AN ANALYSIS OF ERRORS AND MISCONCEPTIONS IN THE 2010 GRADE 12 MATHEMATICS EXAMINATION: A FOCUS ON QUADRATIC EQUATIONS AND INEQUALITIES

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This paper reports on an analysis of 1959 examination scripts to identify errors and misconceptions of learners who wrote the 2010 National Senior Certificate Mathematics examination. It focuses on quadratic equations, quadratic inequalities and linear-quadratic systems of equations. Documentary analysis of the actual scripts of examinees rendered that learners committed careless, procedural, calculation and application errors as described in the literature. The least number of errors was of the careless category and the most errors were found in the procedural category. It is concluded that persistence of errors is not easy to avoid in individualised, time-restricted high-stakes examinations such as the National Certificate Mathematics examination.

INTRODUCTION

There is a general concern by educators, educationists and the general South African public about the poor performance of grade 12 mathematics learners in the final national examinations. This is a perennial unsatisfactory state of affairs in a country which views mathematics as a key subject for the preparation of students for careers in science and technology and for scientific and economic development (Ogunniyi 2005; Rogan & Grayson 2003). Interventions by the Department of Basic Education as well as by provincial education departments do not seem to be achieving much success in arresting, let alone reversing, the decline in the mathematical performance of learners.

From 2008 to 2010, the number of learners who wrote mathematics in the grade 12 examination decreased by 40 000 candidates nationally. The 2010 mathematics results showed that at the 30% pass mark, only 47% of learners who wrote mathematics passed. Also, only 24 % of those who passed obtained a university entrance pass (Department of Basic Education 2011).

This paper stems from a study which sought to identify and analyse the errors which candidates commit in the final mathematics examination paper with a view to use these errors and misconceptions as a tool by teachers for more effective teaching.

AIM OF STUDY

As already indicated above, this paper reports on a study which sought identify the errors committed by examinees in the first question of Paper 1 in the final Grade 12 mathematics examinations of 2010. The focus is on the questions dealing with

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quadratic theory—equations and inequalities and linear-quadratic systems of equations. In the final grade 12 examinations for mathematics this is normally covered in the first question. It constitutes nearly 20% of Paper 1.

The question which is the subject of this paper is the following:

Question 1

1.1 Solve for x, correct to TWO decimal places, where necessary:

1.1.1 $(3-x)(5-x)=3$

1.1.2 $3x^2=2(x+2)$

1.1.3 $4+5x>6x^2$

1.2 Solve for x and y simultaneously:

$3y=2x$

$x^2+y^2+2x-y=1$ (Mathematics: DBE/November 2010)

Errors and misconceptions

By mathematical errors is meant the mistakes which learners commit during calculation or during the solution of mathematical problems. The Oxford Student's Dictionary (2002: 346) defines an error as "the amount of inaccuracy in a calculation or a measuring devise." Luneta & Makonye (2010: 2) refer to it as "… a mistake, slip, blunder or inaccuracy …". Errors, visible in learners' written work, result in learners not obtaining a perfect score in routine assessments or in examinations. Some of these errors occur spontaneously due to carelessness but others are structural and deeply ingrained in the sense of being caused by underlying mental structures which are referred to as misconceptions. While it is relatively easy to correct an error, it is very difficult to eradicate a misconception because it usually becomes embedded in the learners' conceptual map or schema.

Literature Review

A number of studies have been conducted on the nature and causes of mathematical errors committed by learners (Muzangwa & Chifamba 2012; Luneta & Makonye 2010; Seng 2010; Erlbrink 2008; Mason,2000a; Hodes 1998; Olivier 1989; Davis 1984; Orton 1983; Matz 1980). What all these writers agree on is that learners commit errors which are a result of existing conceptual gaps or misconceptions which are entrenched in the learners' conceptual schemes. All these writers categorise errors broadly into unsystematic and systematic errors. Unsystematic errors are unstable "…unintended, non-recurring wrong answers which learners can readily correct by themselves. Systematic errors though are recurrent wrong responses methodically constructed and produced across space and time." (Riccomini (2005) cited in Luneta & Makonye (2010: 2)). Systematic errors, generally referred to as misconceptions, are more resistant to change and hence difficult to eliminate.

It is the usual practice in the study of errors to use a conceptual framework to categorise them. After perusing the literature and from an initial reading of learners' scripts, an analytical framework comprising four categories of errors, viz., careless errors, procedural errors, calculation errors and application errors was devised. We discuss these in turn.

Careless errors

This refers to "unnecessary" errors committed by learners which do not arise from a cognitive or conceptual challenge. These errors, which at times are regarded as "slips or silly mistakes" (Davis 1984), occur when learners incorrectly transcribe questions from the source document such as a question paper or when a learner wrongly transcribes some words or mathematical symbols from his/her own script. The following example illustrates this in a learner's work.

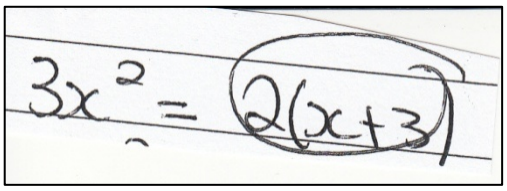

Figure 1: Incorrect transcription.

One can observe that the above is an incorrect transcription of the question directly from the question paper which gives the equation as $3x^2 = 2(x + 2)$. This is a typical example of a careless error. This results in the examinees rendering an incorrect answer although they might obtain the correct answer for the incorrectly transcribed problem.

Procedural errors

This refers to errors in which procedures are applied incorrectly. Figure 2 illustrates this error.

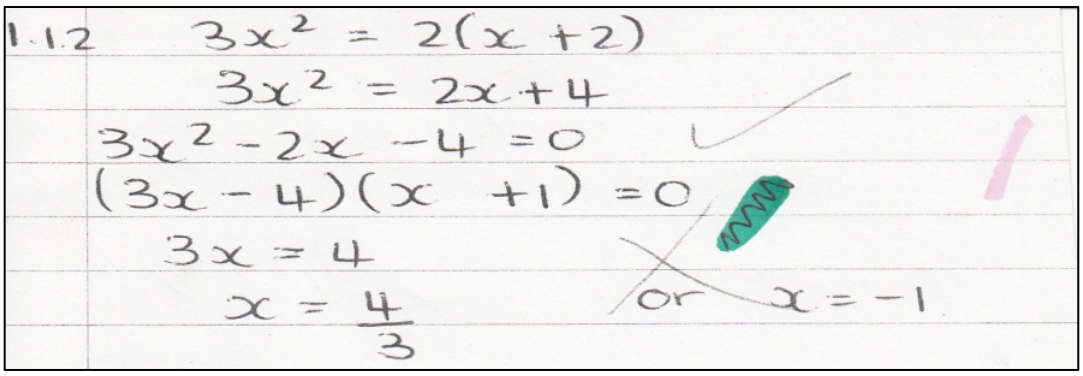

Figure 2. Example of procedural error.

To solve $3x^2 - 2x - 4 = 0$ the quadratic formula had to be applied. The candidate does not use the quadratic formula in this instance but factorises the quadratic equation this to yield the incorrect factors $(3x - 4)$ and $(x + 1)$. The candidate fails to

realise that these factors are incorrect because the products of the two inner terms and the two outer terms do not add up to the middle term, -2*x* in the quadratic trinomial.

Calculation errors

Elbrink (2008) defines calculation errors as errors which occur in the application of the four basic operations namely, addition, subtraction, multiplication and division. In this article we have restricted ourselves to the four basic operations. We are not, for instance, concerned with calculations like finding the factors of a trinomial as in the case above. Figure 3 exemplifies such an error.

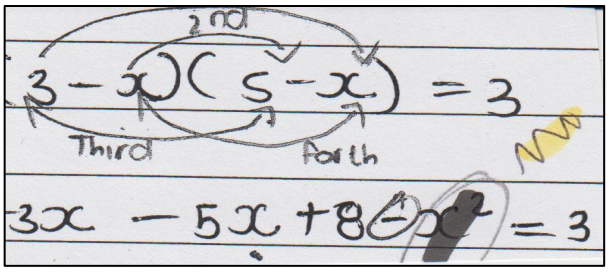

Figure 3: Example of a calculation error by a learner.

To solve the equation it had to be written in the standard form, $ax^2 + bx + c = 0$. This entails therefore the multiplication of the two binomials to yield the product $15 - 3x - 5x + x^2 = 3$ which simplifies to $x^2 - 8x + 15 = 3$. From this, one has to solve the equation. What is observed is that 3 and 5 have been added instead of being multiplied. Also for the term in x^2 the multiplication rendered the incorrect sign.

Application errors

This type of error was identified in instances when candidates factorised correctly but used incorrect signs in the factors of the quadratic trinomial. This type of error is identified by Hodes (1998) as occurring when a learner knows what procedure to apply but applies it incorrectly. This is in contrast to procedural errors where learners apply the incorrect procedure. Figure 4 illustrates one such instance of an application error.

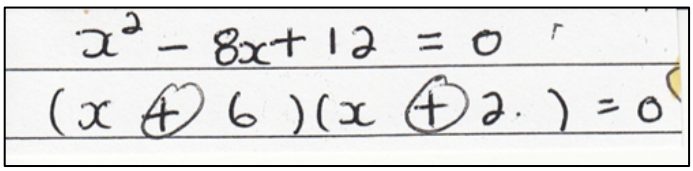

Figure 4: Application error.

The candidate is aware that s/he must factorise and obtains the correct factors for the constant term in the sense of absolute numbers. However, the incorrect application resides in the selection of incorrect signs for the absolute numbers. This again would be corrected by checking the products of the two inner and two outer terms which would not add up to the middle term of -8*x* in the quadratic trinomial.

RESEARCH APPROACH

This study adopted a qualitative research approach with the preferred method of data analysis being documentary analysis. Document analysis is a systematic procedure for reviewing or evaluating documents mostly used to complement other research methods but for this specific study documentary analysis was used only as the method of analysis. Like other methods in qualitative research, document analysis requires that data be examined and interpreted, in order to elicit meaning, gain understanding and develop knowledge (Strauss & Corbin 1990). Document analysis is also systematic, controlled, directed, organized, explicit and objective (Quade1970). For this study, the documents which were analysed were the final examination scripts of grade 12 Mathematics learners of 2010 in the Western Cape Province.

Table 1: Number of schools and scripts in sample.

District	Number of schools	CED	DET	HOD	HOR	IND	WCED	Total
Cape Winelands	11	126	13	0	69	0	0	208
Metropole Central	24	149	72	0	236	143	0	600
Metropole East	11	27	189	0	7	12	82	317
Metropole North	14	151	9	0	81	14	56	311
Eden and Central Karoo	8	57	11	0	31	0	0	99
Metropole South	32	55	38	20	96	11	46	266
Westcoast	5	35	0	0	41	0	0	76
Overberg	4	19	0	0	19	0	44	82
		619	332	20	580	180	228	1959

Stratified sampling was used and the strata were mutually exclusive. There were 17 544 candidates who wrote the 2010 grade 12 final mathematics examination in the Western Cape Province. To acquire a 2% tolerance error with a 99% confidence level, 12% of the total number of candidates was chosen as a sample. This rendered 1959 scripts. Stratification was according to districts and former departments of education per district. Furthermore examination centres were selected due to examination scripts being stored in batches by the Western Cape Education Department (WCED) and requesting learners would have made the process cumbersome. Table 1 presents outcomes of the selection process.

DATA ANALYSIS AND RESULTS

Template analysis was used to analyse the grade 12 examination scripts using the framework constructed above as the template. The total number of errors identified in question one of the sample of 1959 scripts analysed in the first paper of this examination was 2878. The number of errors identified for each of the error categories is presented in Table 2 below.

Table 2: Percentage of errors identified.

Error Category	Total (N = 2878; percentages in brackets)
Careless	232 (8 %)
Calculation	756 (26,3 %)
Application	498 (17,3 %)
Procedural	1392 (48,4 %)

An overarching issue which was displayed is non-completion of solutions. This generally would be of the nature where an attempt is made at a solution but is not carried through to its conclusion. There were 1285 non-completions that are not part of the above errors.

DISCUSSION

The lowest percentage (8%) of errors across the sample was of the careless error category. This percentage is in line with the findings of Yang & Wan (1991) and Elbrink (2008), who concluded that the numbers of careless errors committed by learners generally are small. A further breakdown of the careless category rendered 0.4% as incorrect transcriptions—a symbol or number from the question paper being replaced by one not in the paper; 3.3% incorrect transfers— a wrong result from a previously-obtained result written down and 4.3% were of an arbitrary nature—a number or symbol is replaced by an completely arbitrary one. According to Elbrink (2008) these careless errors are normally the result of short attention span and examination anxiety is also seen to be a possible contributing factor.

Careless errors normally do not result in learners forfeiting a substantial number of marks in high-stakes examinations. This is due to the positive marking regime adopted. Markers of these examination are, for example, advised as follows: “If candidate lets the volume of the first tank be a specific value (instead of a variable) and his/her argument follows correctly, award 4/4 marks” (DBE 2012: 11). With such positive marking attention is also paid to a principle of examinations also serving the purpose of what examinees know rather than what they do not know. However, it is not so that all careless errors get treated in an non-penalised fashion. A cursory glance at memoranda of marking and our experiences of marking and moderating the NSC Mathematics indicate that fairly considered decisions are taken with respect to how marks will be awarded if careless errors are committed. For example, if a

candidate's careless error results in the level of cognitive demand of the question being reduced, then it is unlikely that he/she will be awarded more than 50% of the marks awarded for the examination item.

Although the consequences for making careless errors are not very severe, it still needs to be borne in mind that such errors can be very costly in real-life. One can, for example, just think about the incorrect placement of the decimal point in computerised international financial transactions. It is, therefore, still incumbent upon us as teachers to make certain that learners are aware of checking for incorrect transcriptions and transfers and the arbitrary errors that can creep into their work during examinations.

Seventeen percent of the errors fell in the application error category. These errors occur where learners ostensibly exhibited knowledge of the mathematical procedure at play but used incorrect signs. These errors were basically incorrect factorisations or incorrect substitutions particularly with the substitution required for the linear-quadratic systems of equations. Although errors of the applications nature are not extraordinarily high, it should be a cause for concern. Substitution is a section of school mathematics that is dealt with nearly throughout the curriculum in the Further Education and Training band. Much of it has to do with substitution of numbers in expressions. It seems that learners are not having enough experience with situations where they have to substitute non-numerical entities in another expression. The situation with incorrect factorisation is also of concern. The factorisation at play is mostly that of quadratic expressions. This kind of factorisation is normally dealt with in grade 10. In subsequent grades it is used a tool for other kinds of problems. It appears that remoteness between dealing with factorisation of quadratic expressions as a specific mathematical technique and the use of the technique for a specific objective interferes with the fluency of the use of the technique.

Calculation errors account for 26% of the total number of errors identified across the sample. Of these two thirds were basic calculation errors—mistakes related to multiplication, division, subtraction and addition and a third were calculator errors—incorrect use of a calculator. It is difficult to find plausible reasons for the occurrence of calculation errors. One might be the pressure of time in the high-stakes examination and candidates just moving ahead when they obtained an answer and not checking their calculation. It is a truism that even experts make these kinds of slips but they have the advantage of having time at hand to check and through experience particular mechanisms, which Davis (1984) calls critics, to alert them to the unreasonableness of answers.

The majority of the errors were of the procedural kind. These accounted for 48, 4% of the errors identified for the sample. There were 291 instances where learners resorted to factoring 1.1.2. Since the roots for the quadratic equation, $3x^2 = 2(x+2)$ were not rational, they probably did not have the tools at hand to ascertain whether the factorisation route would be a productive one to pursue. The non-determination of

factorisation as a feasible route is bolstered by sequencing of the solution processes for quadratic equations. The general route followed for solving quadratic equations is by factorisation, completion of the square and use of the formula. However, the formula, particularly using the discriminant gives an indication on whether the factorisation is indeed a feasible procedure to solve through factorisation. Furthermore, learners are not made aware nor do they engage in consolidation exercises in which, particularly, the formula can be used to find the routes for any quadratic equation. Teachers are also not alerted to this in the resources, such as textbooks and other curriculum resources distributed for use. For example, the first question in 2012 NSC Mathematics was "Solve for x in each of the following: 1.1.1 $(2x-1)(x-4)=0$." The expectation was, we assume, that learners would directly write down the roots based on their understanding of the roots of an equation. In the memorandum of marking (DBE 2012) no indication is given that the route of expanding the left-hand side of the equation and use the formula to "solve for x", understood as "finding the roots" is also an acceptable one.

Another way of working that came through in the procedural error category was what we call the *avoidance of fractions* simplification. This way of working was resorted to in the question which required the solution of the linear quadratic system: $3y = 2x$, $x^2 + y^2 + 2x - y = 1$. The solution of this system would require one to first make either x or y the subject of the formula in the linear equation $3y = 2x$. In this case $y = \frac{2}{3}x$, if the candidate prefers to make y the subject of the formula or $x = \frac{3}{2}y$ if the preference for the subject is x. This solution would then be substituted in the quadratic solution for either of the subjects as the case may be. What we observed is that many candidates rewrote the linear equation to obtain one which avoided working with fraction coefficients. Hence they wrote the equation $3y = 2x$ as $y = 2x - 3$ or $x = 3y - 2$. We attribute this kind of error to the fact that in the previous examination papers over at least three years before 2010, all the linear equations of the systems were of the form where the coefficients were of a non-fractional nature. We contend that the candidates who committed this, internalised this kind of question during revision and then simplified $3y = 2x$ to a familiar one where fractions are avoided. There were 426 or about 15% of such errors in this sub-category.

The well-known non-change of the inequality sign when solving inequalities was found in 13% of the total number of errors. For the quadratic inequality $4 + 5x > 6x^2$ candidates correctly transposed $6x^2$ to the left side of the inequality which then transformed into $-6x^2 + 5x + 4 > 0$. When multiplying both the left and right side of the equation by – 1, these candidates did not change the inequality sign from > to <. It is conjectured that this is a kind of overgeneralisation from the solution of equations where the equality sign is left unchanged due to the "syntactical resources

for transforming mathematical expressions, like the rule, 'change sides, change signs'" (Davis 2011) without focussing on the relation.

The 1285 non-completions of solutions can be accounted for from various perspectives. We suspect that advice, such as "do not waste time when you get stuck but move on to the next question", that learners get for writing examinations is a major contributory factor. This should not be construed as "bad" advice since the NSC mathematics examination is a time-restricted one in which learners attempt to attain the maximum possible marks they can muster.

CONCLUDING REMARKS

We have presented one way to identify and classify errors forthcoming from students' work in a time-restricted and high-stakes examination. The lens was a particular set of literature related to the topic. Other lenses such those used by Davis and his collaborators are available (see Davis (2011) for reference to part of this collection of work). They essentially use a pure mathematics lens and provide valuable insights, particularly for teachers and teacher educators, on how to view the productions of examinees from an advanced mathematical point of view. Brodie & Berger (2010: 172), basing their work on Sfard's (2007, 2008) theory of mathematics as discourse, use a discursive framework to analyse learner errors and view learners as "participating in a difference discourse from the teacher …".

As yet there is no golden wand to deal with errors and misconceptions to ensure faultless performance. Current attempts are of the "using common errors" (Mason 2000b: 10) kind of activities. These activities are of the kind where students are presented with exercises containing 'wrong' ways of working which they have to resolve. As far as we could ascertain there are no studies that report on the effectiveness of this strategy and points in the direction that still much work related to examinees' idiosyncratic ways of working in high-stakes and time-restricted examinations needs to be done.

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REFERENCES

Allen, R. (Ed.), (2002). *Oxford Student's Dictionary*. Oxford: Oxford University Press.

Department of Basic Education. (2011). Report on the National Senior Certificate Examination Results. Retrieved from www.info.gov.za/view/DownloadFileAction? Id=138430800/247-6553.

Booth, L. R. 1988. Children's difficulties in beginning algebra. In Coxford, A. F. & Shulte, A. P. (Eds.), *The Ideas of Algebra, K-12. 1988 Yearbook*. Reston, VA: NCTM.

Brodie, K. & Berger, M. (2010). Towards a discursive framework for learner errors in mathematics. In Mudaly, V. (Ed.), *Proceedings of the seventeenth annual meeting of the South African Association for*

Research in Mathematics, Science and Technology Education (SAARMSTE), Durban: University of KwaZulu-Natal, 169-181.

Davis, R. B. (1984). *Learning Mathematics: The Cognitive Science Approach to Mathematics Education*. New Jersey: Ablex

Davis, Z. (2011). Aspects of a method for the description and analysis of the constitution of mathematics in pedagogic situations. In Mamiala, T. & Kwayisi, F. (Eds.), *Proceedings of the 19th Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education*, North West University, Mafikeng Campus, 18–21, January 2011, 97-108.

Elbrink, M. (2008). Analyzing and Addressing Common Mathematical Errors in Secondary Education. BS Undergraduate Mathematics Exchange, 5(1), 2-4.

Hodes, E. (1998). *Winning at Math*. SBCC Math Dept. Santa Barbara City College. Academic Success Press,

Godden, H. J. (2012). An analysis of learners' ways of working in High Stakes Mathematics Examinations: Quadratic Equations and Inequalities. Unpublished Masters dissertation: Bellville: University of the Western Cape.

Luneta, K. & Makonye, P.J. (2010). Learner Errors and Misconceptions in Elementary Analysis: A Case Study of a Grade 12 Class in South Africa. *Acta Didactica Napocensia*, 3(3), 36-46.

Mason, J. (2000a). Asking mathematical questions mathematically. *International Journal of Mathematical Education in Science and Technology*, 31, 97-111.

Mason, J. (2000b). *Practice guide: Teaching Mathematics*. Milton Keynes, UK: The Open University.

Matz, M. (1980). Towards a Computational Theory of Algebraic Competence. *Journal of Mathematical Behaviour*, 3(1), 963-166.

Muzangwa, J. & Chifamba, P. (2012). Analysis of Errors and Misconceptions in the Learning of Calculus by Undergraduate Students. *Acta Didactica Napocensia*, 5(2), 1-10.

Ogunniyi, M. B. (2005). Cultural perspectives on Science and Technology Education. In: Abdi, Ali & Ailie, Cleghorn (Eds.), *Issues in African Education: Sociological perspectives*. England: Palgrave Macmillan.

Olivier, A. (1989). Handling pupils' misconceptions. *Pythagoras*, 21, 10-19.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, 14 (3), 235-250.

Quade, E. (1970). *On the Limitations of Quantitative Analysis*. Santa Monica, California: The Rand Corporation.

Rogan, J.M. & Grayson, D.J. (2003) Towards a Theory of Curriculum Implementation with Particular Reference to Science Education in Developing Countries. *International Journal of Science Education*, 25(10), 1171-1204.

Seng, l. K. (2010). An Error Analysis of Form 2 (Grade 7) Students in Simplifying Algebraic Expressions: A descriptive Study. *Electronic Journal of Research in Educational Psychology*. 8(1), 139-162.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage Publications, Inc.

Yang, C. T., & Wan, S. (1991). Comparison of Selected Bed-Material Load Formulas, *Journal of Hydraulic Engineering*, ASCE, 117(8), 973-988.