose with more than one colour? What if the axes of symmetry had been placed in other positions?

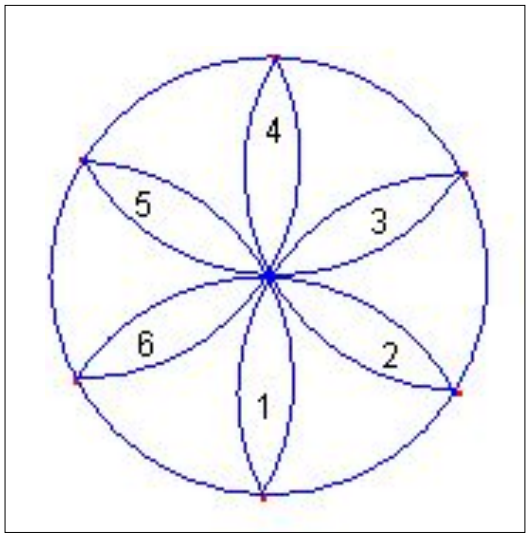

Figure 1: The rose with six petals.

Knowing group theory, the teacher will know that the symmetry group of the rose, before any colouring is done, is the dihedral group D_6. This group is generated by the reflection s (corresponding to the axis s introduced above) and the counter clockwise rotation r of 60 degrees around the centre of the circle circumscribing the rose. The teacher will also know that a composition of two reflections is a rotation through an angle two times the angle between the reflection axes. Hence, the teacher will deduce that composing s with m will give a 60 degree rotation. A more careful analysis will show that indeed $sm = r$ (and $ms = r^5$). Hence, the choice of axes of symmetry made by Pupil A will imply that the symmetry group of Pupil A's figure will contain the generators of D_6, and therefore all of D_6. Composing the two reflections in the figure of Pupil B will on the other hand give r^2 (120 degree rotation) and the symmetry group of Pupil B's figure is isomorphic to D_3.

EPILOGUE

It has been my aim with this paper to show how a seemingly abstract topic as group theory introduced in a course for prospective teachers of mathematics can serve as a

link to topics in both mathematics and mathematics education considered important for teachers. In this final section I will mention another link, very relevant for teacher education, although it has not been pursued in the course I have described. This is the link between mathematics and art, or rather the link between algebra, geometry, and art. Henderson and Taimina (2005: 1- 8) describe four strands along which geometry has developed. One of these strands is the Art/Pattern strand, very much concerned with studying patterns and symmetry and not so much occupied with measurements and calculations. Studying the symmetries of regular polygons is an example of the Art/Pattern strand of geometry, linked to algebra. Extending the study of symmetries to unbounded patterns will give much wider possibilities to invoke exciting examples from art and architecture. A rich source of examples can be found in the decorations of Islamic art where different patterns can be investigated, representing a wide variety of symmetries (see e.g. Rønning 2009). In the case of unbounded patterns, the isometries *translation* and *glide reflection* will appear, in addition to reflection and rotation. A fascinating result in this connection is that there exist exactly 17 non-isomorphic groups describing all possible unbounded patterns, the so-called *wallpaper groups* (see e.g. Martin 1982). Going into this topic will entail engaging with some very nice mathematics, as well as with some interesting examples of art and decorations.

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