

**Proceedings of the 29th National Congress of the
Association for Mathematics Education
of South Africa (AMESA)**

Volume 1

Re-imagining Mathematics in a Digital Space

17 - 21 June 2024



Sol Plaatje University, Northern Cape

Editors: Zingiswa Jojo, Mogalatjane Matabane & Chikiwa
Clemence

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FOREWORD AND ACKNOWLEDGEMENTS

Welcome to the Northern Cape and to the 29th Annual National Congress of the Association for Mathematics Education of South Africa (AMESA). We are thrilled to host you at Sol Plaatje University, a symbol of transformation and progress as the first university established in post-apartheid South Africa. The AMESA Congress theme for 2024, “Reimagining Mathematics in a Digital Age,” offers an exciting platform for teachers, teacher educators, researchers, postgraduate students, and education officials to explore and share innovative practices in mathematics education. Our collective goal is to embrace the opportunities that digital advancements bring to mathematics teaching and learning, fostering environments where both educators and students thrive in this evolving landscape.

Within the South African mathematics education context, integrating technology has become imperative to enhance learner engagement and achievement. The AMESA Congress 2024 is dedicated to ensuring that mathematics classrooms are dynamic spaces where digital tools are harnessed to make learning more interactive, engaging, and relevant to real-world applications. This year, we aim to explore how digital innovations can bridge gaps, build connections, and empower both teachers and learners to navigate the complexities of the digital age.

This year’s congress is enriched by the participation of two international speakers and four local speakers, including two young South African educators who are leading the way in technology integration. These speakers bring a wealth of knowledge and experience, offering diverse perspectives on reimagining mathematics education through digital means. Their presentations, along with panel discussions and workshops, will delve into the congress theme, providing invaluable insights and practical strategies for delegates.

The AMESA 2024 proceedings are presented in two volumes: Volume 1 includes plenary and long papers, while Volume 2 encompasses short papers, workshops, and 'How I Teach' papers. Each paper has undergone a rigorous peer review process, ensuring the highest standards of academic integrity and relevance. Authors have meticulously revised their papers in response to reviewers' comments, with final approval from the Academic Coordinating Team (ACT).

In conclusion, we extend our heartfelt gratitude to the Local Organising Committee, the National Organising Committee, Sol Plaatje University, our esteemed host, the AMESA Congress 2024 sponsors, plenary speakers, presenters, and delegates. Your contributions and participation are what makes this congress a beacon of hope and innovation in the mathematics education community.

Mogalatjane Edward Matabane and Clement Chikiwa

AMESA Editorial Team 2024

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All papers submitted in the congress were sent for blind reviewing. Many thanks to our review team who volunteered their time in reviewing the papers sent to them in a constructive and helpful manner:

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REVIEW PROCESS

Each of the long and short papers, workshops, How I teach submissions, accepted in this volume of the proceedings, were subjected to blind peer review by experience mathematics educators. The academic committee considered the reviews and made a final decision on the acceptance or rejection of each submission, as well as changing the status of submission

Number of Submissions: 88

Number of plenary paper submissions: 6

Number of long paper submissions: 29

Number of short paper submissions: 6

Number of 1-hour workshop submissions: 22

Number of 2-hour workshop submissions: 18

Number of How I teach paper submissions: 7

Number of poster submissions: 0

Number of submissions accepted: 56

Number of submissions rejected: 30

Number of submissions converted to a different format: 0

Number of submissions withdrawn by authors: 2

Thank you to each reviewer

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PROJECT-BASED LEARNING: A PROMISING PEDAGOGY FOR THE DIGITAL AGE

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Introduction

Mathematics and the digital world are truly a topics worthy of consideration as new advances are being made at an increasing pace causing shifts and changes in all aspects of modern life. If we were able to peel back the foundation of these technological advances we would see the mathematical symbols, operations, algorithms and logics that animate these technologies. Essentially, mathematics and digital technologies are intertwined to the point that scholars deem them “inseparable” (Sarinsakova, 2022). However, the intertwining of mathematics and digital technologies is not only unidirectional with mathematics driving technological advances. Rather, the development of novel digital technologies also influences the ways in which mathematics is practiced and developed and therefore, influences the ways in which mathematics is taught and learned. Russian psychologist and learning theorist Lev Vygotsky (Taber, 2020) described this relationship between a person engaged in a practice such as mathematics and the outcome of what he/she is able to accomplish as “mediated” by the tools and resources that are available (see Figure 1). As the technological tools and resources have

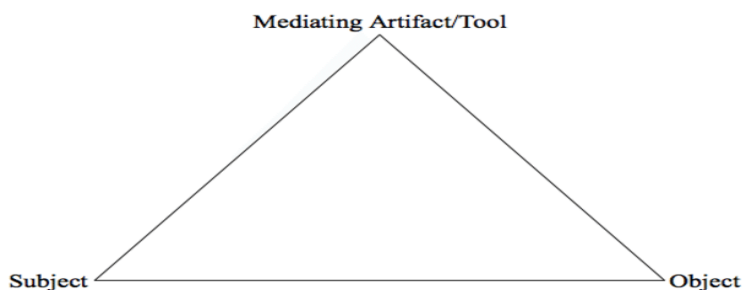


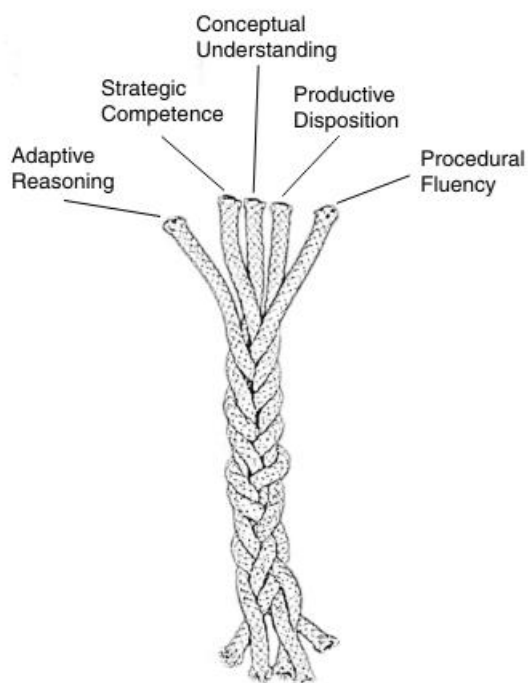
Figure 1: Vygotsky's idea of mediated action (Vygotsky, 1978)

developed over time, so too have the outcomes and possibilities in the fields of both mathematics and mathematics education.

In thinking about the relationship between mathematics and digital technologies, the field of mathematics operates a bit like a “hinge”. Mathematics is vital in the development of these technologies, but at the same time the practice of mathematics is influenced by the digital technologies it helped create. I can think of no other profession that is as profoundly influenced by this “hinge-type” relationship than teachers of mathematics. Mathematics teachers are simultaneously responsible for ensuring that learners have a robust enough understanding of mathematics in order to allow for continued technological innovation, and are simultaneously responsible for incorporating some of these same digital technologies into their teaching repertoire in order to advance their students mathematical understandings. This is not necessarily a brand-new problem as demonstrated from this quote from almost 30 years ago, “We must deal with the fact that computer symbolic algebra and computer interactive geometry are better—far better—tools than paper and pencil for doing many of the manipulations associated with mathematics (Waits & Demana, 1996, p. 712).” However, the array of newer technologies including those incorporating artificial intelligence and the pace of the development of these technologies make this a more pressing matter than ever and one that cannot be ignored or overlooked.

When contemplating the relationships between technology and the teaching of mathematics a myriad of questions arise such as, “When should I use technology in my classroom?”, “What technology should my learners and I use and why?”, “How reliant should my learners be on technology in order to complete calculations?”, “Are there any technologies that I should ban my learners from using and why?”; “How can I keep up with all of the new technological advances?”. While these, and others, are important questions, and I look forward to hearing input from many fine scholars on these, I want to raise a somewhat more fundamental question about mathematics and technology, namely, “what does it mean to do be mathematically proficient in this digital age, and how can I teach learners to reach this mathematical proficiency?” Historically, most school systems have focused on the teaching and learning of procedural fluency with computational algorithms. While this focus is historically understandable given the need for fast and accurate computational skills in light of the lack of technologies that could reliably be accessed, I argue that this sole focus is not sufficient to prepare learners mathematically in the digital age. Rather, we need a more comprehensive understanding of what it means to be mathematically proficient, such as the

view put forth by the United States' National Research Council in one of their seminal publications and mathematics education title "Adding it Up: Helping Children Learn Mathematics." This more comprehensive understanding takes the form of five interwoven and interconnected strands including conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (see Figure 2). This more comprehensive understanding of the nature of mathematical



- conceptual understanding—comprehension of mathematical concepts, operations, and relations
- procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- strategic competence—ability to formulate, represent, and solve mathematical problems
- adaptive reasoning—capacity for logical thought, reflection, explanation, and justification
 - productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

Figure 2: Strands of mathematical proficiency from Findell, Swafford, & Kilpatrick (2001, p. 116-117)

proficiency both prevents an overemphasis on procedural fluency alone, and provides learners with a more robust and well-rounded set of mathematical competencies that are necessary in this digital age. However, if we are going to support our learners in this achieving this more robust mathematical proficiency, we must also have robust pedagogies that allow for the simultaneous development of these competencies. I will focus on one such pedagogy that has demonstrated promise: project-based learning (PBL).

Description of PBL

Project-based learning is not a new concept and, in fact, has roots in the work of the American educator, John Dewey who at the turn of the 20th century was advocating for experiential learning (Pecore, 2015). In particular, Dewey was interested in not only what students knew, but what they could do with their knowledge. This sentiment remains at the core of modern PBL pedagogies and has been gaining in popularity again with the recognition that knowledge acquisition alone is not sufficient in the digital era where knowledge is abundant and easily accessed, but critical thinking and innovation in relation to what to do with this knowledge remains the pressing need of our time.

While a single consensus definition of PBL is not present in the literature, there is general consensus that PBL is more than simply the doing of a project or a hands-on activity. A useful framework that has gained traction is the essential elements of PBL as developed by Larmer, Mergendoller, and Boss (2010, 2015). In this framework (see Figure 3) seven essential elements of project-based learning are identified and include: a challenging question or problem, sustained inquiry, authenticity, student voice and choice, reflection, critique and revision, and a public product all centered on specific student knowledge, skill, and understanding learning goals.

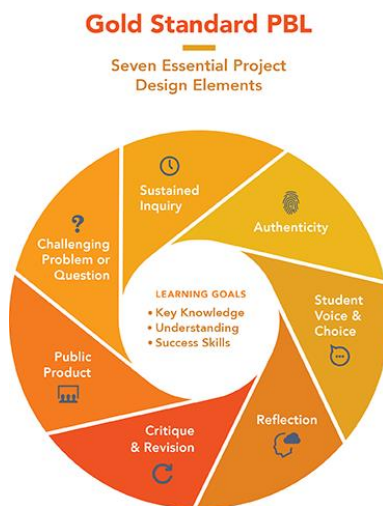


Figure 3: Framework for understanding PBL, from Larmer, Mergendoller & Boss (2015)

The driving question or challenging problem is at the heart of any project-based learning unit. It contains both the “what” of the project, what the project is about and is aligned to the learning goals of the unit and the “why” of the unit in that it should be relevant and of interest and importance to the learners that are going to engage with the unit. Sustained inquiry allows students to develop their knowledge of concepts relevant to the unit’s central

problem/question as well as develop necessary skills. Typically, multiple types of classroom structures such as independent work, group work, teacher-directed and student-centered activities will be utilized over the length of the project and learners will have multiple modalities to access information. Ideally, the PBL unit will be “authentic” in a number of senses of the word including authentic to the real world both in terms of concepts taught and skill developed, authentic to the discipline(s) that are incorporated in the unit, and authentic to the needs of the students and community. Throughout the unit the learners should have some voice and choice in how the project unfolds for example in determining what sources of information they incorporate, what the final product looks like, or structure of the teams. By incorporating learner voice and choice learners must think strategically and critically and if not given a chance to develop these skills in the safe environment of the classroom are less likely to do so in other aspects of their life. Learners also need time to reflect and self-evaluate their learning processes in order to develop the metacognitive skills they will need to be lifelong learners. The learning within a PBL unit is not meant to be a “one-and-done” model. Critique and revision are important parts of the process in that learners learn to both give and respond to intellectual critiques of work and productively use these to improve their understanding. Finally, PBL projects, as they are typically connected to a real-world issue, ideally will have some function or relevance outside the classroom. By having learners share their work and new knowledge with stakeholders of the problem they investigated, learners have a further chance to connect what they are learning in the classroom with important, broader aspects of their community and world beyond.

While this framework presents a somewhat idealized or “gold-standard” version of PBL it is important to realize that, not every PBL unit is suitable for including every aspect of the framework every time. Rather these components should be viewed as goal to be worked toward. Further, while these are worthy goals to aim for, they can be a bit abstract. Therefore, I present two vignettes of PBL units that were developed and taught recently: one at the primary school level and one at the high school level.

Examples of PBL

Vignette 1 – Primary school

Context: This week-long unit was developed for a Grade 1 classroom in a rural community in the western state of Washington in the United States. The location of this school where this PBL unit was taught was in a valley in the foothills of the Cascade Mountains that is known for its strong and frequent winds. Wind farms are scattered throughout the valley to harness the wind for electricity, but the wind also has the disadvantage of sometimes being destructive to trees and buildings in the community.

Driving Question: How can we design a house that will stay strong when it is windy?

Key Math Concepts (from Common Core State Standards):

1.MD.A.2 Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.

1.MD.C.4 Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

Additional integrated subjects: Literacy (3 Little Pigs story) and Science and Technology (engineering design process and properties of materials)

Brief description of the unit: In this unit, learners were first introduced to the project with the reading of the “3 Little Pigs” story. Learners were then asked to think about and share their observations of what it is like on really windy days in the valley, after which they were introduced to the challenge of needing to design and build a model of a structure that would withstand strong winds. Learners were shown a variety of materials such as cotton balls, sticks, leaves, Lego bricks, paper, cardboard, etc that they would have available to build their structure. Learners then worked together to test the materials and see what happened to them in the wind. A fan was set up in the classroom and “runway” was built from blocks of paper that were the same size. Learners held the various materials up to the fan to see how far down the “runway” that particular kind of material was blown. Learners at this point were still using non-standard units of measure, so they counted the number of paper blocks the object traveled and recorded this as their data. With the teachers help, the learners then created a class graph that visually represented how far each type of material was blown. During a whole class discussion learners were asked about the graph and how it connected to the data they had collected as well as comparing and contrasting different materials using the graph. Based on the results from this activity, students first designed, drew, and then built a structure that they thought would best withstand the wind. The structures were then tested with the fan and evaluated for their durability. Based on the results of the structure testing learners were asked to brainstorm modifications they could make to their original structure to make it even more durable in the wind.

Vignette 2 – High school

Context: This 4-5 week unit is designed for Grade 9 students in a community that borders on the North Cascades region of Washington state. Due to hunting and agricultural

development grizzly bears are no longer found in the North Cascades region of the state despite this region historically being prime bear habitat. A debate has arisen in the state regarding whether the grizzly bear should be reintroduced to this region. This particular unit was designed to be a combined math and science unit with students investigating ecosystems concepts in science and working with mathematical concepts in maths.

Driving Question: Should grizzly bears be reintroduced to the North Cascades?

Key Math Concepts (from Common Core State Standards):

Shapes and Centers of Distributions (HSS.ID.A.1, HSS.ID.A.2, HSS.ID.A.3)

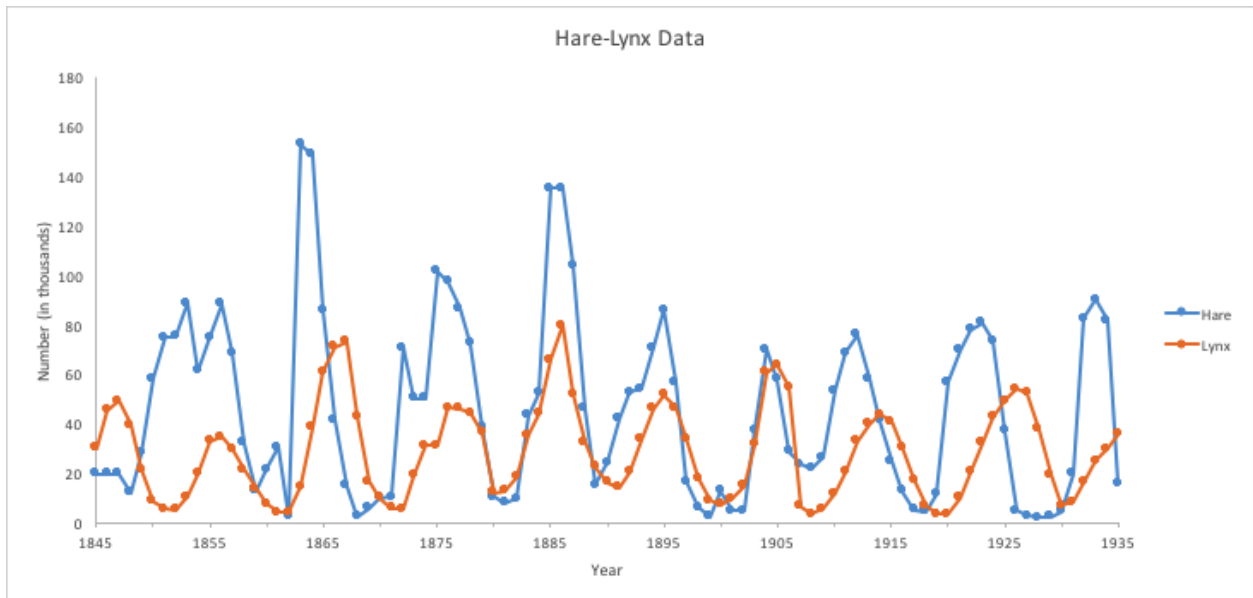
Distributions and their shapes, Describing the center of a distribution, Estimating centers and interpreting the mean as a balance point

Numerical Data on Two Variables (HSS.ID.B.6, HSS.ID.C.7, HSS.ID.C.8, HSS.ID.C.9)

Relationships Between Two Numerical Variables, Modeling Relationships with a Line, Interpreting Residuals from a Line, Analyzing Residuals, Interpreting Correlation

Additional integrated subjects: Life Science (Ecosystems including carrying capacity, predator/prey relationships, and keystone species). Literacy (persuasive writing and public speaking).

Brief description of the unit: At the start of this unit, learners are introduced to the problem, either by inviting a park ranger from the North Cascades to come talk about grizzly bears and their historical presence in the North Cascades or through a short video from a bear biologist that covers the same material. Learners were invited to share their particular knowledge and initial perspectives on bringing back the grizzly bear – this tends to surface both conservationist perspectives and economic perspectives in terms of impact on ranching operations. Learners were then introduced to the case study of the reintroduction of wolves to Yellowstone National Park. They receive actual (messy) population data from a variety of species in the park from before and after the reintroduction of wolves. After being introduced distributions and calculating means (and practicing these skills), learners return to the data to make claims about the impact of wolves on a variety of species including cattle populations surrounding the park. After learners have studied predator/prey relationships in science class they are given data (see Figure 4) to help quantify this relationship and develop equations to model the relationships



between the numbers of predators and prey. Learners are then asked to take a position on whether or not they think grizzly bears should be reintroduced (or assigned if there isn't enough difference) and are tasked with participating in a public debate for the community that addresses both the advantages and disadvantages of grizzly bear reintroduction based on concepts discussed in science class and the data they studied and modeled in maths class.

Efficacy of PBL learning

As educators we are aware that there is no magic teaching strategy that works for all learners and solves all problems in all contexts and situations, and PBL is no exception. Not all situations and learning goals are appropriate for incorporating PBL. However, a brief overview of the literature suggests that PBL is quite effective at promoting particular kinds of learning including a more expansive mathematical proficiency rather than procedural fluency exclusively. In an early study of maths learning using PBL pedagogies, Jo Boaler (1998) found during a three-year case study of two different high schools that students taught using PBL methods scored significantly higher in application tasks and measures of conceptual understanding in addition to higher overall scores on their high school leaving exams. Additionally, students in the PBL high school reported more positive attitudes toward and enjoyment of mathematics and mathematics learning. Also of interest, while there was a gender disparity in mathematical interest and confidence in the more traditional high school with male students reporting higher levels than females, this disparity was not

found in the PBL environment with male and female students reporting equally high interest and confidence in mathematics. Additional studies (see Kokotsaki, Menzies, & Wiggins, 2016 for a thorough review of the literature) have similarly demonstrated the efficacy of PBL pedagogies. As another example, Geier et al. (2008) studied grade 7 and 8 science students in a school district in the United States for three years. Those engaged in the PBL curriculum achieved increased content knowledge and process skills as well as significantly higher scores on the state exams as compared to peers in the district that engaged in the more traditional science curriculum.

While these and other studies show promising results for robust and expansive learning in PBL classrooms, there is another important aspect of learning in many of these studies that worth calling attention to: “who is learning and benefitting from PBL methodologies?” In the Boaler study (2008), PBL methodologies seemed to be supportive of reducing gender disparities that are often found in mathematics. Likewise, Halvorsen, Duke, Brugar, Berka & Brown (2012) found that learners from lower socio-economic status (SES) had less of an achievement gap as compared with their higher SES peers when learning using PBL methodologies and Doppelt (2003) has documented efficacy of PBL learning with students labeled as low achieving. These studies suggest that PBL is a promising pedagogy for both promoting learning and supporting learners that may have been historically underserved in traditional classrooms.

Conclusion

As we continue to think about what it means to teach and learn mathematics in the digital age, we must recognize that learners today, largely driven by rapidly developing new technologies, are going to face work and life situations unlike those any other generation has faced. As we think about preparing learners for this future, we need to identify and implement pedagogies that are most likely to help prepare them for this complex and somewhat uncertain future. While not all of our learners will go on to seek employment as a math or STEM professional, all of our learners do and will continue to operate in a digital world. In order to make informed decisions as voters, consumers, healthcare patients, workers, parents, etc, it is incumbent on us to utilize teaching methods that are going to help them navigate and make sense of the many quantitative situations they will encounter throughout their lifetimes. PBL, while not an educational panacea, is one such pedagogy that shows enormous promise in supporting the mathematical proficiency needs of today's learners.

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ENSURING INCLUSIVE LEARNING OPPORTUNITIES IN THE DIGITAL SPACE- DIGITAL MATHEMATICS FOR ALL

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There is no denying that technology changed the state of education. This presentation explores the unique learning needs and strengths of right-brained learners in the context of mathematics education. While traditional approaches to teaching mathematics often emphasize logical and analytical thinking, some learners are naturally inclined towards creativity, intuition, and visual-spatial reasoning—hallmarks of right-brained thinking. Drawing on current research this talk will examine how to integrate technology in mathematics to accommodate and empower right-brained learners in the mathematics classroom and make it more meaningful and interesting to everyone. We should all embark on a journey to nurture the next generation of innovative mathematicians.

INTRODUCTION

Mathematics is one of the most important subjects a learner can learn. Proficiency in mathematics opens doors to a wide range of career opportunities in STEM-related industries, where quantitative skills are highly valued and sought after. In our rapidly evolving world, the integration of digital technologies has reshaped virtually every aspect of our lives. From communication to commerce, from education to entertainment, the digital revolution has brought about unique opportunities and challenges. In the field of mathematics, this change is equally significant.

Imagine a classroom where students engage with mathematical concepts not just through textbooks and lectures, but through interactive simulations, virtual manipulatives, and immersive experiences. Picture a world where mathematical problem-solving is not confined to the pages of a workbook, but extends into virtual environments where students collaborate, experiment, and explore in real-time. This is the promise of reimagining mathematics in a digital space.

However, in my view, transformation can only occur when we grasp the concept of the 4—and 8-dimensional brain, which Dr Kobus Neethling explains through his work with The Neethling Brain Instrument (NBI) (Neethling et al., 2021). In Table 1 below, Dr Neethling highlights the varying dominance of left and right brain functions among learners, with some favouring one hemisphere over the other and others maintaining a balance.

Table 1. 4- and 8-Dimension Brain Summary

L1	R1
Teaching: textbook, precise instruction, logical arguments, opportunity to analyse, discuss technical aspects. Learning: at desk, supplies handy summaries, research. Environment: neat, info available, blackboard work accurate, quite.	Teaching: holistic view, spontaneous participation, visual, associate contents with past/future, fun, experimentation, new concepts Learning: seldom at desk, diagrams, learning games, tries new methods, rarely timetable, mind wanders. Environment: colourful, challenging info, humorous exhibitions, artistic, changing
L2	R2
Teaching: textbook, chronological & well-planned lessons, clear instructions, repetition, clear lesson objectives, formal lessons, Learning: at desk, detailed summaries, practices subject-matter, practical application, timetable for studying. Environment: programs etc. handy, detailed blackboard work.	Teaching: group work, movement, music, associate contents with personal experiences, show emotions, acting as teacher Learning: amongst people, emotional about contents, music, talks loudly to memorise, non-verbal communication, role play. Environment: room for movement, music, person-oriented, exhibitions, friendly, colourful, comfortable.

Overall, a whole-brain conceptualization of mathematics education recognizes the interconnectedness of cognitive processes and seeks to optimize learning by engaging both hemispheres of the brain in a balanced and integrated manner. By incorporating creative activities, multisensory experiences, real-world contexts, and digital resources, educators can create enriching mathematics learning experiences that empower students to become confident, proficient, and lifelong learners.

Traditional mathematics education often emphasizes left-brain functions such as logical reasoning and analytical thinking. By integrating creative activities, visualizations, and problem-solving approaches, educators can activate the right hemisphere of the brain, fostering a more holistic understanding of mathematical concepts.

Therefore, it becomes imperative to adapt our teaching methods by flipping classrooms to accommodate the diverse learning styles of all students, especially in the field of mathematics. Every child possesses unique talents and abilities. It is the responsibility of teachers to recognize and harness the latent potential within each child.

According to (Hoyles and Lagrange, 2010) substantial body of research has been done on how particular teachers in particular settings have integrated particular technologies within their classroom settings. Numerous studies and trends also indicate a growing inclination towards integrating technology in the teaching of mathematics. “We need technology in every classroom and in every student and teacher’s hand,” says David Warlick, “because it is the pen and paper of our time, and it is the lens through which we experience much of our world.” Amongst other authors many years ago John Dewey also mentioned that “If we teach today as we taught yesterday, we rob our children of tomorrow.”(Dewey, 2011). Chamberlain as cited in (Al-Haj, 2020) indicates that “Education is evolving due to the impact of the Internet. We cannot teach our students in the same manner in which we were taught. Change is necessary to engage students not in the curriculum we are responsible for teaching, but in school. Period.”

These statements are affirmation that we are in the right track to want to digitalize the teaching of Mathematics. Digital classrooms allow both educators and learners to grow beyond linear, textbook-based learning.

EXAMPLES

Few examples will be shared to try and demonstrate the challenges that learners can encounter if taught a concept without any visual demonstration:

1. Let us consider fractions.

In a traditional classroom setting, fractions are often taught using textbooks, chalkboards, and worksheets. The teacher introduces fractions as parts of a whole, drawing circles or rectangles on the board and dividing them into equal parts to represent fractions. Learners are asked to identify fractions, compare their sizes, and perform basic operations such as addition, subtraction, multiplication, and division of fractions using pencil and paper. The teacher demonstrates fraction concepts using static diagrams and encourages rote memorization of fraction rules and procedures. Some learners may struggle to grasp abstract fraction concepts or visualize fractions as they rely solely on static representations and verbal explanations.

Common mistakes learners make when dealing with fractions:

- Learners may forget to find common denominators when adding or subtracting fractions with different denominators, resulting in errors in calculations. Without a common denominator, it is challenging to accurately combine or compare fractions.

We can assist those who might struggle to fully understand the concept by incorporating digital tools and resources that provide interactive and multisensory learning experiences. The teacher brings in educational software, online simulations, and interactive whiteboards to engage students in exploring fractions in dynamic and interactive ways. Students use virtual manipulatives, such as fraction bars or pie charts, to visually represent fractions and manipulate them to understand fraction concepts intuitively.

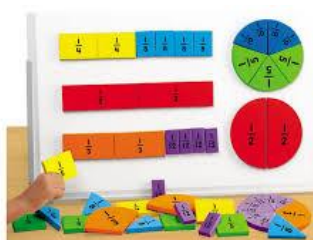


Figure 1

They interact with animated demonstrations that illustrate fraction operations and concepts, such as equivalent fractions, improper fractions, and mixed numbers, in real-time. Digital platforms provide immediate feedback, hints, and scaffolding to support students as they practice fraction skills and solve problems collaboratively. Students can access online tutorials, instructional videos, and interactive games that cater to their individual learning styles and pace. Technology facilitates differentiated instruction, personalized learning pathways, and collaborative learning experiences that promote deeper understanding and mastery of fraction concepts for all students.



Figure 2

Ensuring that we assist learners in understanding the concept of fractions better by using digital tools is important because, in real life, they will encounter numerous situations where this knowledge is essential. Whether they are managing money, calculating time, measuring ingredients in recipes, or comparing prices, a solid grasp of fractions is necessary to navigate these everyday tasks effectively. Digital tools can provide interactive and engaging ways to learn fractions, making it easier for students to understand and apply these concepts in practical scenarios.

2. Limits

Limits are of fundamental importance in mathematics, particularly in calculus, and they play a crucial role in various mathematical disciplines and real-world applications. Limits serve as the foundational concept of calculus, which is one of the most important branches of mathematics. Calculus relies heavily on the concept of limits to define derivatives, integrals, and other key concepts that are essential for understanding rates of change, accumulation, optimization, and modelling in mathematics and science. Limits are used to define derivatives, which represent rates of change of functions. Derivatives play a crucial role in calculus, physics, engineering, economics, and many other fields, as they provide insights into how quantities change with respect to one another and enable the modelling and analysis of dynamic systems.

Informally a limit can be defined as a value that a function or sequence approaches as its input approaches a certain value, but it may not actually reach that value. It's like getting closer and closer to a destination on a map, but never quite arriving at the exact spot. The tricky part here is getting *closer and closer and closer to but never reach that value*.

Example1: Consider this example and its graph $f(x) = \frac{x^2-4}{x-2}$. This graph assists in visualizing the idea of getting closer and closer without actually arriving. The hole emphasizes that this function is undefined at $x = 2$ because it would involve dividing by zero, which is not allowed.

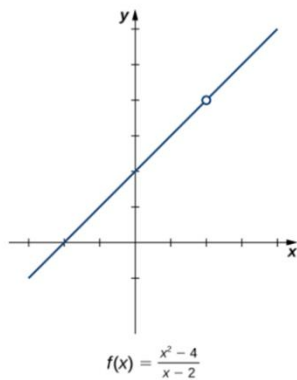


Figure 3. The graph shows the behaviour of $f(x)$ around $x=2$

Example2: Suppose you wanted to cross a bridge safely and smoothly. What three things would you need? Would you be able to do so in the situation depicted in the picture? What is missing?

(The answer will be demonstrated on the presentation)

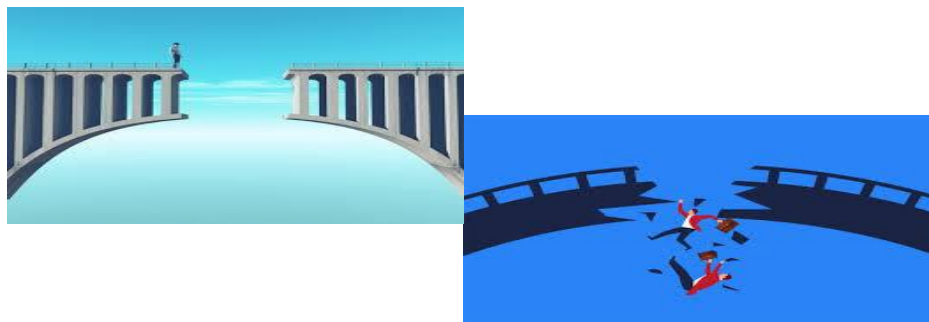


Figure 4. Broken bridge to demonstrate the concept of Limits.

3. First Principle of Differentiation

The first principle of differentiation provides a systematic approach for finding the slope of a curve at a specific point, allowing us to analyse the behaviour of functions, solve optimization problems, and model real-world phenomena in various fields such as physics, engineering, economics, and biology.

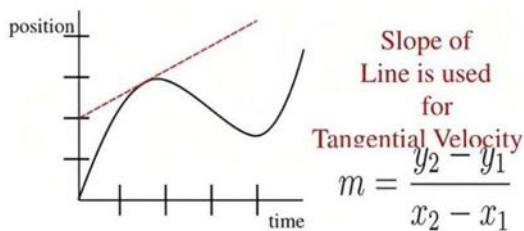
Understanding the theory behind first principles differentiation is crucial for developing a solid foundation in calculus and mastering the concept of derivatives. It forms the basis for more advanced techniques and applications of differentiation in mathematics and its related disciplines.

Example 1: Let us consider a car traveling along a straight road. We want to find the velocity of the car at a particular instant t seconds after it starts moving.

Let us denote the position of the car at time t seconds by the function $x(t)$, where x is the distance travelled (in meters) and t is the time elapsed (in seconds). We measure the position of the car from a fixed point on the road.

We want to find the velocity of the car at a specific moment, which is the rate of change of its position with respect to time. We can use the first principle of differentiation to calculate this velocity. The difference quotient is given by v_{inst} in **Error! Reference source not found.**:

•Instantaneous velocity



$$v_{ave} = \frac{\Delta x}{\Delta t}$$

$$v_{inst} = \lim_{t \rightarrow 0} (v_{ave}) = \lim_{t \rightarrow 0} \left(\frac{\Delta x}{\Delta t} \right) = \frac{dx}{dt}$$

Figure 5. Instantaneous Velocity

Instantaneous velocity is important in various real-world scenarios, such as monitoring vehicle speed, analysing motion in sports, predicting traffic flow, and designing transportation systems. Understanding how to calculate instantaneous velocity using first

principles differentiation allows us to make informed decisions and solve practical problems in physics, engineering, and everyday life.

Solution: By taking the limit as Δt approaches zero, we obtain the instantaneous velocity of the car at time t . This represents how fast the car is moving at that exact moment, without any averaging over a time interval.

Example 2: Optimization using first principles in calculus is finding the maximum or minimum value of a function, which often arises in various real-world scenarios. Let us consider a simple example:

Suppose a company manufactures and sells a product, and the total profit (in rands) generated by selling x units of the product is given by the function:

$$f(x) = 2x^3 - 15x^2 + 24x - 7$$

This function represents the profit as a cubic function of the number of units sold x . The company wants to determine the number of units it should produce and sell to maximize its profit.

Solution:

- *Thus, the company's maximum profit is 4 rands when selling 1 unit, and the minimum profit is -23 rands when selling 4 units. The negative profit value reflects that the costs associated with producing and selling 4 units outweigh the revenue generated, leading to a financial loss.*
- *The visualization will be done using the digital tool “Desmos.”*

By using Desmos, we can quickly and accurately visualize the profit function and determine that the maximum profit of R4 occurs when 1 unit is sold. This digital tool not only helps in solving the problem but also enhances understanding through interactive visualization.

CONCLUSION

Ensuring inclusive learning opportunities in the digital space underscores the imperative of creating an equitable educational landscape where every learner, regardless of background or ability, can thrive in the digital space of mathematics. This presentation explored the transformative power of digital technologies in democratising access to mathematical learning. From interactive simulations to collaborative online platforms, these digital tools hold the potential to engage, empower, and inspire learners of all ages and abilities.

By adopting inclusive and innovative approaches that recognize and celebrate the strengths of right-brained learners, educators can create enriching mathematics learning experiences that foster creativity, curiosity, and mathematical proficiency for all learners.

However, we must also acknowledge the inherent challenges and disparities that exist within the digital learning environment. We must remain vigilant in addressing issues of accessibility, digital literacy, and socio-economic barriers to ensure that no learner is left behind. Moloi and Matabane suggest that we should also imagine mathematics classrooms in digital space not only to provide sustainable education, but also to promote core values and qualities of social justice (Moloi and Matabane, 2020).

Moving forward, it is essential that educators, policymakers, and stakeholders work collaboratively to design inclusive digital mathematics experiences that cater to diverse learning styles, preferences, and needs. By embracing innovative practices, enhancing digital literacy, and promoting an inclusive culture, we can harness the full capabilities of digital mathematics. This approach will create a future where all learners can excel, prosper, and make significant contributions to society. Together, let us commit to the vision of "Digital Mathematics for All" and strive towards a more inclusive, equitable, and empowering educational landscape in the digital age.

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IS ABSTRACT MATHEMATICS ABSTRACT?

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Abstract mathematics is a very active field where mathematicians refine and create new concepts within mathematics. The field is extensive, with many well-studied areas and branches. One of the earliest known areas of abstract mathematics is Euclidean mathematics, specifically the axioms of plane geometry taught in high schools worldwide. Additionally, high school mathematics classes cover abstract concepts such as matrices, linear algebra, topology, polynomials, cartesian coordinates, and other mathematical objects. This helps students understand geometric and algebraic structures to solve theoretical or real-world problems. However, for many students, mathematics remains too abstract and appears as a disorganized collection of rules, tricks, and curiosities. We investigate whether some abstract mathematics problems can be explained using real-life problems to make them more accessible and help students appreciate very abstract concepts. According to Dean Schlicter, “Go down deep enough into anything, and you will find mathematics.” We delve into the daily activities of a fruit market to explain an abstract problem in topology at tertiary level. Our goal is to determine if this approach can be applied to as many abstract mathematics problems as possible. Finally, we track down and related primary and high school mathematics problems which can be related to the daily activities of a fruit market.

Introduction

What is mathematics? Students may answer this question differently depending on their educational level. In the early stages, students might view mathematics as a subject dealing with numbers, then later as involving shapes and numbers, and eventually as encompassing numbers, shapes, logic, quantity, and arrangements. When traditional numbers are replaced by other mathematical structures, the gap between abstract mathematical concepts and real-world problems widens, making mathematics seem inaccessible to many.

It is interesting to note that while the tangible results produced by engineers or the emotive impact of music, though based on abstract concepts, are easily appreciated and understood, the beauty and significance of abstract mathematical concepts might not be immediately evident. This discrepancy in appreciating abstract mathematics, compared to fields like engineering or music, often stems from its perceived disconnect from everyday life. The role of a mathematics educator is to bridge this gap, making mathematical concepts more tangible and relevant to students.

One method is to contextualize abstract mathematical concepts within familiar scenarios or practical applications. Demonstrating how abstract ideas like matrices or topology can be applied to solve real-life problems or model everyday phenomena helps students develop a deeper appreciation for the power and versatility of mathematics. A second method involves explaining the similarities between a mathematical problem and a real-life problem solved without mathematical tools. This approach makes students more curious and engaged, encouraging them to solve the problem and perhaps even devise their own solutions.

In this paper, we demonstrate the second approach by exploring activities in a fruit market. We examine the problems being solved and the tools used. We then introduce a topological structure called a knot and show how the challenges faced by knot theorists are similar to those encountered by a fruit seller, requiring similar tools for resolution. In addition, we give a very brief highlight, how this fruit market problem is similar to a problem in primary school mathematics. We conclude by questioning, “Is abstract mathematics really abstract?”.

Fruit Market Problem

Consider a fruit shop that has ordered various fruits from a farm. The farmer delivers a container full of mixed fresh fruits to the shop, as shown in the following picture.



FIGURE 1: MIXED FRUITS DELIVERED TO THE MARKET

It is obvious that the first problem the fruit shop has to deal with is sorting the fruits to make them accessible to buyers. There are many approaches to sorting these fruits, and some may not be the best. The best method might not group all fruits perfectly. The first approach is to sort by the color of the product. For example, in a green basket, you might find green apples, green grapes, some pears, etc. However, most buyers will not be satisfied with this method. The second approach is to sort by name, placing all apples in one basket, all grapes in another, and so forth. But this raises a question: Should red apples and green apples be in the same basket? Many buyers would also find this approach unsatisfactory. Consider this: Sorting fruits may seem like a straightforward task, yet it presents a very abstract problem that requires careful thought to achieve satisfactory results, as illustrated in the following picture.



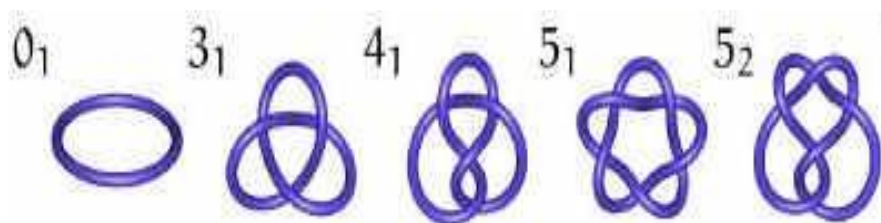
FIGURE 2: SORTED FRUITS AT THE MARKET

Consider the same problem, but now with a twist, the fresh fruits delivered to the fruit shop are wrapped so that one cannot see the fruit inside the wrapper. How do we sort these wrapped fruits? Thinking outside the box is necessary to solve this problem. One potential solution is designing a chemical test that reacts with the fruit and changes the color of the wrapper. Let's assume we develop two tests: one test that predicts the color of a fruit, say, red, green, yellow, or orange and another test that determines whether a fruit is citrus or non-citrus. After testing, wrapped fruits with the same color will be placed in one basket. However, similar to the first problem, sorting by color or citrus type is not entirely satisfying.

What have we learned from these two problems? When thousands of fruits are delivered, the first task for the shop is to sort the fruits according to the preferences and specifications of the shop. It is evident that fruits in different baskets are distinct. However, the fact that fruits are in the same basket does not mean they are identical. Therefore, the shop needs to conduct additional tests to further distinguish the contents of each basket. Not all items delivered and sorted were fruits.

Mathematical Knot Problem

Consider a simple piece of string. Tying a knot and then gluing the two loose ends together forms what is called a knotted loop in mathematics. A mathematical knot differs from the common notion of a knot, which typically involves a piece of string with free ends. When two or more disjoint knots are combined, the result is referred to as a link. To ease notation, in this paper, a link will just be called a knot. We recommend the reader refer to [1] for a comprehensive introduction to knot theory.



We can compare the challenge of sorting fresh fruits with the task of sorting knots to make the theory of knots more accessible to students. Just as a truckload of fruits from a farm presents a sorting challenge, a knot theorist faces millions of knots that have been cataloged by mathematicians. The first step, similar to that of the fruit shop, involves sorting the knots in a way akin to addressing the wrapped fruit problem. Knot theorists must think creatively to develop tools for distinguishing knots. These tools, known as knot invariants, help differentiate knots according to specific criteria.

However, just because a knot invariant produces the same result for two knots does not imply that the knots are identical. Similarly, just because two fruits share the same color does not guarantee they are the same type. We direct the reader to references [2, 5] for a concise overview of knot invariants and the algorithms used in their determination. One popular way of sorting knots is placing knots with the same number of components in the same basket. Thus, component number is one of the well-known knot invariants in topology

The knots shown in Figure 1 have different component numbers. By inspection, it is clear that one can count the number of components by traversing the knot and counting the number of closed strings. This is possible because the knots in Figure 1 are very small and traversing the knot is possible making the problem similar to the unwrapped fresh fruit problem. Imagine if the knot is very big like a fisherman's net, traversing the knot will not be efficient and straight forward. Thus, the problem is similar to the unwrapped fresh fruit problem. Hence, there is a need to develop a test which will give the number of components of the knot. We give a brief outline of one of the tests developed in knot theory to count the component number. For full details, we refer the reader to [3].

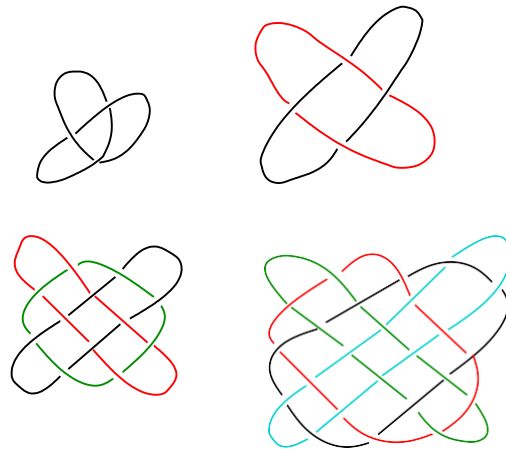


Figure 4. Example of Knots with different component numbers.

A graph G consists of a non-empty set of elements, called vertices and a list of unordered pairs of these elements, called edges. It has been shown in the literature that each knot diagram has a corresponding planar graph, see [1]. Thus, we can extract a planar graph G from a knot diagram denoted $K(G)$ as illustrated in Figure 2.

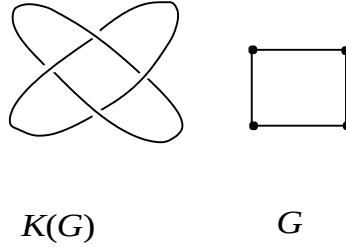


Figure 5. A knot diagram $K(G)$ and the corresponding planar graph G .

Mphako [3] introduced an algorithm which is used as a tool for calculating the component number of a knot by defining a special number for each knot which the component number can be extracted from.

Definition 3.1. We define, for an unoriented knot diagram $K(G)$, a number $L(K(G)) = (-1)^{|E(G)|}(-2)^{c-1}$, where c is the component number of $K(G)$ and $E(G)$ is the edge set of G . The algorithm starts with a knot, extract the planar graph of the knot and then apply any of the operations outlined in the following three theorems depending on the extracted graph.

Theorem 3.2. *Let G be a planar graph. Then*

- (a) $L(K(G)) = (-2)^{n-1}$, if G is a graph with n vertices and no edges, thus
 $L(K(G)) = 1$, if G is a single vertex;
- (b) $L(K(G)) = -1$, if G is a bridge or a loop;

Theorem 3.3. *Let G be a planar graph. Then*

- (a) $L(K(G)) = -L(K(G \setminus e))$, if e is a loop;
- (b) $L(K(G)) = -L(K(G/e))$, if e is a bridge; and
- (c) $L(K(G)) = L(K(G \setminus e)) + L(K(G/e))$, if e is neither a loop nor a bridge.

Theorem 3.4. *Let G be a planar graph. Then,*

- (a) $L(K(G)) = L(K(G/e/f))$ if e and f are a series pair; and
- (b) $L(K(G)) = L(K(G \setminus e/f))$ if e and f are a parallel pair.

Brief Application of the Fruit Market Problem to School Mathematics

We can compare the challenge of sorting fresh fruits with the task of sorting different mathematical structures at various educational levels. The progression of sorting tasks in mathematics education become more complex as students advance. At each level, learners encounter new types of sorting tasks that build upon their previous knowledge. Just as sorting fruits involves categorizing them based on different criteria such as size, color, or type, sorting mathematical structures involves categorizing them based on properties like parity, sizes, or geometric attributes.

As an example, in primary school, learners are given a set of integers just like the fruits delivered from a farm to a fruit market, then the sorting game begins. The sorting at this level can take different forms depending on the objectives of the lesson. A teacher can ask learners to identify and place same integers in the same basket i.e. placing all 1's in one basket, 2's in another basket etc which is similar to placing fruits of the same type in the same basket the sorting in the fruits market problem. This is visual just like the unwrapped fruits problem. Fast forward, the sorting of integers in primary school continues to even and odd numbers, whole numbers and fractions, multiples of 2's, 3's, and many more. This is similar to the wrapped fruit problem in the fruits market problem. Hence, the mathematics educators design different tests which learners can use to sort the integers accordingly.

An example of sorting problem in high school geometry is giving learners a set of triangles to identify right-angled triangles and place in one basket and non-right-angled triangles in another basket. But this is not the best sorting because isosceles triangles will be placed in different baskets just like the red and green apple problem.

Overall, the analogy of the fruit market problem highlights the connectedness of mathematical concepts and the importance of progressively building upon foundational knowledge to tackle more advanced problems effectively. It also highlights the need for educators to carefully design sorting tasks that challenge learners while promoting a deeper understanding of mathematical concepts.

Conclusion

Mathematics content is categorised using certain aspects of the problems being solved. If one looks deeper, a big chunk of mathematics problems solved from grade one to undergraduate level fall in the same categories, only that at each level learners deal with different mathematical structures. If each category of mathematics problems can be translated or related to a real-life problem which can be solved by a layman without any mathematical tools, this will be a game changer. Mathematics educators will be well informed of the problems they are teaching at each level then communicate to learners with ease. This is just one way abstract mathematics will be appreciated. In addition, mathematics educators will be introducing mathematics in a way that highlights the continuity and connections between different levels which will certainly enhance students' understanding and appreciation of the subject. By framing problems as variations or extensions of concepts they have encountered before, educators can help students build a more cohesive mental framework for mathematical thinking. Regarding the accessibility and enjoyment of mathematics, making these connections explicit could indeed be beneficial. Recognizing that they are building upon familiar ideas can boost students' confidence and motivation. Moreover, understanding the underlying structure of mathematical problems can help students see the beauty and elegance of the subject, making it more enjoyable. As for the question of whether abstract mathematics is truly abstract, it depends on how one looks at it, considering that most of it has roots in tangible concepts from grade one and can ultimately be understood in terms of more concrete ideas.

Acknowledgments

I would like thank AMESA for the invitation and support to speak at the 29th Annual National Congress, 2024. This is an exciting opportunity for me, not only for sharing insights but also connecting with other mathematics educators and researchers. This has led me to step out of my comfort zone and open my eyes to new avenues for exploration and growth in my research and professional development.

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FINDING X AND Y USING AI

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There is no doubt that the dawn of Artificial Intelligence (AI) has brought with it as much excitement as it has anxiety as all spheres of life are challenged to think about what is possible with the development of new technologies. The impact of AI in society is a certainty although it remains to be seen whether the effects will be positive or negative, Kiemde and Kora (2022). This uncertainty of the impact is behind the mixed emotions about this new era of modern world living. Professionals across global industries are being called to consider the core elements of their work by looking carefully at what can be automated and what can not. Teaching, as one of the oldest professions, is not exempt from this progression. This analytical research paper seeks to explore what teachers can do to ensure they stay abreast of technological advancements and harness AI to enhance Mathematics teaching in the classroom.

Fundamentals of AI

Artificial Intelligence has been a topic of discussion from as early as the 20th century, Turing (1950). Its definition and understanding remain in refinement but broadly it is believed to be the branch of computer science that relates to machines that can evaluate how to perform effectively in a wide scope of situations, Russell & Norvig (2021). Early researchers such as Turing considered AI the process of machines learning what they have been taught and the machine in turn reproducing and recreating that outcome repeatedly. This process of input and refinement makes the machine a “better student” and allows it to work from a broader range of data to confirm its algorithm, specifically in the generative pre-trained transformers (GPTs). Artificial intelligence has been developed across fields outside of computer science from general learning and reasoning to particular tasks such as proving Mathematical theorems, operating a vehicle Russell et al., (2021), or even writing a research paper on AI for a Mathematics conference.

In the field of Medicine, Artificial Intelligence has contributed to research in disease treatment and prevention. Although a new branch of medicine, AI-related healthcare gained prominence during the COVID-19 pandemic, when the United States of America invested four times as much financial resources in AI-driven medical and healthcare research in 2020 than in 2019, Saboowala (2021). One of the principles of AI is the ability to solve problems by searching and analysing many possible solutions Russell et al., (2021). There exist two categories of search generation in AI namely, state space search and local search. The purpose of a state space search would be to work through possible states to find the goal state. State space search has application in AI planning where planning algorithms search through trees of goals to find the path that leads to the target goal.

Russell et al., (2021), see binary search tree below. Local search entails the use of mathematical optimization to solve a problem. The process begins with a form of estimation and approximation and is then refined incrementally until the desired solution is reached Russell et al., (2021). AI's ability to search and analyse solutions for large data sets at unprecedented speed has allowed researchers to accelerate treatment in diseases like Parkinsons" to name a few. This bodes well for further development in global dread disease research and overall health and well-being management of the human population.

Figure 1 - Search tree. (2024, January 6). In *Wikipedia*. https://en.wikipedia.org/wiki/Search_tree

The rate of development of AI

There is no doubt that AI is growing rapidly, and the potential seems limitless. There is large-scale development in the research of AI as well as in its consumer uptake. It has been seen as the most transformative technology of the 21st century Verma (2023). In a Reuters article of 2023, platforms such as ChatGPT averaged about 13 million unique visitors per day two months after its launch. This made ChatGPT the fastest-growing consumer application in history. Experts in AI such as Kai-Fu Lee predict that the impact of AI will be “more than anything in the history of mankind” Russell et al., (2021). The potential of AI lies in its advancement across various fields including medicine as detailed above as well as entertainment, education, and transportation Verma (2023). This wide application and use of these advanced technologies, is was has enabled AI to be integrated into society.

AI in education

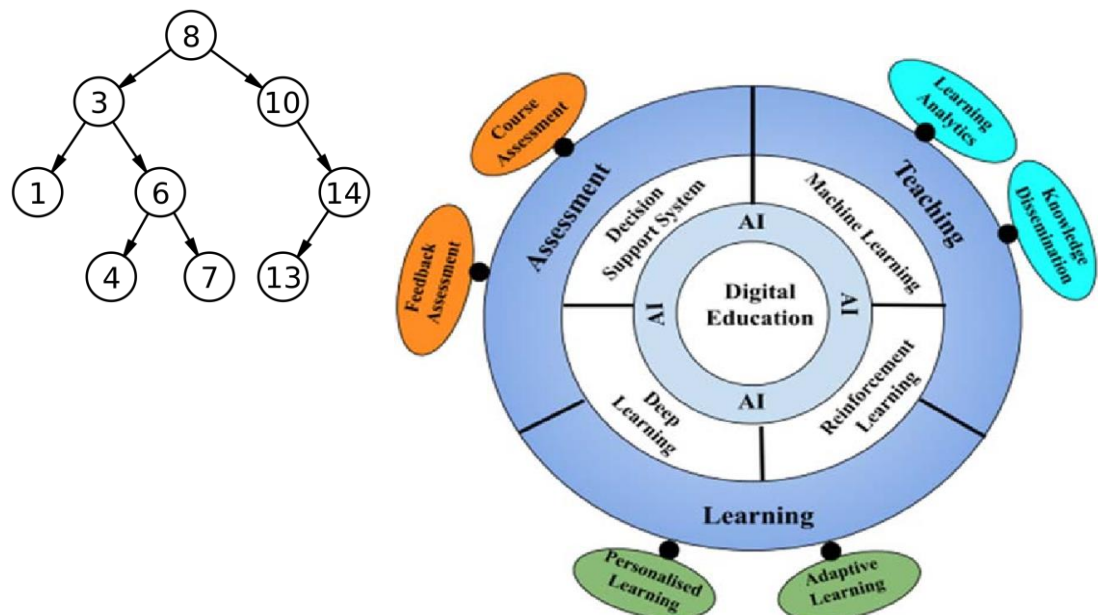


Figure 2 Role of AI in education. AI, Artificial intelligence. Masengu et al., (2023)

As stated earlier, teaching is not exempt from technological changes in society. AI has a direct impact on education policies linked to assessment, administration, and teaching. In recognition of the role of AI in education, UNICEF developed the Global Insight Policy on AI for children (2021). The document looks specifically at the role of key stakeholders in ensuring equitable access and protection of the rights of children. In their research, Holmes &

Tuomi (2022) presents Artificial Intelligence in Education (AIED) as a network of tools and applications that can be classified into three different yet overlapping categories of student-focused tools, teacher-focused tools and institution-focused tools. Teaching and learning should embrace AI with the values of the UNICEF policy in mind. The diagram above, by Masengu et al., (2023) illustrates the potential of AI to support educational processes such as teaching, assessment and learning. The inherent adaptiveness of AI enables dynamic teaching and learning.

AI in Mathematics Education

Since the dawn of the internet, teachers have been worried about the integrity of student work. Search engines such as Google have enabled effortless access to information for students since 1998, Carr (2006). This highlights that students have had some form of non-human-assisted learning for the last 20 years and more, thus the rapid development of AI in education should be considered a natural progression from the last 25 years of similar technologies as opposed to a new threat to teaching that teachers should fear. Although there is evidence to support that learning can be aided with the use of technology, research has shown that the role of the teacher cannot be replaced.

With the assurance that their role in the classroom is not under threat, teachers should look to embrace the opportunities presented by AI to further the potential for teaching and learning while challenging the philosophy, skills and content, and assessments in Mathematics education. The AI tools that I have engaged focus on simplifying complex Mathematical concepts which begs the question, what are we trying to achieve when we teach Mathematics? The philosophy of Mathematics as well as the skills and content necessary in a modern world are conversations that teachers and policymakers should be having.

AI as an enabler

UNICEF defines AI as machine-based systems that when given a set of human-defined objectives can make predictions, recommendations, and decisions, UNICEF (2021). Minsky (1969) concluded that AI is research that develops technologies that would require intelligence if done by humans. These views provide alternative thinking when it comes to AI technologies as we recognise that they are not machine learners but that we are their human teachers. The objectives, predictions, and recommendations are defined by humans, placing human intelligence at the centre of artificial intelligence. To enable our students to engage in AI, as teachers and schools, we need to equip them with the knowledge and skills of how it works, Luckin and Holmes (2016). AI development provides a branch of computer science that should

be explored as a school subject. In an increasingly digital world and job market, the skill of digital literacy is paramount to student success.

AI serves as an enabler by allowing students with barriers to learning to access knowledge. Students who require differentiation in the classroom can access AI tools that enable learning that suits them. Applications such as speech-to-text support those with speech, language and hearing impairments. Generative Pre-Trained Transformer programmes such as ChatGPT allow students who struggle with idea conceptualisation to gather information, those who struggle to articulate their ideas succinctly to have their work summarised and rephrased and those who are not English first language speakers to convey their work in a grammatically correct manner. Where there are knowledge gaps in subjects such as Mathematics, students can go back and reinforce concepts. The material can also be scaffolded to match the student's skill set.

AI as a disabler

Emergent research suggests that AI impacts brain development and cognition. Holmes et al., (2022) present work that questions whether the various cognitive and behavioural challenges such as attention deficit disorders are a result of technology use restructuring parts of children's brains. As children's brains are still developing, these are important considerations for teachers as they consider AI and technology integration in the classroom.

John Searle (1992) considers the neural function of the brain as a collection of simple cells that leads to thought, action and consciousness. Thinking is the purpose of the brain. When one enlists the help of AI with academic tasks such as writing a speech, summarising a piece of text, or doing Mathematics homework, the neural pathways that would be developed if the task was done by the human remains undeveloped as the task is done by the computer. This limits the human's potential to engage with the content, develop the skill and transfer it to different context.

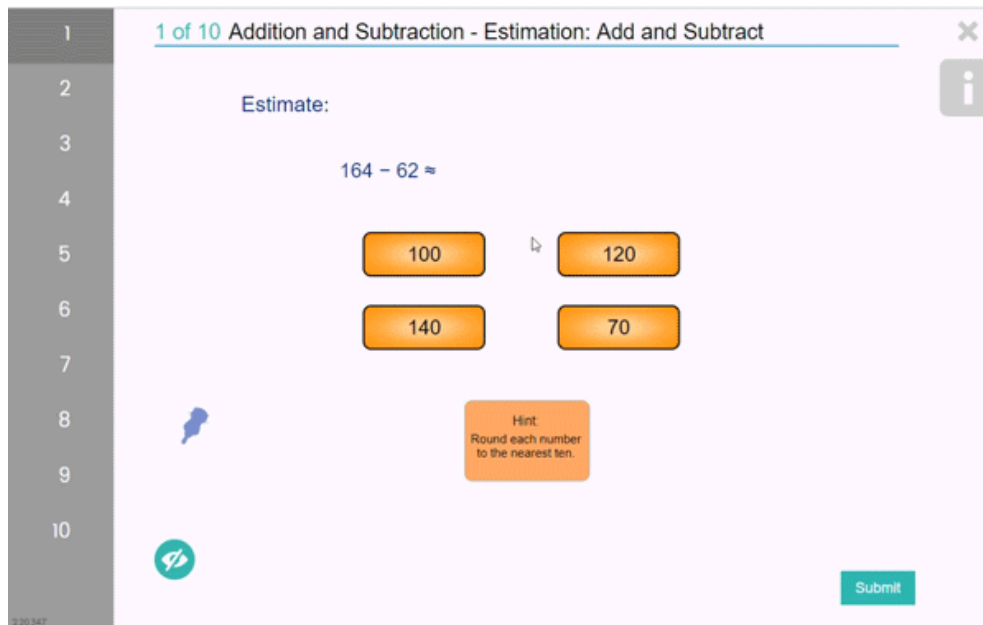
AI tools in the Mathematics classroom

Gamification

For most teachers, their first introduction to AI was in the form of Mathematics gamification. Applications such as Mathletics and Monster Maths to name a few provide an interactive platform for repetitive engagement on Maths tasks, building skill and confidence in a topic. The platforms are designed to be responsive, providing scaffolding and extension in topics based on the student's level of mastery. The feedback is instant and students can access additional explanations and resources when they experience difficulty. Most of the applications also rely on the concept of appreciative learning, Fitria (2021) which focuses on engagement through fun activities. The development of digital currency as a reward in several gaming platforms serves to further increase engagement, enabling students to interact with the Maths problems for long without boredom.

Figure 3 - Mathletics Addition & Subtraction quiz (<https://www.mathletics.com/>)

Content Assistance



Intelligent Tutoring System, Fitria (2021), and applications such as PhotoMath use AI-powered technology to provide worked solutions to problems that are typed out or hand-written. This new-age technology serves as an instant tutor to students, allowing them to access the steps and notes to a worked-out solution. In June 2023, AI-driven, MathGPT launched, providing instant solutions to complex problems in Maths, Physics, and Accounting. The main focus of MathGPT and other applications like it was post-COVID-19 curriculum recovery for students who did not have access to adequate teaching during the pandemic.



Figure 4 - MathGPT solutions -
(<https://app.mathgptpro.com/>)

These applications help students visualise solutions by providing worked-out examples that can be further expanded with the click of a button, enabling deeper understanding and instant clarification of misconceptions.

Teacher Administration

There is no doubt that there are heavy administrative responsibilities placed on teachers. The simplification of the administration has the potential to allow teachers more time to focus on creating engaging lessons that increase student engagement and ultimately their performance. Student engagement is an important aspect of inspiring a love for Mathematics. Applications such as Grammarly and ChatGPT help teachers refine and paraphrase their teaching notes. Owlift enables teachers to simplify their notes further or generate new ideas for concept explanation. Lesson planning tools such as Angel Solutions' 5 Minute Plan and Lesson Plan AI provide instant lesson plan ideas.

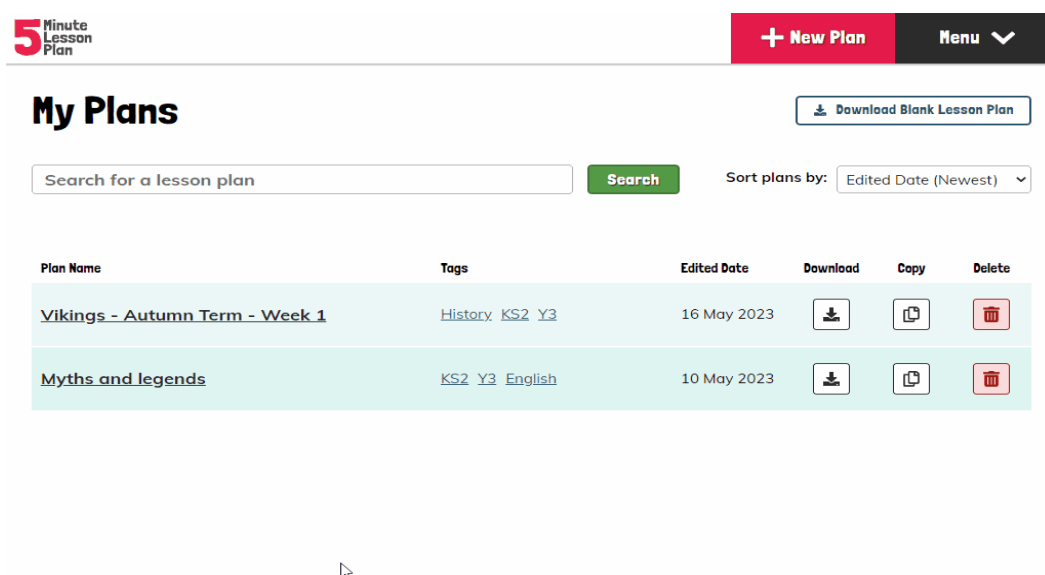


Figure 5 - Angel Solutions 5 Minute Lesson Plan

(<https://www.angelsolutions.co.uk/>)

Student assessment

Assessment is an important part of the learning process. AI-powered assessment tools enable instant grading which allows students to address misconceptions earlier in the learning process. This in turn allows teachers to adjust their teaching where necessary. Data-informed instruction allows personalised teaching that meet the students where they are by creating profiles that highlight student areas of strength and further development. This helps increase student self-awareness and drive.

As stated earlier, since the early days of Google, teachers have worried about student authenticity when it comes to their work, and with developments in AI technologies, this concern is further compounded. For as much as AI could facilitate students plagiarising their work, it also helps teachers check the originality of their submissions using apps such as Quillbot, Turnitin and other plagiarism detection applications.

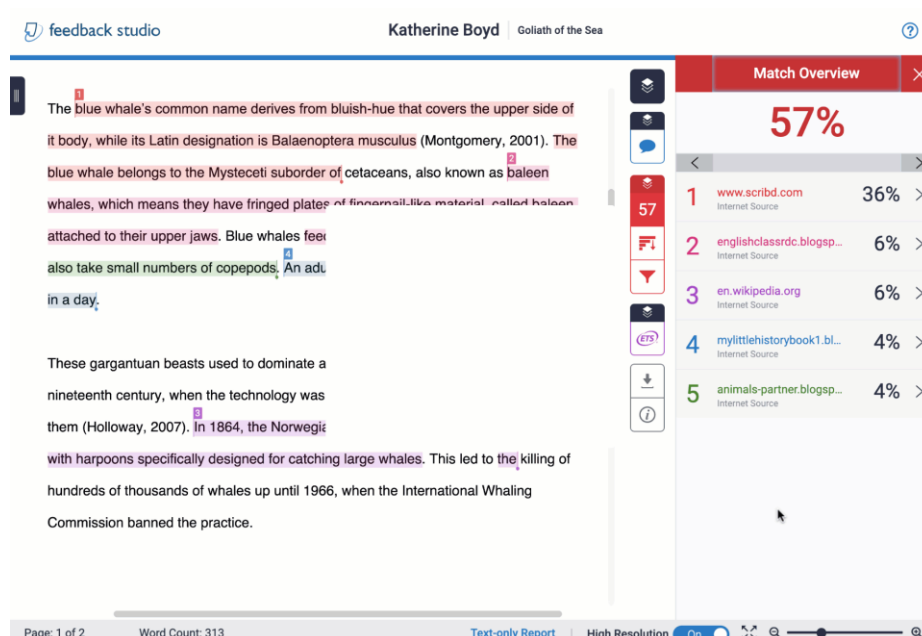


Figure 6 - Turn it in Demo (<https://turnitin.com>)

Dynamic assessment opportunities are necessary if deep learning needs to occur. Teachers will need to consider alternate assessment methods such as investigations. Teachers can also use AI-generated solutions to ask students to generalise the steps for working out.

Considerations for AI in the Mathematics Classroom

All new technologies have presented ethical challenges and AI is no different. As teachers and students are empowered to embrace and use AI technology, there also needs to be caution against overreliance. This overreliance could detract from the learning process. Bloom (1956) defined the learning process as consisting of three domains, namely, knowledge, skills and attitudes. The ability to store and recall data is diminished with the fingertip access to information as the drive to recall is redundant.

Cognitive	Affective	Psychomotor
knowledge	attitude	skills
1. Recall data	1. Receive (awareness)	1. Imitation (copy)
2. Understand	2. Respond (react)	2. Manipulation (follow instructions)
3. Apply (use)	3. Value (understand and act)	3. Develop Precision
4. Analyse (structure/elements)	4. Organise personal value system	4. Articulation (combine, integrate related skills)
5. Synthesize (create/build)	5. Internalize value system (adopt behaviour)	5. Naturalization (automate, become expert)
6. Evaluate (assess, judge in relational terms)		

Figure 7 - Bloom, B (1956) *Taxonomy of Educational Objectives Book 1: Cognitive Domain* New York: Longman

AI bias is an emerging research topic with many users finding bias and stereotyping in AI programmes. The knowledge of the shortcomings and limitations of AI is important Kiemde (2022). Learning is as much a social experience as it is cognitive and teachers need to guard against AI technologies driving division through tunnel vision and echo algorithms that perpetuate narratives about groups of people, concepts and ideologies. Classrooms that foster a sense of belonging and deep care are the places where the most learning can occur, with or without AI.

Conclusion

The novelty of new global developments allows a societal shift in the way we do things. It is important that Mathematics teaching can respond to these changes in a way that increases student love of learning the subject. Mathematics is taught as a school subject to enhance students' problem-solving abilities, ability to consider problems from different perspectives, to think critically and to transfer skills across a number of contexts. These values and principles of Mathematics teaching and learning need to remain true regardless of the medium of instruction and assessment. The greatest benefit of the skills taught in Mathematics is that those skills match almost exactly, how AI works. The teacher is the human element that AI technologies are working so hard to imitate.

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MATHEMATICS EDUCATION AND THE HARKNESS METHOD: ENHANCING LEARNING THROUGH TECHNOLOGY

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This paper explores the definition and aims of mathematics education as outlined in the Curriculum and Assessment Policy Statement (CAPS) document and draws parallels to the Harkness teaching method. It discusses how integrating technology can enhance the teaching and learning of mathematical concepts. Mathematics is a language of symbols and notations, emphasising its role in developing critical thinking, problem-solving skills, and understanding the world. The Harkness method, which emphasises student-centred discussion and collaborative learning, aligns well with the aims of mathematics education. Technology can be a powerful tool to support these educational strategies, making mathematical concepts more accessible and engaging.

Introduction

Mathematics is a fundamental discipline crucial to understanding and interpreting the world. According to the CAPS document, mathematics is a language using symbols and notations to describe numerical, geometric, and graphical relationships (Department of Basic Education, 2011). It involves observing, representing, and investigating patterns and relationships, fostering logical and critical thinking, accuracy, and problem-solving skills. The Harkness method, with its emphasis on student-led discussions and collaborative learning, offers a compatible pedagogical approach that can be enriched through technology.

Definition of Mathematics

Language of Symbols and Notations

Mathematics is a language that employs symbols and notations to articulate various relationships. This symbolic language enables the representation of numerical, geometric, and graphical relationships, facilitating understanding complex concepts and patterns (Department of Basic Education, 2011).

Human Activity and Mental Processes

Mathematics is not just an abstract discipline but a human activity that involves observing and investigating patterns in physical and social phenomena. It contributes to developing mental processes that enhance logical and critical thinking, accuracy, and problem-solving abilities. These skills are essential for effective decision-making and creative thinking (Department of Basic Education, 2011).

The following are the aims of Mathematics Education:

Fluency in Computation and Real-Life Application

The specific aims of mathematics education include developing computational fluency without reliance on calculators, emphasizing mathematical modelling, and incorporating real-life problems. Contextual problems should be realistic and encompass various societal issues such as health, social, economic, cultural, scientific, political, and environmental topics (Department of Basic Education, 2011).

Methodical Thinking and Generalization

Learners should be encouraged to think methodically, make conjectures, and justify or prove them. This involves understanding and working with number systems, appreciating the history of mathematics, and promoting the accessibility of mathematical content to all learners (Department of Basic Education, 2011).

Problem-Solving and Cognitive Skills

Mathematics education aims to develop problem-solving and cognitive skills by focusing on the "when" and "why" of problem types, not just the "how." Understanding the relevance and application of mathematical procedures and proofs is crucial for preparing learners for further education, training, and the world of work (Department of Basic Education, 2011).

Essential Mathematical Skills

To develop essential mathematical skills, learners should use the language of mathematics correctly, analyze and organize quantitative data, and use mathematical process skills to solve problems creatively and critically. Additionally, spatial skills and the properties of shapes and objects should be utilized to address problems responsibly and effectively (Department of Basic Education, 2011).

The Harkness Method and Mathematics Education

Student-Centred Learning

The Harkness method, characterized by its round-table discussion format, emphasizes student-centred learning and collaborative problem-solving. This approach aligns with the CAPS document's emphasis on developing critical thinking, problem-solving, and communication skills (Maheshwari, 2017).

Collaborative Problem-Solving

In the Harkness method, students work together to explore mathematical concepts, discuss various problem-solving strategies, and justify their reasoning. This collaborative environment fosters a deeper understanding of mathematical principles and encourages students to think creatively and critically (Maheshwari, 2017).

Role of the Teacher

In the Harkness method, the teacher acts as a facilitator, guiding discussions and encouraging students to explore different perspectives. This role is crucial in helping students develop a deeper understanding of mathematical concepts and their applications (Maheshwari, 2017).

The Harkness Method at Phillips Exeter Academy

Phillips Exeter Academy in the United States is renowned for its implementation of the Harkness method across various disciplines, including mathematics. This prestigious institution has adopted the Harkness method to foster a more interactive and student-centred learning environment. At Exeter, the Harkness table symbolizes a collaborative learning space where students engage deeply with mathematical concepts through discussion and peer teaching. This approach not only enhances understanding but also develops critical thinking and communication skills, preparing students for both higher education and real-world problem-solving (Phillips Exeter Academy, 2023).

Integrating Technology in Mathematics Education

Enhancing Understanding and Engagement

Technology can enhance the teaching and learning of mathematics by providing interactive and engaging tools for exploring mathematical concepts. Tools such as graphing calculators, dynamic geometry software, and online collaborative platforms can facilitate a deeper understanding and engagement with the material (Drijvers et al., 2016).

Supporting Collaborative Learning

Technology can support the Harkness method by enabling virtual collaboration and discussion. Online platforms can provide spaces for students to share ideas, solve problems together, and receive immediate feedback from peers and instructors (Drijvers et al., 2016).

Real-Life Applications

Technology can help incorporate real-life problems into the mathematics curriculum by providing access to data sets, simulation tools, and modelling software. These resources can help students understand the relevance of mathematics to real-world issues and develop practical problem-solving skills (Drijvers et al., 2016).

Teaching and learning using Demos applications.

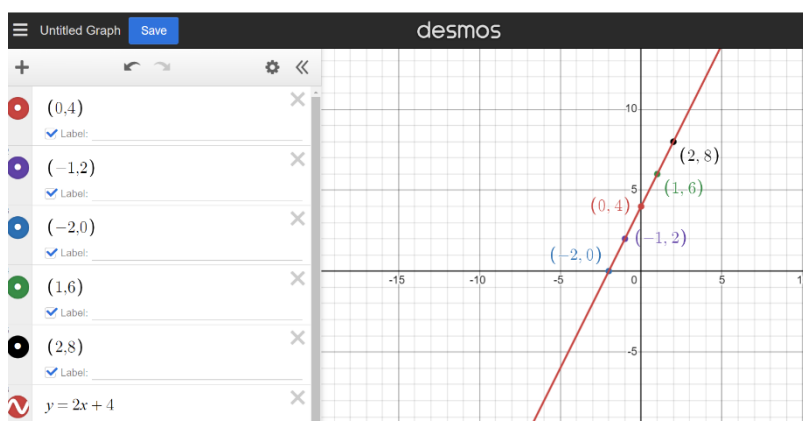


Figure 1: Grade 7-9: Functions and relationships

A demo application can plot points on a Cartesian plane to illustrate mathematical concepts. For example, given the equation $y = 2x + 4$, if $x=1$, the coordinate is (1, 6). By joining the points with a linear graph, students can visualise the relationship between x and y . The class can interact with the graph by clicking on the x -axis and y -axis to discuss the x -intercept and y -intercept, enhancing their understanding of these critical concepts.

Grade 10: Functions

Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effect of the parameter which results in a vertical shift and that which results in a vertical stretch and /or a reflection about the x -axis (Department of Basic Education, 2011).

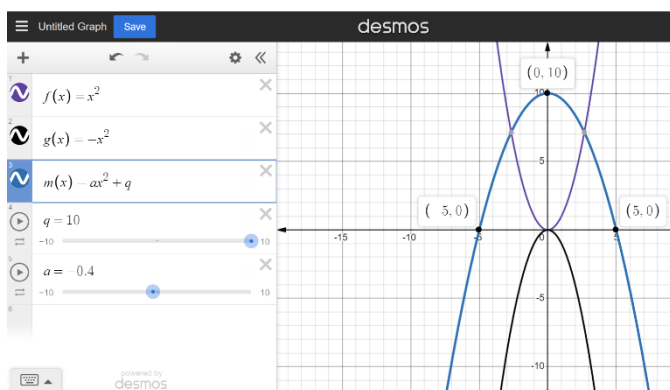


Figure 2: Grade 10 Functions

At the Grade 10 Mathematics level, we can use the Desmos application to teach the function $f(x) = ax^2 + q$. Students can investigate the effects of changing the values of a and q independently, allowing them to discover these relationships without direct instruction. Additionally, by clicking on the x-intercepts within the application, students can link these concepts to quadratic equations and learn how to find the x-intercepts independently.

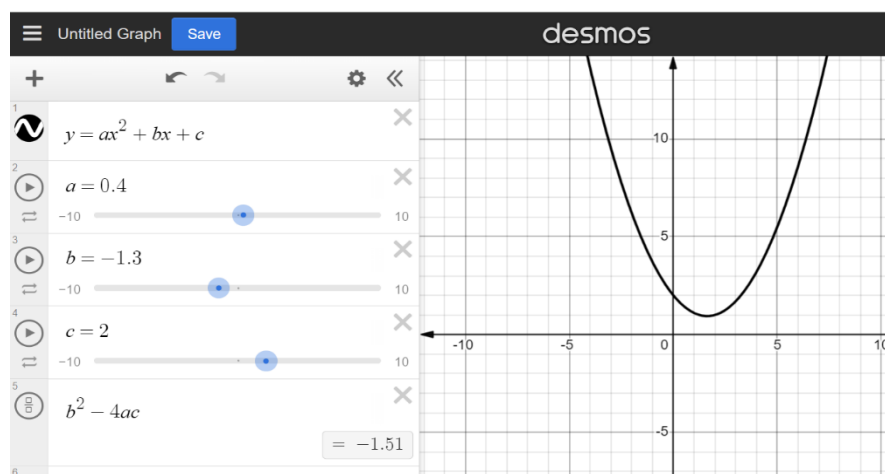


Figure 3: Nature of Roots

We can also utilize the Desmos application to teach the Nature of Roots for conceptual understanding. By graphing $f(x) = ax^2 + bx + c$, students can investigate the effects of a , b , and c on the graph, as well as explore the discriminant $\Delta = b^2 - 4ac$ to determine the nature of the roots.

Grade 11 and 12: Maximum and Minimum problems

Given that $f(x) = -x^2 + 8x + 20$ and $g(x) = 4x + 8$ EF is parallel to the y - axis.

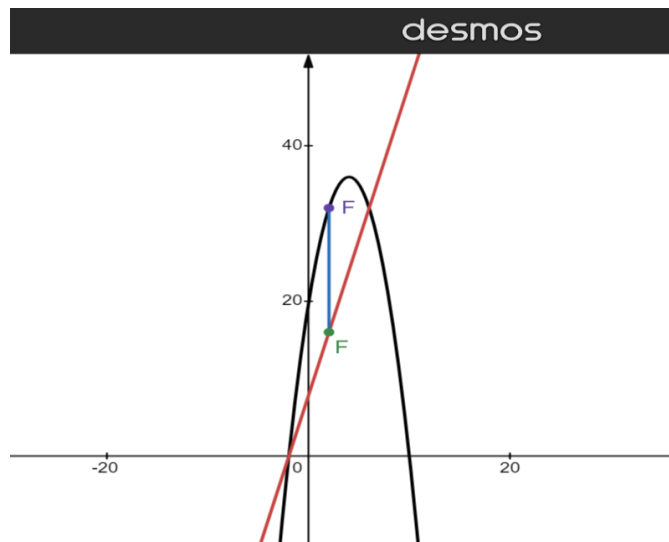
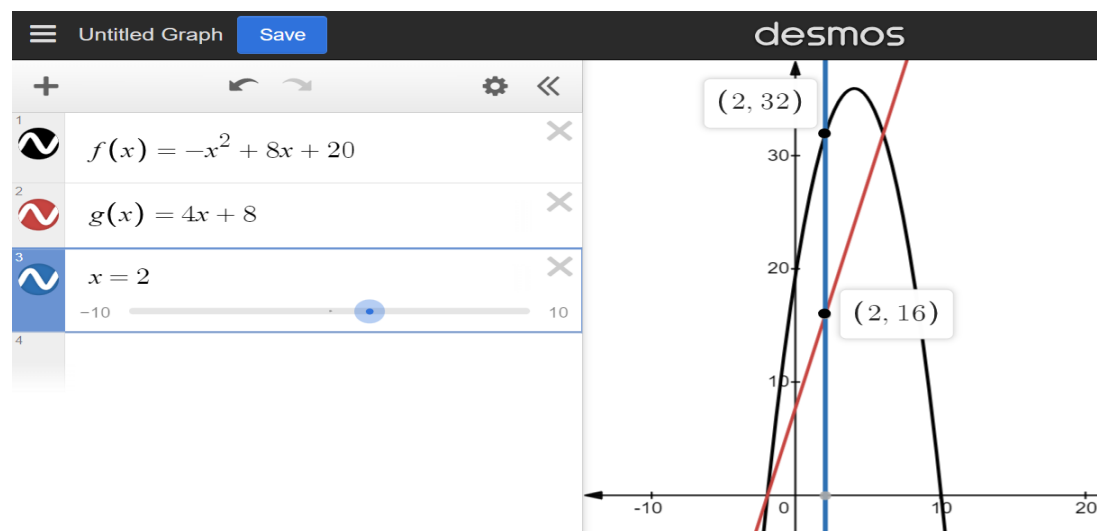


Figure 4: Calculation of the maximum length of EF.

We can calculate the maximum length by using the Desmos application without starting with the more routine procedures, such as completing the square or using calculus. By drawing $x = k$, we can slide and investigate where the maximum occurs and understand why by analysing the top graph minus the bottom graph.



Distance = Top Graph – Bottom Graph

$$EF = -x^2 + 8x + 20 - (4x + 8)$$

$$= -x^2 + 4x + 12$$

$$x = \frac{-b}{2a} = \frac{-4}{-2} = 2$$

$$\begin{aligned} \text{Maximum length of EF} &= -(2)^2 + 4(2) + 12 \\ &= 16 \text{ units} \end{aligned}$$

Conclusion

Mathematics education, as defined by the CAPS document, aims to develop critical thinking, problem-solving skills, and a deep understanding of mathematical concepts and their applications. The Harkness method, with its emphasis on student-centred discussion and collaborative learning, aligns well with these aims. Integrating technology into mathematics education can further enhance these learning experiences by providing interactive tools and supporting collaborative learning environments. By combining the principles of the CAPS document, the Harkness method, and technological advancements, educators can create a rich and engaging mathematics curriculum that prepares students for the challenges of the future.

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AN ANALYSIS OF AFFORDANCES FOR LEARNING PROBABILITY IN A COMMONLY USED MATHEMATICAL LITERACY GRADE 12 TEXTBOOK

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The study set out to examine the affordances for learning that are made available to Grade 12 learners when engaging with the Platinum Mathematical Literacy Textbook. The Platinum Mathematical Literacy textbook was chosen because it is a commonly used textbook across schools in South Africa. An analysis of the topic of Probability in the Platinum Mathematical Literacy textbook using the Dyadic Framework for Comprehensive Textbook Analysis reveals that the layout of the textbook is quasi-deductive. Consequently, the study attempted to investigate the nature of the exemplification in the textbook. The results of the study show that the textbook thoroughly exposes the critical features of each core idea through using multiple presentation formats; however, the patterns of variation were limited when variation theory is used as a lens to understand the choice and the sequence of examples included in the textbook. Recommendations are made on elements that should be considered when selecting or developing Mathematical Literacy textbooks.

Introduction

This paper reports on a research that investigated the affordances for learning Probability in a commonly used Mathematical Literacy textbook. This research concept is influenced by the awareness that improvement of Mathematical Literacy is an ongoing endeavour in South African Education. Mathematical Literacy is still a new subject when compared to other subjects in the South African High school curriculum (Bansilal, Webb & James, 2015). The ongoing development of the subject has resulted in only few universities offering Mathematical Literacy as a subject of specialisation for new teachers. Therefore, many Mathematics teachers have to learn Mathematical Literacy through watching videos and consulting textbooks and other materials.

The National Senior Certificate Report published in 2024 by the Department of Education (DoE 2024) as informed by the Matric results from the year 2021-2023 and shows statistics on the total number of students who have registered for Mathematical Literacy. In 2021, 441 067 learners wrote the Mathematical Literacy final exam compared to the 259 143 who registered and wrote the Mathematics exam. In 2023, there was still a similar picture as there were 421 835 learners who sat for the Mathematical Literacy Exam compared to the 262 016 learners who sat for the Mathematics exam. This data is an indication that a very significant population is taught Mathematical Literacy in schools although it is still regarded as a new subject when compared to other subjects in the country (Bansilal, Webb & James, 2015).

The number of learners who are registering in Mathematical Literacy has increased over the years; this is when compared to those who register for Mathematics at Grade 12 level ([DoE], 2018). This growth is attributed to concerns that are similar to the ones that resulted in the introduction of Mathematical Literacy in South African schools (DoE, 2006). Some of these

concerns were firstly the poor results in Mathematics nationally and secondly the vestiges of the Outcomes-Based Education (OBE) curriculum that came before the year 2005, when it was not compulsory to register for Mathematics in the FET phase, thus only a few learners opted to register for Mathematics (de Clercq, 2020).

Since 2006, it has become obligatory for learners in South African schools to register for either Mathematical Literacy or Mathematics. This is because basic calculating skills needed for the world of work remain essential (Jojo, 2019). Mathematical Literacy can be regarded as a tool that can be used to redress the past political differences that have existed in South Africa; this is done through increasing the number of students who obtain a matric qualification (Jojo, 2019). This is done through reducing the number of subjects that learners are struggling with, such as Mathematics, while still offering students the necessary mathematical skills (Jojo, 2019).

Studying the nature of classroom practice remains necessary, especially within the changing dynamics in education globally. Acknowledging the implication that the global pandemic has had on teacher and learner interactions is essential to understanding the teaching and learning issues that have emerged over this period. A textbook is an essential tool that facilitates the learning process in and out of the classroom (Basar *et al.*, 2021). It offers a widespread range of information, different types of examples and activities that are represented in multiple formats (O'keeffe, 2012). Moreover, textbooks illustrate representations and explanations that are pertinent to understanding mathematical topics (Ceretkova & Sedivy, 2008). In essence, textbooks mediate the curriculum from the intended, to the implemented and finally the attained curriculum (Valverde *et al.*, 2002).

Using textbooks as a resource for learning is advantageous as they can be used in and outside of classrooms. This is why continuous exploration of textbook quality is a significant and imperative research imperative. Peppin & Haggerty (2001) argue that “*Concerns have been expressed about the quality of the textbooks, the way they are written, and their persuasive influence*” (p. 74). This statement justifies the significance of purposefully selecting textbooks for the intention of improving prospects of the learners to learning for understanding.

The purpose of our study of Platinum Mathematical Literacy textbook was to analyse the topic of probability within a Grade 12 Mathematical Literacy textbook to ascertain what affordances are made available for learning on this topic. Multiple factors were considered when considering which textbook to analyse in this study. One of the considerations was the first publication date. Platinum Mathematical literacy is one of the textbooks that were published shortly after Mathematical Literacy was introduced. It is important to note that only few Mathematical Literacy textbooks were published with resources that cover Grades 10, 11 and 12. Some textbooks have only focused on Grade 10 and 11 material. Within the national curriculum statement of South Africa, probability constitutes around 8.6 % of the total workload that is to be taught Grade 12 Mathematical Literacy (DoE, 2019). Although it may seem as if it constitutes a small percentage of the overall annual teaching plan, it is a very important section. Concurring with this, the study by Taylor (2011) asserts that studying the concept of Probability is essential as it helps with the development of mathematical reasoning

for learners. It also allows for an integration of Mathematics with the real world (Brijlal, 2014). The following served as guiding questions for our investigation:

1. How is the topic of Probability presented in a commonly used Mathematical Literacy textbook?
2. Which patterns of variation can be identified within the examples and activities found in the topic of Probability?
3. What affordances for learning are made available through engaging with the textbooks?

The first question considered the structure of the textbook. The patterns of variation were identified within the examples and activities presented within the chapter of probability. The affordance for learning was analysed through observing the structure and the presence of the patterns of variation.

Theoretical background

Two frameworks – *Variation* theory (Watson & Mason 2006) and the framework for Mathematics *Textbook* Analysis (Valverde et al, 2002), inform this study. In what follows, we engaged with both theories and thereafter developed a justification for the need for a combination of both frameworks.

Marton and Booth (1997) originally proposed the Variation theory as a strategy intended to understand the choice of examples through exposing the critical features of the object of learning in order to encourage their discernment of these features. Within a set of questions, variation theory provides a lens to determine what changes/varies or what remains the same/invariant (Watson & Mason, 2006). The concept of variation allows teachers to purposefully draw learners towards focusing on the idea that teacher intends the learners to ‘master’ (Adler, 2007, p. xii).

Every lesson must have an object of learning. The object of learning is the focus of attention. In mathematics, it is the mathematical focus of attention. An object has multiple features and these features can be perceived in various ways (Essien, 2021). Focusing on particular features influences the perception of the object of learning (Essien, 2021). When planning to teach a particular object of learning, it is important to start by identifying the critical features in order to facilitate teachers understanding and to help deal with student’s differences (Lo, 2012). In that way, the examples selected are purposeful and ordered. This purposeful selection leads to understanding the patterns of variation. “To discern the critical features of novel situations, learners must experience for themselves certain patterns of variation and invariance of these features” (Lo, p.2012). This statement explains the importance of the patterns on variation in lessons.

Four patterns of variation have been proposed by the theory. These patterns are: Contrast, similarity, generalisation and fusion (Marton & Booth 1997). Each of the patterns are explained in the Table below:

Table 1: The patterns of variation.

Similarity	The decision to keep certain things constant, while varying others (Watson & Mason, 2006).
Contrast	It allows students to see the object through grounding this judgement on an observation of what the object cannot be in order to discern contrast, the critical features of one object must be compared with the critical features of another objects. (Lueng, 2012)
Fusion	An integration of critical features. In this pattern, there is more than one aspect varied at a particular time. Fusion ought to come after contrast or separation because these two focus on individual features while fusion occurs when learners' attention is on more than one feature (Watson & Mason, 2006).
Generalisation	An activity of making a conjecture based on the pattern that has been identified through multiple examples that may be done using anyone of the above-mentioned patterns of variation. In essence, it is a shift from the particular to the general (Mhlolo, 2013).

While variation theory enables the analysis of how the object of learning can be exposed through varying the critical features according to the different types of the patterns of variation, what it does not do, is provide a framework that engages with the layout of mathematical content through a *general structure*. The textbook analysis framework by Valverde et al (2002) focuses on this and is therefore brought in to complement variation theory in our study.

The purpose of using the Framework for Mathematics textbook analysis in this study is to bridge the gap between what O'keeffe (2012) refers to as intended and implemented curriculums. The framework for Mathematics textbook analysis considers the role of the textbooks in education before expanding on how to analyse them.

Over the years, some researchers further refined the framework of the analysis of Mathematics textbooks (O'keeffe, 2011) to "*allow for a finer grained analysis*" (p. 5). The focus is still on these three main tenets which are: the structure, content and performance expectation:

Structure is the sequence or the layout of the text. The order in which presentation formats are arranged creates a pattern that informs how a person engages with the text. The structure can be analysed through evaluating the general approach that is used within the textbook (Ensor, et al. 2002). Quasi-deductive and Quasi- inductive are the two types of approaches presented in the theory.

- Quasi-deductive occurs when the textbook or teacher uses examples and definitions before giving the learners activities to engage in or apply the information they have attained (Leshota, 2017).
- Quasi-inductive on the other hand is the direct opposite of quasi-deductive. It occurs when learners work through investigative means to make a general conjecture or discover the theory that has informed the activity (Leshota, 2017).

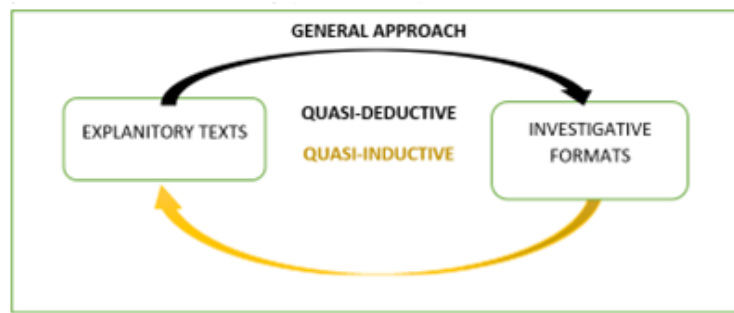


Figure 1: The general approach used in textbooks. Source: Ensor et al. (2013).

When analysing the structure, the focus is about presentation formats that have been purposefully included in the text for meaning making. The types of formats found in mathematics textbooks are pictures, graphs, tables, activities, information boxes etc. **Figure 1** above represents the general approach used by textbooks. The black arrow describes the pathway of quasi-deductive approach. This emphasises that explanations and definitions are the foundation of learning. The examples and exercises that follow are dependent on the definitions that are initially provided. The yellow arrow describes quasi-inductive approach. In the deductive approach “*Activities lead to definitions and from this point opportunities may be provided for students to practice*” (Ensor et al., 2013, p. 23):

Our comprehensive textbook analysis framework is an integration of the two frameworks that focus on important features of a textbook. In Variation theory, examples are the core idea, while in the Framework Mathematics textbooks analysis, the broader presentation formats are important. Our framework makes provision for analysing textbooks at the macro (presentation formats) level, but also at the microstructure (patterns of variation) level.

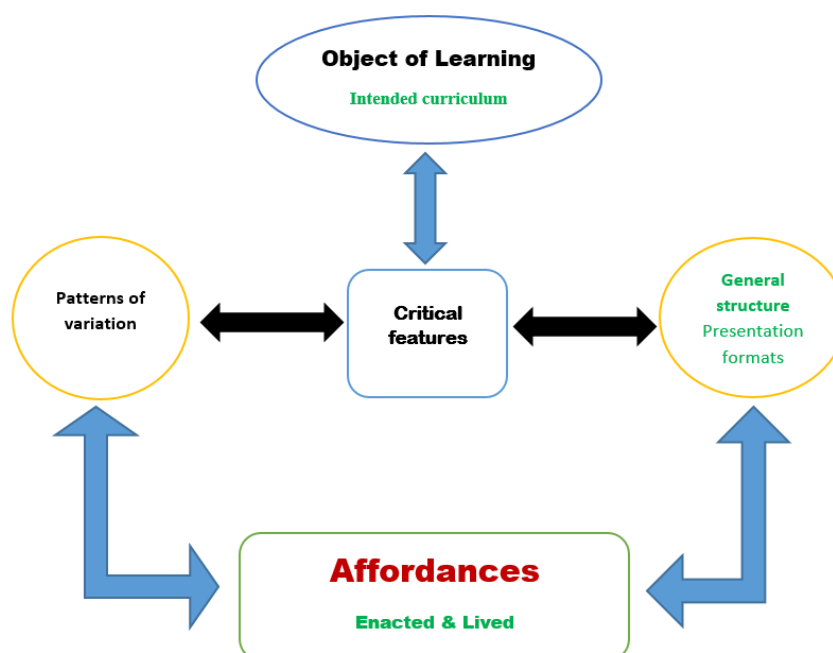


Figure 2: Dyadic Framework for a Comprehensive Analysis of Textbooks

Using the Dyadic Framework for Comprehensive textbook Analysis

The dyadic framework for a comprehensive analysis of textbooks informed the structure of the method of the study and the analysis. Analysis of the ML textbook followed this order: First, analysis of the general approach – which engaged with the order in which the presentation formats appear, to deduce the general structure of the textbook; second, analysis of exemplification – which identified the core ideas and its critical features to focus on the extent to which the exercise examples attended to the core ideas and its critical features. The initial section of the analysis responds to the first research question on determining the general structure of the mathematical literacy textbook. Through an analysis of the general approach of the textbook, this structure is made explicit. Thereafter, the analysis draws on two important tenets of the variation theory, the first tenet is the *critical features* presented in each of the units of the chapter, and the second tenet is the *nature and order of examples* presented.

The general approach of the textbook

This phase of analysis focuses on determining the General approach of the textbook. As indicated in the framework, the approach can either be Quasi-deductive which is regarded as a narrative approach, or the Quasi-inductive regarded as an investigative approach (Valverde et al. 2002). To illustrate the order in which the presentation formats appear, the figures below are from Unit 3 (a subsection of the chapter of probability).

The following figures show the presentation formats identified in Unit 3.

Unit 3 Determining possible outcomes

If a couple has a baby, they know that the outcome will be either a boy or a girl. However, in other cases, the outcomes are not as obvious. For example, if a couple plans to have four children, the possible outcomes are not as clear. In such cases, it is useful to represent the data using a tree diagram or a two-way table to see all the possible outcomes.

Key Words

tree diagram – a visual way to express outcomes of an event, particularly when there are more than two variables or events.

two-way table – a visual way to express outcomes of an event that can only be used when you have two variables or two events.

WORKED EXAMPLE

1 Use a tree diagram to help you list all the possible outcomes of a slot machine that has the first two spinning wheels as shown in the diagram alongside.

2 Do you think you are equally likely to get two 7s as you are to get two cherries? Give a reason for your answer.

SOLUTION

1

Event 1	Event 2	Possible Outcomes
7	7	7, 7
7	cherry	7, cherry
7	lemon	7, lemon
7	orange	7, orange
cherry	7	cherry, 7
cherry	cherry	cherry, cherry
cherry	lemon	cherry, lemon
cherry	orange	cherry, orange
lemon	7	lemon, 7
lemon	cherry	lemon, cherry
lemon	lemon	lemon, lemon
lemon	orange	lemon, orange
orange	7	orange, 7
orange	cherry	orange, cherry
orange	lemon	orange, lemon
orange	orange	orange, orange

2 Although two 7s and two cherries are both outcomes, the possibility of obtaining two 7s is less likely because there is only one 7 on each wheel, whereas there are three cherries on each wheel. There is a 1 in 4 chance that you will spin a cherry on either wheel.

If the first wheel was a cherry and the next wheel was a 7, is that any different than the first wheel being a 7 and the second wheel a cherry? The outcomes are the same despite the order, since in this case, order does not matter.

WORKED EXAMPLE

Use a two-way table to list all the possible outcomes of a recent questionnaire given to a sample of people waiting at a bus rank. The first question asked the respondents their gender. The second question asked whether they were employed or unemployed (giving the options in the table below for the employed/unemployed answers).

Gender	Employed	Unemployed
Male	12	8
Female	15	10

Introductory text

Worked example 1

Key word box

Worked example 2

Figure 3: Determining possible outcomes from multiple events

The first presentation format in **Figure 3** is a narrative format to introduce the concept. This is followed by an investigative format of a worked example that exposes learners to questions and answers on the topic. A key word box on the side defines the key words that learners will be exposed to in this section. A second worked example precedes the exercise that learners must complete to check their understanding of the topic. Through observing the pattern above, we can deduce that the textbook is largely quasi-deductive. This is because the narrative presentation formats precede the investigative formats. When engaging with the textbook it is evident that there is a narrative introduction of the content. The application come in once a foundation is presented.

SOLUTION

The possible outcomes from the questions are listed as the first row and first column of the two-way table, and the inside of the table represents all the possible responses.

	Employed full time (FT)	Employed part time (PT)	Occasionally work (O)	Unemployed and looking for work (UL)	Unemployed and not looking for work (UNL)
Male	Male, FT	Male, PT	Male, O	Male, UL	Male, UNL
Female	Female, FT	Female, PT	Female, O	Female, UL	Female, UNL

You could have also designed your table like this:

	Male	Female
Employed full time (FT)	FT, male	FT, female
Employed part time (PT)	PT, male	PT, female
Occasionally work (O)	O, male	O, female
Unemployed and looking for work (UL)	UL, male	UL, female
Unemployed and not looking for work (UNL)	UNL, male	UNL, female

In these examples, the order of the events does not matter. However in some cases, the order of the events may be important. For example, if you want to know the possible outcomes of having two children, the order does not matter. But, if the eldest child must be a boy, then the order of the outcomes is important.

EXERCISE 6

- Most car insurance companies evaluate your driving history to determine how much money you have to pay for car insurance. They check two things: if you have any speeding fines with categories 0 tickets, 1 ticket, 2 tickets, 3 tickets, 4 tickets or 5+ tickets, and if you have had any accidents (yes or no).
 - Create a two-way table listing all the possible outcomes that the insurance company looks at.
 - What do you think the category 5+ represents?
 - Why do you think there is a category 5+?
 - Which outcome do you think would have the cheapest insurance? Give a reason for your answer.
- Consider the diagram alongside:
 - Draw a tree diagram to represent the possibilities of spinning the spinner and flipping a coin.
 - How many possible outcomes do you have from your tree diagram?
 - What is the probability of spinning red and flipping heads?
 - What is the probability of flipping tails?

Unit 3 Determining possible outcomes 245

Figure 4 determining possible outcomes cont.

What is common in all the units on probability is that the narrative presentation formats constitute the introduction of the core idea presented in the chapter, and after that comes exercise examples that are intended to evaluate the learner's ability to apply the knowledge that they have acquired through engaging with the text.

Every approach has its benefits and limitations. The benefits of using the quasi-deductive approach are that it allows for scaffolding. This means that the content pieces have been partitioned into simpler smaller narrative formats before engaging with investigation. The limitation of using this approach is that the end goal is for the textbook users to complete the exercise, thereafter there is rarely explanation. The advantage of a quasi-inductive approach is that new information is learnt through deduction. This requires higher order thinking because learners learn through drawing conclusions.

Exemplification

This section of the analysis intends to use the theory of variation as a theoretical lens to analyse exemplification within the Platinum Mathematical Literacy Grade 12 textbook. According to Valvede et al (2002), textbooks are designed to provide opportunities to learn. This statement addresses the purpose of this study, which is to analyse the affordances for learning that may be inferred through engaging with the Platinum Mathematical Literacy textbook. Throughout the exemplification section, the core ideas in each unit are lifted determine the extent to which the textbook attends to the critical features.

The presentation formats as explain in previous sections of this paper are grouped into two categories. These are either narrative or investigative. In this part of the analysis, the gist is to thoroughly analyse the exercises, which is a type of investigative presentation formats. The narrative formats are used to determine the core idea and its critical features. Thereafter, the exercises are considered to firstly analyse the extent to which the critical features are made explicit to the learners and secondly, considering the how a variation informed approach would improve the understanding of the topic. Due to space limitations, we have used the example of Unit one to illustrate our point.

Analysis of Unit 1: Expressions of Probability

Expressions of probability is the core idea presented in the first unit. The textbook defines probability as the likelihood or chance that an event will occur (Frith et al, p.238-251). This is expressed as a formula that is used as a mathematical representation.

$$Probability = \frac{\text{favourable outcomes}}{\text{total possible outcomes}}$$

The critical features of the expressions of probability are inferred from the above-mentioned formula. These features are the ability to express probability in three ways as a fraction, decimal and a percentage. The contextual meaning of the value of probability is the second critical feature which represents the likelihood of an event occurring. To illustrate this Figure 5 below is an extract from the textbook to illustrate likelihood through a probability number line. The number line uses probability terminologies that express the chances of an event occurring, where 0% is impossible and 100% is certain.

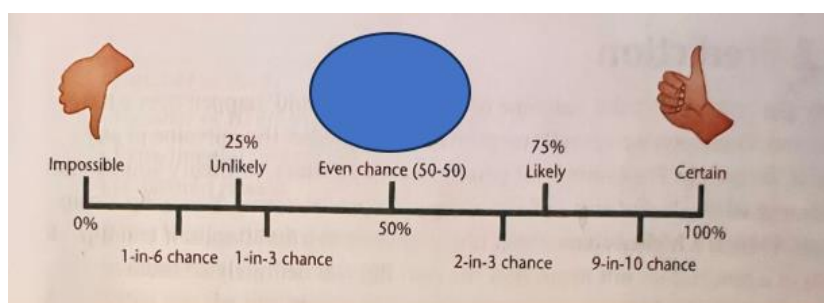


Figure 5: Probability Number line

The table below represents a summary of the critical features of the core idea of expressions of probability. In order to evaluate the extent to which the critical features attend to the core idea, the critical features must be clearly outlined.

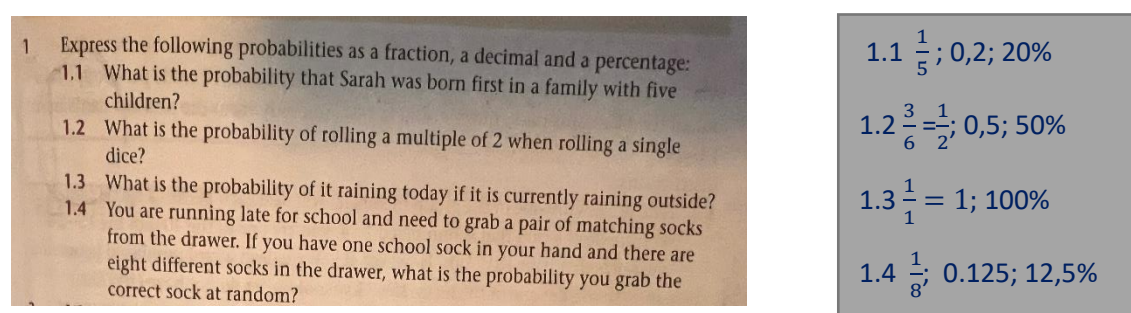
Table 2: The critical features of expressions of probability

Core Idea	Expressions of probability
Critical features	• The representation of probability as a fraction, decimal and as a percentage • The representation of likelihood on a probability number line • Probability increases from 0 to 1

The critical features of the expressions of probability are centred on the formula stated above. Table 2 above indicates the different notations that can be derived from a fraction, this can be a simplified decimal or percentage. From the fraction, decimal and percentage the value of the probability can be interpreted to explain the chances of the event happening.

In the next section, we engage with how the examples and the sequencing of examples attend to this critical feature(s).

Variation as a lens in Exercise 1



1 Express the following probabilities as a fraction, a decimal and a percentage:	
1.1 What is the probability that Sarah was born first in a family with five children?	1.1 $\frac{1}{5}$; 0,2; 20%
1.2 What is the probability of rolling a multiple of 2 when rolling a single dice?	1.2 $\frac{3}{6} = \frac{1}{2}$; 0,5; 50%
1.3 What is the probability of it raining today if it is currently raining outside?	1.3 $\frac{1}{1} = 1$; 100%
1.4 You are running late for school and need to grab a pair of matching socks from the drawer. If you have one school sock in your hand and there are eight different socks in the drawer, what is the probability you grab the correct sock at random?	1.4 $\frac{1}{8}$; 0.125; 12,5%

Figure 6: questions and answers of exercise 1

Mathematical Literacy questions are phrased using long sentences, the answers can either be in the form of mathematical expressions or sentences. In order to use Variation as a theoretical lens, we provided the answers to the questions **Figure 6** above. What we noted through analysing the whole of question 1 was that the gist of the questions was on the core idea, which is the Expressions of probability. The requirement of the questions is to express probability either as a fraction, decimal or as a percentage.

The observation is that the numerators of all the simplified answers is 1, but the denominators vary according to the total possible outcomes. This leads us to reflect on the critical features of expressions of probability. Firstly, probability can be expressed in three different ways, as a fraction, decimal and a percentage. The other feature is that the larger the denominator, the lower the probability and that decreases the likelihood of an event occurring.

From the answers in **Figure 6** the grey textbox, we can deduce that there is no particular pattern used to order the questions. The example space does not show an increase or decrease in probability through the order in which the denominators vary. The example space created does not seem intentional. A variation informed textbook would have reordered the questions such that there is gradual increase of the dominator to show the decrease in probability. In that way, the focus is not only on expressing probability in three ways, but it is also about the meaning that is drawn from intentionally ordering questions. If the questions can be reordered from 1.3 to 1.2 then 1.1 and 1.4. This change would allow students the opportunity to discern the decrease in the likelihood of an event occurring. A variation informed textbook would order the question such that all the critical features are made more explicit through the questions asked.

Discussion of results

Table 3 below provides insights into the potential benefits that could arise from integrating variation theory into a textbook. The Dyadic Framework for Comprehensive Textbook Analysis considers the General Approach and the extent to which the textbook attended to the critical features of the Object of learning. The table below is a deduction that can be used to generalise the possible affordances that may be attained through engaging with a Dyadic framework theory informed textbook.

Table 3: A theory informed textbook

General approach Variation	Deductive	Inductive
Variation informed	a) Deductive + Variation	b) Inductive + Variation
Non-variation informed	c) Non-variation informed + Deductive	d) Non-Variation informed Inductive

There are four types of textbooks that can be identified from the table above. These are;

a) A quasi-deductive textbook that is Variation theory informed

The affordances of a deductive textbook are that the examples have the potential of providing information that is necessary to understand the topic before they engage with questions that require them to apply the knowledge. The affordances of a variation informed textbook are that it allows learners to discern the critical features of the object of learning. Therefore a textbook that is quasi-deductive and variation informed is intentional about all the information presented whether it is the narrative or investigative presentation formats. Therefore, the narrative texts and examples are sequenced through the patterns of variation to create discernment of the critical features.

b) A quasi-inductive textbook that is Variation theory informed

An inductive textbook is more investigative which requires thoroughness from the learners. The inductive and variation informed textbook would have well-structured questions that build

up to generality. The questions would allow students to discern the critical features of the object of learning as they engage with them. This will encourage in-depth reasoning that is structured towards the attainment of new information.

c) A quasi-deductive textbook that is not Variation theory informed

A textbook that is deductive and not informed by variation is a narrative textbook that may be informed by other theories. The focus of this textbook is to give foregrounding information that is necessary for the learners to understand the object of learning. If examples are varied, they are not intentionally structured to bring out the different critical features. *The Platinum Mathematical literacy textbook is quasi- deductive and not informed by variation.*

d) A quasi-inductive textbook that is Variation theory informed

A textbook that is investigative and not informed by variation may not necessarily be structured to bring out the critical features but the textbook will encourage higher order thinking from the learners. The difficulty with this textbook is how the critical features may be exposed by the investigative questions if they are not structured.

Conclusion

It is important to consider the general approach of a textbook to encourage a particular type of reasoning. If there is a need to encourage investigative thinking, one should consider a textbook that is quasi-inductive. However if one wants to encourage deductive reasoning, a teacher should consider using a quasi-deductive textbook. For the purpose of this research, the Platinum textbook was identified as a quasi-deductive textbook.

The quasi-deductive approach is relevant to the structure of schooling in South Africa, where learners are allowed to change from Mathematics to Mathematical Literacy until Grade 11. A deductive textbook is effective in encouraging a narrative explanation of the content which helps to bridge the gap between the knowledge that learners should have learnt in Grades 10 and 11.

An understanding of critical features should be encouraged in order to create better opportunities to learn. For learners to discern the critical features of an object, the features must firstly be exposed. This would mean that effective learning material must expose its critical features. The Platinum textbook does expose critical features of probability. However, the only concern is that the exercises are not intended to allow learners to engage with all the critical features and secondly there are very few discernable patterns of variation.

When analysing the patterns of variation, a lens is cast on the order in which the critical features vary. Therefore, an awareness of the critical features is an important aspect to consider when selecting a textbook. Secondly, when choosing examples to use for teaching, intentionally sequencing of example is vital to expose the critical feature to encourage discernment of these features.

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HELPING MATHEMATICS TEACHERS DEVELOP THROUGH AN ADAPTED PROFESSIONAL LEARNING COMMUNITY APPROACH: THE CASE OF 1+9 MODEL IN LIMPOPO

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The time for teachers to determine their own development trajectories is overdue. Research has amply pointed out to both the rationale, efficacy and modalities of teacher “self-directed professional development”, otherwise herein repackaged as Professional Learning Communities (PLCs). This paper sheds light on an adapted professional learning community-type teacher capacity development of FET Mathematics teachers in the Limpopo province in 2023. The 1+9 model of teacher development was implemented provincially in 2023, with groups of twenty (20) teachers meeting once every nine days for planning and preparation.

Mathematics teachers planned, presented, discussed and strengthened nine lessons each through intensive interaction and interrogation with peers in a session facilitated by a lead teacher or subject advisor. The paper outlines the gains and benefits of this approach. The discourse on contextual relevance and resources, both human and physical, in the African context is one that deserves frequent attention with a view to interrogation and improvement. This paper posits a proposal that localized, regular, coherent and teacher-led professional development holds greater potential to improving mathematics classroom practice (teaching and learning), and thereby enhance teacher confidence, competence and job satisfaction, resulting in improved learner outcomes.

Keywords:

Introduction (PKD)

This paper sheds light on an adapted professional learning community-type teacher capacity development of FET Mathematics teachers in Limpopo in 2023. In its quest to support high school Mathematics teachers across the province, the Limpopo Department of Education initiated an adapted form of a professional leaning community (PLC) programme for teachers to meet every nine days (commonly known as the 1+9 model) to plan, to develop and to grow professionally together.

Background

The collaboration between South Africa and Cuba is longstanding and dates back to the days of struggle for liberation in South Africa which Cuba supported. A collaboration agreement was established between the two countries which saw a group of mathematics and science specialists deployed to work in South Africa. Limpopo is one of the provinces which received Cuban specialists, and one of the authors of this paper is working there. So this paper is a blend of tools tested and tried in both countries but contextualized and researched in this paper.

Perturbed by the dismal Grade 9 Mathematics learner performance in the Annual National Assessment (ANA) in 2014, the Department of Basic Education (DBE) introduced a mathematics teacher development programme called the “1+4 Model” in 2015 in order to support the Senior Phase (Grades 8 & 9) Mathematics teachers. The programme involved the coming together of teachers one day a week (usually a Monday) to plan together the week’s programme, and then to go back to their schools and teach the next four days.

The Director-General of Basic Education said “the 1+4 model is a **special training programme** targeting Senior Phase Mathematics teachers to provide support to them in dealing with endemic, chronic and perennial weaknesses of learner performance. [This programme will] create a platform and opportunities for teachers to **share** with one another best methods, ideas, techniques, expertise and resources to **improve** learner performance and thereby enhancing the effectiveness of Subject Committees and Professional Learning Committees” (29 April 2015).

While the programme added value in supporting teachers, it could not be sustained due to some technical reasons. Some provinces, however, modified the concept to suit their context, and thereby continue to provide ongoing support to teachers. One such province was Limpopo which resuscitated, adapted and re-directed the approach to support the FET Mathematics teachers in 2023 in what was renamed the “1 + 9 Model”. This paper reflects on the nature and effect of this model of supporting Mathematics teachers. To do this, some theoretical considerations are necessary.

Theoretical Considerations (PKD)

Thiessen (1992) indicated that teachers are seldom involved in decisions about the content and structure of the workshops that they have to attend. They are expected to change their behaviour after brief demonstrations of what is required in workshops which are generally few and far in between. Thiessen then argues “that teachers should develop themselves. It should be less a matter of determining what ‘to do to them’ or on their behalf, and more a matter of teachers inventing what to do ‘*with others*’ or *by themselves*” (Thiessen, 1992:85, my emphasis). Clark (1992) calls it teacher “self-directed professional development”. Clark argues that to the question, “what can we do to make professional development programmes work for professional teachers?” the answer should be “give the responsibility... to teachers themselves” (p77). This is the essence of professional learning communities (PLCs). The ISPFTED (2011:14) describe PLCs as “communities that provide the setting and necessary support for groups of classroom teachers, school managers and subject advisors to participate collectively in determining their own developmental trajectories, and to set up activities that will drive their development”. The PLCs must be *professional*, and be characterized by *learning* within the context of the *community* of practitioners (DBE, 2015).

The World Bank (2018) advised that for effective teacher professional development, it has to be designed and targeted according to key concepts, *followed by coaching* around pedagogic strategies. The World Bank cited “programs in South Asia that provided long-term coaching [which] led to sizeable learning gains” (p22)

The “1+9 model” as adapted by Limpopo involved subject advisors and lead-teachers. While the locus of control was not entirely in the hands of teachers, the teachers were given enlarged space to discuss and shape their own lessons plans and approaches. This is elaborated on in the methodology used.

Methodology (RMG start, and PKD comes on board)

With the aim to improve the teaching and learning of mathematics in the FET, the Limpopo province revisited the old “1+4 model” and reconceptualised into a “1+9 model” for Grade 12 teachers. The model entailed the coming together of teachers in groups of twenty once every nine days to plan for the delivery of the lessons according to the schedule of the annual teaching plans (ATPs). With agreed upon topics, each teacher was asked to come with 9 prepared lessons to the session.

The programme started in 2022 with under performing schools, with the perceived improvements of the schools was expanded to the whole province in 2023. While the programme was implemented across the province, three groups of twenty teachers: two in Capricorn south and one in Sekhukhune East were observed over a period of three months in 2023. Observation was done by subject advisors and the Cuban Mathematics specialist. Journal notes were kept. An online questionnaire was also administered in six districts to gather teacher perceptions about the programme.

The lesson plans that teachers produced incorporated the teacher’s context, competencies and beliefs. The sessions were facilitated by a lead-teacher or a subject advisor in a professional learning community mode. One or two teachers *were randomly* selected to present a lesson per topic, after which the group gave input with the sole aim of strengthening the lesson for improved delivery. The inputs in the few presented lessons, helped the group members to each modify his or her own lesson in line with the group’s input.

Back at school, a teacher recorded in a journal mode what worked well and what was a challenge. Subject advisors too supported, coached and monitored the schools as they implemented. At the next “1+9 session”, feedback was given by the participants before starting with the planning for the next session. The following describes the “1+9 model” process



1+9 model” process

Findings

From the implementation of the 1+9 model, a number of crucial findings became evident. Observing three communities or “societies” (2 Capricorn South and one Sekhukhune South) over a period of three months in “1+9 sessions” revealed crucial observations. On the basis of these observations, it can be inferred that the 1+9 model is one of the key ways of helping teachers develop personally and professionally.

(a) A culture of collaborative teamwork was established

The Collins English dictionary define “culture” as “the ideas, customs and art produced and/or shared by a particular society”, “development or improvement by special attention or training”, “knowledge or refinement resulting from an interest in a particular activity”. Initially it was not easy for teachers to be open about their strategies. The tendency to be defensive about one’s work was very prevalent when the peers gave input and critique of presented lessons. This tendency diminished over the three months as teachers realized that the purpose was development and not criticism.

The sessions increasingly became “safer” for participants share views and strategies and to receive constructive feedback from their peers. Evidence of collaborative culture began to emerge with time and indeed clear traces of ‘*development or improvement by special attention*’ began to show among the three groups.

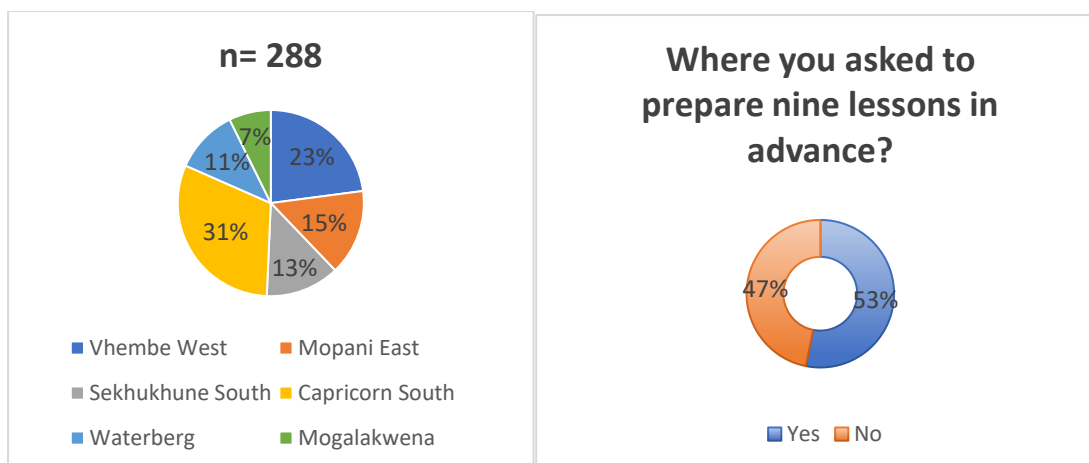
(b) Professional Growth Enhancement

Meaningful teacher professional engagement was evident throughout the sessions. Prior preparation of lesson plans coupled with the readiness to present the lessons to colleagues meant that teachers did not come “empty” to the PLC. The random selection for presentation on any one of the nine lessons meant substantive work was done before the sessions. For a teacher to ‘open up’ his/her lesson delivery to twenty other professionals meant that the platform created a “safe space” for teachers to engage professionally. Positive input and critique by one’s peers, followed by pruning of one’s work meant professional growth for participants.

Every “1+9 session” started with feedback from implementation of the previous cycle. Most teachers shared how their lesson delivery benefitted from input from peers and sometimes from monitors and coaches, resulting in **improved classroom practices**. These included, among others, techniques for differentiated teaching, learner-centered approaches and the integration of technology in the classroom.

(c) Teacher perceptions of the programme were positive

A questionnaire was administered online to six districts (including the three groups which were observed) about the teacher perceptions of the “1+9 programme”. There were 288 respondents to the questionnaire.



Two hundred and eighty eight (288) participated in the questionnaire.

- Fifty three (53) percent of them agreed to have been asked to prepare and bring nine lessons to the 1+9 sessions.
- Eighty nine (89%) brought all the nine lesson plans which were used for discussion and refinement.
- 72% said they found the 1+9 sessions helpful
- 80% said they are willing to participate in the program in 2024
- 65% believe the programme assisted to build teacher-to-teacher professional relations among the groups.

From the foregoing statistics, it is apparent that Mathematics teachers find the 1+9 programme useful. They are willing to have it continued.

(d) Establishment of mentorship through Partnership with Kagiso Trust was realized

In the area of Sekhukhune South, there are some mining operations. Kagiso Trust is involved in the area. As soon as they got to know and understand the 1+9 programme initiative they offered to be involved. They offered to hire some Mathematics coaches, usually experienced and/or retired Mathematics teachers to be part of the programme as coaches or mentors. Indeed, for the professional learning community to run well there has to be “knowledgeable others” as part of the facilitation. This is the role that lead teachers are already doing.

Pulling it Together and Insights for Further research

On the basis of the “1+9 session” observations as well as the questionnaire, it can be inferred that the 1+9 model carries huge potential for helping teachers develop personally and professionally. The adapted programme happened frequent enough and with a clear focus on planning and preparation guided by the ATPs. This opened a space for “continuing” professional development in a PLC-type format. This contrast the usually once-off and far in between teacher programmes commonly done in South Africa. The sessions kept alive the concepts of professional growth, professional learning, feedback and reflection, peer critique all towards developing together as communities of professionals.

Clark (1992:76) describes the *emerging* picture of a teacher as a reflective professional in a developmental process that begins with initial teacher education, grows and transforms through

practice towards a *maturing* professional teacher. The process involves subject content, the cognitive organization of that content, as well as contextual delivery of that content in ways that are useful to the teacher and his students. This paper argues that the “1+9 model” can contribute substantially Clark’s notion of progression from *emerging* to *matured* reflective Mathematics professional.

Going forward, the research on the 1+9 model can do well to focus on the following areas which could not be covered by the current paper. These are (i) to investigate how the model can help novice Mathematics teachers; (ii) to research and record what happens in the actual classroom and (iii) if, and in what ways, the model does improve learner performance.

Conclusion

The UNESCO (2024) Global report on teachers “highlighted the importance of reimagining the teaching profession as a collaborative profession, [and] recognizing professional development as a life-long learning journey [for teachers], page 16. This paper’s central tenet resonates with UNESCO’s report, in believing that the “1+9 model” provides a regular, coherent and systematic way of helping Mathematics teachers develop personally and professionally.

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INVESTIGATING THE MEANING OF THE EQUAL SIGN IN GRADE 1 DBE MATHEMATICS WORKBOOKS

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Researchers in mathematics education have long been concerned about the meaning of the equal sign and how best to teach the concept of equivalence. Across the world, studies suggest that many children understand the equal sign in operational, as opposed to relational terms, and that this understanding persists well into secondary schooling. Research further suggests that children's understanding of mathematical equivalence is highly influenced by how the equal sign is introduced in the early grades. In 2009, Essien reviewed three learner workbooks available to Grade 1 teachers. Concluding that the workbooks reinforced an operational as opposed to relational view of the equal sign. He recommended that workbooks be developed to better support teachers to introduce a relational view of the equal sign and mathematical equivalence from the early grades. This paper undertakes a systematic review of the Grade 1 DBE Workbooks, currently distributed to learners in South Africa across system scale. The paper concludes that the workbooks introduce and maintain the equal sign subsumed within the patterns of operational arithmetic. Further, the words used to speak to the equal sign encourage teachers and learners to adopt more operational as opposed to relational terminology in both isiXhosa and English. Noting the strong uptake of the DBE Workbooks as an instructional tool across the system, improving the workbooks in this area remains an important opportunity to support teachers to establish a more relational view of the equal sign in the formative early grades.

INTRODUCTION

Researchers in mathematics education have long been concerned about the meaning of the equal sign and how best to teach the concept of equivalence. A wide body of literature, across diverse country contexts, suggests that many children retain an *operational* as opposed to *relational* view of the equal sign, and that the persistence of an operational understanding of equivalence works against the development of algebraic thinking (Stephens et. al., 2013; Knuth et. al., 2008; Alibali et. al., 2007;; Knuth et. al., 2006; McNeil and Alibali, 2005).

Confronting the assumption that a child's understanding of the equal sign simply reflects individual cognitive capacity, research suggests that children's understanding of the equal sign is highly dependent on the way the equal sign is introduced in the early grades (Stephens et. al., 2013; McNeil & Alibali, 2005.) The meaning of the equal sign is dependent upon both the nature of the tasks as well as the tools and representations available to learners in the early grades (Kieran, 1981; Stephens, et. al., 2013; Knuth et. al, 2006; Essien & Setati, 2006; McNeil & Alibali, 2005).

Essien (2009) undertook an analysis of how the concept of the equal sign was introduced to Grade 1 learners, exploring three available Grade 1 textbooks at the time. He concluded that the materials reflected an operational understanding of the equal sign, subsumed within the introduction of the addition and subtraction signs. He recommended that future learning

material for Grade 1 support teachers to introduce the equal sign in a way that supports a more relational view of mathematical equality.

The authors of this paper bring together teacher leaders and educationists concerned about mathematics education in the primary phase in African language dominant classrooms. We have become interested in how the equal sign is communicated, especially in the context of isiXhosa instruction.

In 2011 launched an initiative to distribute learner workbooks for core primary subjects, including mathematics. Each year, a set of two learner workbooks are distributed to every Grade 1 learner in the South African public school system. As emphasised by McKay and Spaull (2022: 43), the workbooks have become the “de facto curriculum tool in South African schools,” establishing a new “minimum floor” of text resources.

Building on the work of Essien (2009), in this paper explore how the equal sign and the notion of mathematical equivalence is framed in these important national resources.

RESEARCH QUESTION

The investigation was framed by the following research question: How is the equal sign and notions of mathematical equivalence communicated in the DBE Grade 1 Workbooks in mathematics?

LITERATURE REVIEW

The Meaning of the Equal Sign: Operational vs Relational

As the most used notation in mathematics, there is a wide body of research on the meaning and interpretation of the equal sign in mathematics education (Kieran, 1981; Stephens et. al., 2013; Alibali et. al., 2007; Knuth et. al., 2006; Knuth et. al., 2008; McNeil, 2014; McNeil and Alibali, 2005; McAuliffe et. al., 2020; Essien, 2009; Essien & Setati, 2006).

Mathematical equivalence emphasizes sameness and substitutive. Mathematical equivalence is the relationship between two quantities which are both the same and interchangeable (Kieran, 1981; McAuliffe et. al., 2020; McNeil, 2014.) As emphasized by McAuliffe et al (2020), mathematical equivalence is a relation which is reflexive ($a=a$), symmetrical ($a = b$ and $b=a$) and transitive (if $a = b$ and $b = c$ then $a = c$.) An understanding of the equal sign with an emphasis on sameness and substitutive is known as a *relational* view of the equal sign.

However, across country settings, researchers have found that learners’ understanding of the equal sign is more *operational* than *relational*. That is, learners interpret the equal sign as signaling “do something” or “work out the answer” or “the answer is” (Kieran, 1981; Alibali et. al., 2007; Knuth et. al., 2006; Knuth et. al., 2008; McNeil, 2014; McNeil and Alibali, 2005; McAuliffe et al 2020). This unidirectional view is known as an *operational* view of the equal sign.

Teaching An Operational View of the Equal Sign

It has long been recognized that many children struggle to gain a relational view of the equal sign and mathematical equivalence. Students' difficulties in understanding the relational significance of the equal sign were historically attributed to low levels of mathematical proficiency amongst learners themselves. It was assumed that children who maintained an operational view would pick up a relational view over time (McNeil, 2014).

A wide body of literature confronts the assumption that the persistence of an operational view of the equal sign reflects the capacity of individual children. The literature suggests, rather, that children struggle to develop a relational view of mathematical equality because of the way the equal sign is introduced and developed in the early primary grades.

In their paper entitled in part, "Why won't you change your mind?" McNeil and Alibali (2005) formulate a "change resistant" account of mathematical learning (McNeil, 2014; McNeil & Alibali, 2005). Learning is a process of detecting and extracting (often subconsciously) patterns that are regularly encountered. The equal sign (and notions of equality) in the early primary grades are widely introduced within a narrow format of traditional arithmetic (Stephens et. al., 2013; McNeil, 2014; McNeil and Alibali, 2005; Essien, 2009; Essien & Setati, 2006; Prediger, 2010). There are two interrelated concerns. First, the equal sign (and the notion of equality) is often not introduced systematically and in its own right but rather subsumed within the introduction of operations, and in particular addition and subtraction. Further, the equal sign is introduced and maintained within the narrow operational patterns of arithmetic whereby: a) operations are on the left-hand side of the equal sign; b) the equal sign is followed by a space to write an answer; and c) the problem structure follows the pattern of "operations = answer" ($a + b = c$). Children establish a mental map through these consistent early encounters whereby the equal sign is unidirectional and signals "do something" or "total" or "the answer is."

These early mental models are deeply engrained, and resistant to change (McNeil, 2014; McNeil & Alibali, 2005). McNeil and Alibali (2005) conducted two experiments supporting this claim. In one experiment, children (aged 7 to 11) completed tasks focusing on operational patterns common to arithmetic including the "operations = answer" problem structure and the concept that the equal sign means "the total." Children were then given a lesson on equations outside of the structure of operational arithmetic and asked to solve a number of problems. They found a negative relationship between a child's adherence to operational patterns and learning of equations that were more relational in nature. In a second experiment, they found that when college students' operational knowledge was activated, they did less well on working with more relational equations (McNeil & Alibali, 2005).

The tenacity of an operational understanding of the equal sign is known to cause significant problems, especially when students transition from arithmetic to algebra (Stephens et. al, 2013; McNeil, 2014; Knuth et. al, 2008; Knuth et. al, 2006). When a child moves into middle school, virtually all manipulations of equations require understanding that the equal sign represents a relation (Knuth, et al. 2006). Reviewing research on instruction of the equal sign across phase of schooling, Knuth et. al. (2008) observe that after the initial introduction to the equal sign in

the early grades, little if any, instructional time is explicitly spent on the equal sign and the concept of equality in the later grades. Alibali, Knuth, Hattikudur & Stevens (2007) investigated the relationship between middle school students' understanding of the equal sign and their performance solving equivalent equation problems. They concluded that students' performance on equivalent equations varied "as a function of when they acquired a sophisticated understanding of the equal sign" (p. 221).

South African Teaching, Learning and the Equal Sign

While much of the research literature on the interpretation of the equal sign is derived from the Global North, there is an emerging and rich body of literature focusing on teaching and learning in South Africa.

There are a number of studies that explore the meaning of the equal sign among South African learners. Investigations into learners' interpretation of the equal sign in Grade 8 and 9 conclude that learners interpret the equal sign as a "do something" unidirectional symbol, in essence signaling to write the answer (Essien & Setati, 2006; Machaba and Makgakga, 2016)

McAuliffe et. al (2020) investigated 410 Grade 4 learners understanding of equivalence across six (6) primary schools in the Western Cape, where English was the language of teaching and learning. They highlighted several findings. They found that most students view the equal sign as an operational symbol, with most students rating "the equal sign means the answer to the problem" as a good definition and over half of learners choosing this as the best definition. That said, they also recognized that learners could hold different views of the equal sign (relational and operational) at the same time.

Essien (2009) undertook an analysis of three available Grade 1 learner workbooks in South Africa in 2009. He confirmed that all of the workbooks subsumed the introduction of the equal sign within the task of teaching traditional addition ($a + b = \text{---}$) and subtraction ($c - d = \text{---}$). He concludes that even if teachers attempt to introduce the equal sign with words such as "the same as", learners are likely only to appreciate the equal sign as simply a tool for enacting an operation ("do something") and writing the answer.

Providing recommendations for future curriculum and workbook design, especially from the earliest primary grades. Essien (2009) suggests both that the equal sign must be introduced in its own right before traditional additive relations and suggests the importance of providing a number of representations and problem settings that emphasise the relational meaning of the equal sign from Grade 1.

Instructional Documents: Curriculum and Learner Workbooks

From 2011, the Department of Basic Education introduced a new curricular framework (known as CAPS), a national assessment strategy, and designed and distributed national learner workbooks to all learners in the primary phase. The curriculum and the workbooks have become an enduring policy scaffolding over the past decade.

McKay and Spaul (2022) review the early ideas, development, and possible impact of the DBE Workbooks. Currently, over 65 million workbooks are distributed to learners in more than 24,000 schools (McKay & Spaul, 2022, p. 25). In mathematics, the workbooks take the form of 2 workbooks per year, each workbook made up of approximately 128 worksheets (4 terms, 8 weeks per term, 8 pages per week, and supplemental pages).

McKay was appointed to oversee the development of the workbooks (across subject areas, from Grade 1 to 6) in June 2010. By January 2011, they were designed, versioned into all 11 official South African languages, and distributed in over 24,000 schools. A development team made up of curriculum developers (largely working as consultants) were contracted for approximately one month of work per grade per book.

Using data from the School Monitoring Survey of 2017, McKay and Spaul report on the massive uptake of the workbooks in the system. 32% of teachers reported they focused all of their time (100%) on the workbooks. Another 36% said they use the workbooks for more than two thirds of classroom time. As emphasised by McKay and Spaul (2022, p. 43), the workbooks have become the “de facto curriculum tool in South African schools,” establishing a new “minimum floor” of text resources.

EXPERIENCE REVIEW: ISIXHOSA AND THE EQUAL SIGN

As discussed above, children’s understanding of the equal sign is highly dependent upon both the nature of the tasks as well as the tools and representations available to learners in the early grades. One of the most important *tools* available to teachers is our instructional language; the words we use to help learners make sense of a concept. As emphasized by Mostert (2021), and summarized by Barton (2012: 227), “we bring mathematics into existence by talking about it.”

This paper emerges from a growing interest amongst the author team about how teachers communicate the idea of “equal” in primary mathematics in isiXhosa. We are interested in the words teachers use to both “read” and describe the equal sign and the notion of mathematical equivalence. Accepting the language is discursive and interpretive, we are interested in how task design and language use mediate each other, and how this influences learners’ understanding of the equal sign.

There are a number of words primary school teachers use when they speak to the equal sign in English. As we began this study, we identified several words commonly used by primary school teachers to “read” the equal sign in English: *equal to*, *is*, *same as*, *make(s)* (Essien, 2009; Stephens et. al., 2013; Prediger (2010).

There has been less research about how teachers speak to and about the equal sign in isiXhosa (and other African languages). Without a wider research base, we turned to our collective empirical experiences. We identified three words that are commonly used in isiXhosa to talk to the notion of equal: *ilingana* (equal to); *ifana* (the same as); and *zenza* (make(s)).¹ Accepting

¹ As an agglutinated language, noun stems combine with verbs into a single isiXhosa word. The verb stem - *enza*, can be expressed, for example as *zenza* [it-makes] or *benza* [they-make]. In this paper, we present the isiXhosa words as they are most commonly used by teachers to talk about the equal sign. When the words take

the importance of mediation, we hypothesised that words derived from *-enza* lend themselves to a more operational view of the equal sign, while words derived from *-ilingana* and *-ifana* lend themselves to a more relational view.

Some of these words lend themselves to a more relational view of equivalence (equal, same as) while others lend themselves to a more operational view (make(s)). It is not suggested that words, in isolation, are responsible for facilitating a more relational or operational meaning of the equal sign. Rather, the meaning of the equal sign emerges from how language and tasks mediate each other.

METHOD

The study adopted a document analysis method (Bowen, 2009). Document analysis is a systematic procedure for evaluating print or electronic documents. Atkinson and Coffey (1997) refer to documents as social facts, which are produced, shared and used in socially organized ways. As described by Bowen (2009), document analysis entails the systematic examination and interpretation of both text and images.

At the centre of the study were Grade 1 Workbooks distributed by the Department of Basic Education (commonly referred to as the “DBE Workbooks” or the “Rainbow Workbooks.”) Mathematics for Grade 1 is presented in two workbooks, one allocated to each semester of work. The workbooks are translated into all official South African languages. For the purposes of this study we examined the isiXhosa and English versions.

The study applied mixed methods. We started with a quantitative analysis, counting the frequency of the occurrence of key terms, and checking each occurrence to ensure it related to the equal sign or a discussion of equivalence.

When we began, we identified three primary terms used in isiXhosa² (*zenza*, *-ilingana*, and *ifana*) and three primary terms used in English (*make(s)*, *equal*, and *the same as*) to “read” or describe the equal sign in Grade 1 curricular documentation. Without claiming linguistic equivalence, we observed a pattern in meaning of these sets of words. We paired words in isiXhosa and English based on apparent shared meaning. As we worked through the workbooks, we found an additional paired set of words (*-indinika* : *it give(s)*); we extended our list and counted the frequency. Further, in English we found evidence of using the word “is” to replace the equal sign. We interpret the shift from “makes” to “is” as a step toward a more relational expression of “is the same as.”

We coded each term by whether the term lends itself to a more operational or more relational understanding of the equal sign. Two word pairs (*zenza* : *make(s)*) and (*indinika* : *give(s)*) were coded as lending themselves to a more operational understanding of the equal sign. Two word pairs (*ilingana* : *equal/equivalent*) and (*ifana* : *the same as*) were coded as lending themselves to a more relational understanding of the equal sign. Using simple descriptive statistics, we

slightly different forms do to either the noun or the tense, we make notes where we think this has an impact on the meaning of the word.

² As an agglutinated language, the subject is attached to (modifying) the verb.

present the percentage of words lending themselves to an operational versus a relational understanding of the equal sign, across languages.

After a simple “count” of language use, we adopted a qualitative stance. We examined the workbooks to understand how the equal sign is communicated, and how task design mediates the meaning of the equal sign. We purposefully selected typical examples from the workbooks to illustrate and discuss how the meaning of the equal sign is mediated in this text.

Findings and discussions

Introduction

Like workbooks for other grades, the Grade 1 Mathematics workbooks are organized into two full colour books, roughly allocated to each semester. Each book is organized into roughly 64 2-page “lessons.” In terms of content design, the workbooks, across language versions, present the same material; the only difference between the texts is the language used.

Frequency of Use of Terms

Table 1 presents the frequency of the words used in reference to the equal sign across the isiXhosa and English in the Grade 1 Workbooks. The table presents the terms that were used organized into pairs that appear to share a common meaning, and whether the words lend themselves to a more operational or relational understanding of the equal sign. Figure 1 compares the overall use of “operational” versus “relational” oriented terms in the isiXhosa and English versions of the workbooks.

Overall, across both language versions of the workbooks, words that lend themselves to a more operational understanding of the equal sign dominate. In the isiXhosa workbooks, 85% of the references to the equal sign use operational terminology (*zenza*, *indinika*). In the English workbooks, operational terminology accounts for just under 75% (*makes*, *gives*). The fact that words derived from *-ilingana* account for only 2% of the terms used in reference to the equal sign stands out.

isiXhosa Term	English Term)	Operational / Relational	isiXhosa		English	
			N	%	N	%
zenza ³	make(s)	Operational	83	79	59	57
ilingana ⁴	Equal(s)	Relational	2	2	15	14
ifana ⁵	the same as	Relational	14	13	7	7
	Is				6	6
Indinika	give(s)	Operational	6	6	17	16
Totals			105	100	104	100

³ *zenza* [it-makes]; *benza* [they-make]; *unokwenza* [to be making]

⁴ *ilingana* [equal]; *izilingana* [that is equal]; *nokulingana nelinye* [to-be equal-to the-one]

⁵ *elifana* [that is the same]; *afana* [that are the same]; *efanayone* [that is the same as]

TABLE 1: Frequency of key words related to ‘equal’.

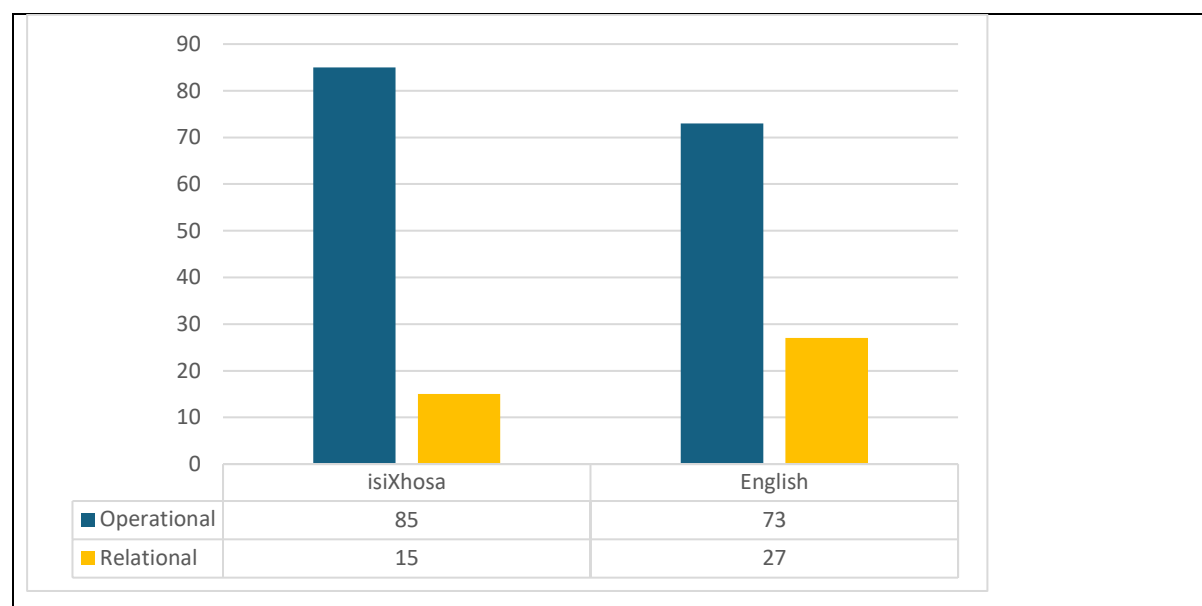


FIGURE 1: Use of Words: Operational and Relational (%)

Introduction of Equality and the Equal Sign

In the first 15 lessons of the workbook, two items make explicit reference to notions of equality, and a few additional items ask children to match items that have a common attribute (colour, position etc.) In the first explicit reference to equality, learners are asked to find a picture that is the same (-ofana.) In the second, learners are asked to tick the blocks that have the same number, introducing the word stem *-ilingana* for the first time in the isiXhosa texts.

Khangela umfanekiso ofana nalo ukwibhokisi yokuqala.

Phawula ibloko okanye ibloko ezinenani elilinganayo lezinto.

☐

☐

☐

☐

FIGURE 2: Early language of sameness .

The use of the equal sign *in words* is introduced in the 15th lesson. See Figure 3. This is a vital moment of instruction, as it begins to set up the format of a mathematical equation, following the conventions of arithmetic, it is presented from left to right. From this lesson through to lesson 30, the workbooks use the word “zenza” and “indinika” (and in English “makes” and

“gives”) in place of the equal sign. In the third example below, the equal sign is essentially read as *fakele impendulo* or fill in the answer.

From Lesson 22, there is a subtle shift in the English workbooks. See Figure 3, exemplar D. While the isiXhosa workbooks continue to primarily use -zenza (and to some extent -dinika), the English Workbooks shift from “makes” and “gives” to the word “is.” To the extent that the use of ‘is’ may be building toward ‘is the same as’, the English version veers closer to relational terminology.

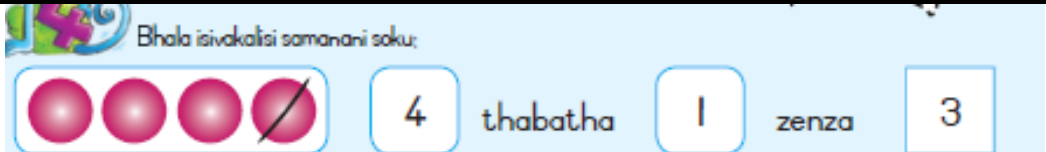
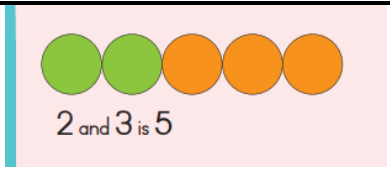
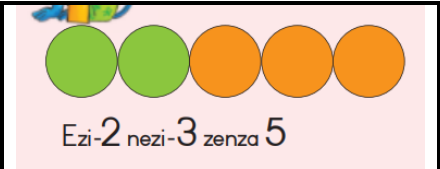
A	 Instructions in English: Complete the following by making a drawing: [Page 35]
B	 Instructions in English: Write a number sentence for: [Page 35]
C	 Instructions in English: Add these and then fill in the answer. [Page 36]
D	 Instructions in English: Complete the following:
E	  Shift to equal as “is” in the English Workbooks (Page 48).

FIGURE 3: ‘Equal’ as a word introduced as -zenza and ‘dinika [makes / gives] .

Use of the Equal Sign as a Symbol (=)

The first use of the equal sign *as a symbol* is presented in Lesson 29. This is the first time that children are exposed to an equation that reads more relationally, where the operations are on the left-hand side of the equal sign.

The equal sign is not presented (either as a symbol or in words) after this page until Lesson 43, at least 3 weeks later. It is at this point that examples shift from iconic pictures to number symbols. In this same lesson, children are presented with a page of addition problems, all presented from left to right, following arithmetic conventions. The following two lessons continue to present addition and subtraction, presented within this convention ($a + b = c$; $a - b = c$). We see this standard arithmetic equation continued across lessons for repeated addition (see Lessons 50, 52-55).

The second workbook (supporting the second semester) continues to use the equal sign only as it relates to operations. The equal sign is used at least once on 35 pages, roughly one quarter of the pages. On all of these pages, the equal sign is subsumed within pages that focus on operations – addition, subtraction, repeated addition, sharing, double and half. In all cases, across this second book, the equations are presented in the conventions of traditional arithmetic, with the operations on the left, followed by an equal sign, followed by space for an answer.

English Instructions: Share the objects between the two boxes. Draw the two groups in the separate boxes. [Page 65]	
English Instructions: Add these sweets and then fill in the numbers.	
English Instructions: Try these.	

FIGURE 4: Use of the Equal Sign as a Symbol.

The Notion of Equality in Other Content Areas

While the equal sign is primarily used in the context of number work, the notion of equality can be leveraged across wider mathematical content areas. Outside of the very early comparison work presented above, there were three examples where notions of equality or sameness were used in other content areas in the workbooks.

In the first semester book, the notion of equal was leveraged to engage with volume. See Figure 5, Exemplar A. The lesson provides a high value representation for mathematical equality, supported using -ilingana, which lends itself to a relational understanding. The page itself focuses attention on “more than” and “less than” as key concepts rather than placing emphasis on the relational notion of “equal to.”

One of the notions that helps children think about equality, as suggested in the research literature, is the notion of mass, and specifically thinking about balancing a scale. In the second workbook, there is one lesson that presents pictures of scales, with items to help learners think about scales. See Figure 5, Exemplar B. Across the two pages, there is only this one (complex) item focusing explicit attention on the task of equalizing the scale. The language in the isiXhosa version, using a derivative of -ilingana (kulinganiswe) provides better language than the English version which speaks about “needing to balance.”

The final use of “same as” is presented in a late year lesson on symmetry, where children are asked to draw a line to show two equal halves. The words of -ilingana (alinganayo) and “equal” are used in the isiXhosa and English worksheets respectively.

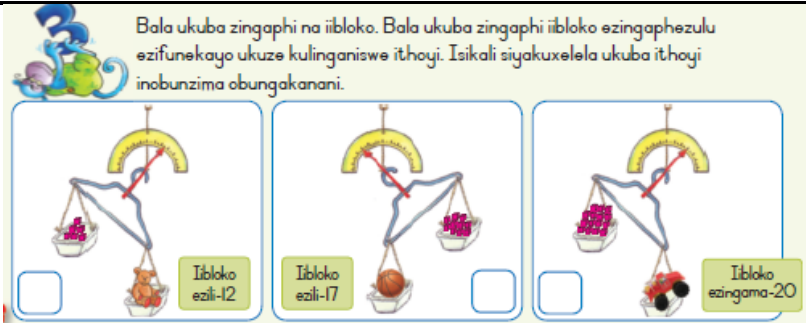
A	 English Instructions: Circle more then, less than, or the same as. Page 88.
B	 English Instructions: Count how many blocks there are. Then work out how many extra blocks are needed to balance the toy. We tell you how many blocks the toy weighs.

FIGURE 5: The Notion of Equality in Other Content Areas

Summary

As we reviewed how the equal sign is introduced and developed across the Grade 1 DBE Workbooks, several observations stood out. Across the Workbooks, the equal sign is communicated to teachers and learners primarily signaling operational terms. Operational terms (*-enza* and *-dinika*) account for 85% of the references to the equal sign in the isiXhosa workbooks, and over 70% of the references in the English workbooks.

In the early lessons of Grade 1, there is some work that lends itself to developing learners' relationship with equality, for example by identifying an object that "is the same as." See Figure 2. While this important early work introduces the more relational term of *ilingana*, it is neither highlighted or well developed. It is unlikely that teachers will appreciate this work as an opportunity for developing children's relationship with mathematical equivalence.

When learners are first exposed to the formal notion of the equal sign in words, it is presented within the traditional operational pattern (see Figure 3). Supporting Grade 1 teachers to help children listen and think about the equal sign in (spoken) words before introducing the more abstract symbol is extremely valuable. The opportunity presented by these items to communicate a more relational starting point for the understanding of equality is weakened both by confining these items to a unidirectional operational pattern and restricting terminology to operational terminology (*zenza* (make(s)) / *indinika* (give(s))).

Arguably the most interesting page of the workbooks for this investigation appears in Lesson 29, where the equal sign as a symbol is first introduced. See Figure 4. On this important page, the equal sign is not represented in a more relational pattern ($4 = 2 + 2$), that works against a narrow operational understanding of the equal sign. This is the first time learners are exposed to the equal sign and the first time learners are exposed to an equation outside of the traditional operational pattern. Given a lack of explicit signalling, teachers are unlikely to appreciate the potential opportunity created by this page to discuss a relational view of the equal sign.

Unluckily, the use of the equal sign from this page forward retreats back into the traditional operational format. From this page forward, the equal sign is presented within operational equations restricted to the traditional arithmetic pattern ($a + b = c$).

While there is some reference to the notion of equality working with volume, mass and symmetry, the references are at best incidental. Without signaling the notion (or terminology) of equality as a central concept, these opportunities are unlikely to be used by teachers to deepen children's relational view of the equal sign.

Conclusion

There is a rich body of literature exploring the meaning of the equal sign in South Africa (McAuliff et al, 2020; Essien, 2009; Essien and Setati, Machaba & Makgakga, 2016). Taken as a whole, the literature suggests that like other country contexts, South African learners adopt an early operational view of the equal sign which persist well into secondary schooling.

One of the most powerful contributions of research in this area is the suggestion that children's development of a relational understanding of mathematical equivalence is highly influenced by how the equal sign is introduced in the early grades. When the introduction of the equal sign

is both subsumed and maintained within the patterns of operational arithmetic, the equal sign is communicated as a unidirectional signal to “do something” (“total”) or simply “the answer is.” A narrow operational view of the equal sign has been demonstrated to be difficult to change.

In 2009, Essien made recommendations to develop learner workbooks that assist teachers to provide a more relational view of equality starting from the formal introduction of the equal sign in Grade 1. In this paper we review the Grade 1 DBE Workbooks, distributed at system scale from 2011. Two overarching findings stand out. First, the equal sign continues to be subsumed within the introduction of operations and maintained overwhelmingly within the patterns of operational arithmetic. Second, the words used in the workbooks encourage teachers and learners to adopt more operational (*zenza* (make(s)); *indinika* (gives)) as opposed to relational terminology in both isiXhosa and English. The nature of the tasks mediates instructional language, reinforcing an operational as opposed to relational understanding of the equal sign.

The review supports the recommendations originally made by Essien (2009) that instructional materials should be purposefully designed to help teachers communicate a relational understanding of the equal sign from the early grades. Given the strong uptake of the DBE Workbooks as an instructional tool across the system (Spaull & McKay, 2022), improving the workbooks in this area represents an important opportunity to support teachers to establish a more relational view of the equal sign in the formative early grades.

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CLASSROOM DESIGN INTERVENTION STRATEGIES TO IMPROVE GRADE 10 LEARNERS' COMPREHENSION OF SIMILAR TRIANGLES.

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Some South African mathematics teachers face several challenges in devising creative instructional strategies that can effectively aid their students in comprehending geometry concepts. This presents a significant concern for the nation, as learners' capacity to understand problem-solving abilities in the domain of similar triangles, a subsection of Euclidean geometry, is notably low. In this paper, we evaluate the effective intervention strategies that improve Grade 10 learners' comprehension of similar triangles. The purpose of this mixed-method paper is to investigate the effectiveness of specific classroom intervention strategies in enhancing the understanding of similar triangles among 43 grade 10 learners. The quantitative phase was used to assess the learners' foundational knowledge of similarity and to measure the impact of the intervention on their comprehension during the qualitative phase. After the intervention session, the post-test was conducted to compare the results with the baseline assessment, evaluating both the effectiveness of the intervention and the extent to which the learners' understanding of similarity had evolved. The results indicate that learners' performance in geometry assessments dropped below the expected level, with their content knowledge equivalent to below grade 7 standards. To elucidate the impact of the modification, descriptive and statistical data analysis were employed. The study findings reveal that 50% of the learners in this group demonstrated improvement based on performance metrics related to the understanding of similar triangles when comparing similar items from both tests. The findings suggest that the implemented intervention has the potential to be an effective strategy for imparting knowledge, fostering deeper learning, and enhancing student engagement.

Keywords: Mathematics, Geometry, Similar triangles, Intervention, triangle.

Introduction and background

The majority of mathematics in South African classes have a negative view of similar triangles (Naidoo & Kapofu, 2020; Ngirishi & Bansilal, 2019; Van Putten et al., 2010). In Grade 10, misunderstandings, and difficulties seeing the abstract character of geometry topics are common obstacles to comprehending similar triangles (Ngirishi & Bansilal, 2019). Errors are frequently the result of learners' inability to grasp geometry's deeper conceptual knowledge (Makhubele, 2014). The study by Bansilal and Ubah concentrated on the difficulties that learners had while naming and recognising similar triangles using symbolic representations (2019).

Furthermore, understanding the sequence of vertices in triangles that are in similar relationships, as well as the specialised similarity notation ($\sim$), proved to be challenging for several learners. These misconceptions resulted in inaccurate representations of similarity connections. Based on erroneous reasoning, learners mislabelled the triangle. The study emphasised the significance of fluency in the transition between geometrical symbolic and visual representations. Similarity can enhance the ability of learners to visualise, critically

think, intuition, problem-solve, conjecture, deductive reasoning, logical argument, and proof (Jones, 2002).

Compared to other mathematical subtopics, similarity requires more advanced cognitive thinking (Atebe, 2008). Learners cannot frequently use figure attributes during geometric reasoning (Mabotja et al., 2018). The findings of their study indicate that learners typically seek aid from teachers in folding back and accessing attributes required for proof. In addition, while manipulating and proving axioms, learners require assistance in mixing pertinent ideas. The difficulties that learners confront indicate knowledge gaps relating to the second precept (explaining and linking) and the third tenet (axiom formation) (Usiskin, 1987). Thus, using geometric representations and thinking can aid learners in understanding other areas of mathematics and applying the skills they have learned (Jones, 2002).

Similarity is one of the important concepts linking many mathematical areas, such as fractions, ratios, proportions, and the congruency of triangles. Specific real-world scenarios are required for several geometry standards related to triangles (Stevens, 2023). Stevens continued by saying that the topics covered in this strand include congruence, similarity, trigonometric ratios, and special right triangles. These can be applied to a variety of real-world triangular shapes, including buildings, homes, flagpoles, maps, trees, hair braiding styles, and any triangular arrangement of people or objects. In Grade 12, Similarity reappears as the relationship between triangles with specific properties.

The conceptualisation of similarity is based on understanding number sense and involves a comparison of the relationship of proportionality between corresponding lengths in similar shapes. Sanwidi and Swastika (2018) state that even if similar shapes have the same shape, it does not mean that the length must be the same size or equal. A crucial element of conceptualising similarity is a clear understanding of ratio and equality, which can be transcended to geometry applications. According to Misnasanti et al. (2017), the idea of ratio and proportion are key in many fields of knowledge and are significant in mathematics. Similar triangles of the same shape with corresponding different sizes can be classified as proportional triangles. The ratios of linear measures in one triangle will be equal to the corresponding ratios in the other triangle.

Other studies indicate that research on similarity suggests that abstract ideas, like deductive inquiry, are often ineffective in practical teaching methods (Steele, 2013). A hands-on geometry instruction can enhance learners' mathematical thinking, communication abilities, and creativity (Bankov, 2013). Furthermore, Chiphambo (2011) suggests that learning through doing helps retain mathematical concepts, and Mabotja et al. highlight that learners should be able to apply figure qualities to solve geometric issues (2018).

Conceptualising geometric knowledge requires an educational process that aligns elementary figures and concepts through intervention tactics and well-incorporated teaching materials (Bussi & Frank, 2015). To improve geometrical knowledge, proposed interventions should include both visual and verbal representations, (Alex & Mammen, 2018; Pule & Ngoveni, 2024), in their argument. Riccomini, Smith, Hughes, and Fries (2015) emphasise the need for a flexible and fluent vocabulary foundation (Mabele & Pule, 2023), including numbers, symbols, and diagrams, and an understanding ability to communicate mathematically. To

enhance and conceptualise similar triangles, an intervention addressing early childhood years and promoting interaction is designed (Fujita & Keith, 2003). The purpose of this paper is to investigate strategies for classroom design intervention that can help Grade 10 learners comprehend similar triangles.

Problem statement

According to Biber's research, learners typically struggle to solve problems with overlapping similar triangles (2020). They do, however, far better when handling problems involving adjacent or distinct triangles. Students also have trouble recognising the corresponding angles between two identical triangles. The Angle-Angle-type questions get the highest success percentages. According to Biber (2020), there are three minimal requirements for two triangles to be similar: they must meet the Side-Angle-Side (S.A.S.), Side-Side-Side (S.S.S.), and Angle-Angle (A.A.) similarity rules. It is an area that has to be addressed to assist learners develop abstract thinking, intuition, and visualisations to help them tackle daily practical difficulties, just to mention a few (Chiphambo & Feza, 2020).

The 2018 National Senior Certificate mathematics examinations reported that learners performed the lowest in Euclidean geometry questions of similar triangles in mathematics (DBE, 2019). It was found that several factors, including teachers' teaching methods, learners' learning methods, geometric terminology, visualising skills, gender differences, poor reasoning skills, insufficient teaching time, a limited school curriculum, and learners' lack of proof skills, all impede and have bad effects on learners in learning similar triangles (Aysen, 2012; Mashingaidze, 2012; Uduosoro, 2011). Against the backdrop, the purpose of this study is to investigate the classroom intervention strategies to improve Grade 10 learners' understanding of similar triangles. The main research question is "How can the classroom intervention strategies be designed to improve Grade 10 learners' understanding of similar triangles"?

Conceptual framework: A learning constructivist environment

This paper is hinged around the constructivist approach to learning. It argues that learning should occur in a constructivist classroom, with teachers and learners embracing the theory of constructivism. As per the findings of Miller-First and Ballard (2017), Constructivist teachers actively seek and value students' viewpoints, organising classes to challenge preconceptions. This approach places a strong focus on learner engagement by empowering them to construct their understanding through interactions and experiences with the subject matter. Constructivist teachers prioritise learner comprehension and involvement to foster a dynamic, collaborative learning environment that cultivates critical thinking and problem-solving skills. Mvududu and Thiel-Burgess (2012) describe constructivism as a method for assessing a learner's level of knowledge and demonstrating its potential for advancement to higher levels of thinking.

Constructivist teaching practices shift the focus from passive information reception to actively empowering learners to construct their knowledge. Through collaboration, learners can engage in problem-solving within a supportive learning environment. According to Kim (2005), Fernando and Marikar (2017), Miller-First and Ballard (2017), and Zajda (2021), it fosters a collaborative learning environment where students work together, support each other, and enhance their problem-solving abilities. For instance, in the context of comparable triangles,

constructivist teaching methods involve learners in similarity problem-solving activities using tangrams and pattern blocks to deepen their understanding of geometric shapes and their relationships (Sales, 1994).

In constructivist classrooms, learners engage in a learning approach called learning by learning, where they both learn from and instruct their peers. This pedagogical method aligns with the principle of learner-to-learner interaction (LLI) and has been effectively used in this study to enhance the comprehension of similarity triangles. The peer learning strategy is versatile and can be implemented in various teaching and learning formats, as evidenced by the work of Miller-First and Ballard (2017).

Intervention

Design intervention, as defined by Halse and Boffi (2014), is a unique approach aimed at creating impactful experiences and addressing emerging issues within a specific context. The goal of design intervention is to elicit thoughtful action within the setting of interest (Halse & Boffi, 2014) which is the grade 10 mathematics classrooms.

In this study, the authors facilitated meaningful interaction between teachers and learners to foster a constructivist learning environment, with a specific focus on engaging the learners actively in the learning process. In the study conducted by Lin et al. (2020), the results demonstrate a strong and positive correlation between teacher leadership, learners' freedom styles, and their mathematical learning outcomes. This research study used both teacher-to-learner interaction (TLI) and learner-to-learner interaction (LLI) but concentrated more on LLI.

Teacher-to-learner interaction (TLI)

TLI allows learners to take an active role in their mathematics learning. During the intervention, learners are given an open-ended question to discuss among themselves or with the authors. Such queries enable learners to be themselves and to voice their ideas about the diagram or what they see. By pointing learners in the direction of solutions to the issues, the teacher serves as a facilitator to enhance learning. This method encourages learners to participate more actively while gaining self-assurance and mathematical proficiency. According to Lin et al. (2020), the power of positive teacher-learner interactions in classroom settings cannot be overstated. Furthermore, they have demonstrated that these interactions can significantly impact learner motivation and emotions. As such, teachers must build strong relationships with their learners, fostering an environment of trust and respect that can lead to improved academic outcomes.

In Figure 1 below, the “Happy Face emoji” concept was used to demonstrate similar triangles and was characterised as a way to determine the equivalent sides of similar triangles. The three guiding principles of the “Happy Face” method are “eye for an eye, eye for an eye, and mouth for the mouth”. In triangles $\triangle ABC$ and $\triangle PQR$ which are similar, AB is the eye, PQ is another eye, BC is the eye, QR is another eye, AC is the mouth and PR is another mouth. Therefore, it implies that AB and QR, BC and PQ, and AC and PR are corresponding sides of each other, where the ratio of $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$ will be proportional to each other.

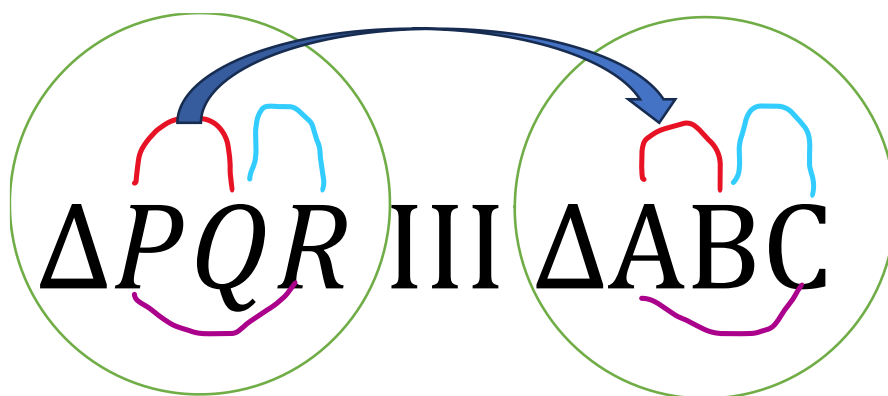


Figure 1: “Happy Face” concept

Learner-to-learner interaction (LLI)

As the education system continues to evolve, it is crucial to adopt effective approaches. LLI, for example, promotes social interaction by facilitating the sharing of problem-solving strategies and concepts among learners. This approach supports the constructivist philosophy and can benefit both reserved and bright learners. For instance, reserved learners can benefit from increased social interaction, while bright learners can help struggling learners understand concepts better. According to Lin et al. (2020), learner-centeredness is widely regarded as one of the most promising approaches in education and policy discussions today. By focusing on putting learners at the centre of education, we can witness the positive impact it has on their learning experience. In the study by Sato and Storch (2022), learners were questioned before participating in peer interaction tasks in the classroom. The findings revealed that learner views were not connected, not only to their interactional actions but also accounted for assessed second language progress.

Research Method

The interpretations of this research study create a subjective understanding of reality. The Van Hiele levels of geometric thought were used to assist in determining and identifying grade 10 learners’ geometric level and understanding, through the use of a self-designed baseline test to measure learner geometric level. The quantitative approach was used with baseline tests and post-tests as data collection instruments. The population comprised 43 FET band students who are mathematics at the selected school. The baseline tests were used to determine the learners’ present knowledge of geometry. After the designed intervention, the post-test, in comparison with the baseline test, was used to determine the effectiveness of the intervention and the change in understanding of the learners’ concepts of similarity. Mamali (2015) points out that data analysis is a means to minimise gathered data into a manageable subject, generating summaries and interpretations. To effectively showcase the impact of an intervention, tables were used to display data understandably and concisely illustrate study findings through the use of descriptive statistical analysis.

Discussion of findings

The research findings are sourced from the participants based on the two questions that follow:

Question 1: *What fundamental knowledge of Euclidean geometry and similar triangles do learners in the 10th grade have?*

We use general concepts to determine foundational knowledge.

- The hour-hand clock moves a quarter way from noon. On which number will it now point? Through what angle will it have moved?
- An artist drew the Union Buildings in Pretoria. An observer said that the artist got the proportions wrong. What does he mean?

The study evaluated learners' foundational knowledge using the questions above. Results in Table 1 showed that 6.98% did not attempt to answer questions one and two, while 37.20% and 83.72% attempted but answered incorrectly; 32.56% partially answered question 1 and none for question 2, and 23.26% and 9.30% correctly answered question 2.

Table 1 *Summary of learners' marks*

CODE	Descriptions	The number of learners in question 1	Percentage (%)	The number of learners in question 2	Percentage (%)	Total learners in both question	Percentage (%)
-1	No attempt	3	6,98	3	6,98	6	6,98
0	Incorrect	16	37,20	36	83,72	52	60,46
1	Partial Correct	14	32,56	0	0	14	16,28
2	Correct	10	23,26	4	9,30	14	16,28
TOTAL		43	100	43	100		100

The data visualisations below depict learners' responses to question 1. Learners who fall into code -1: this indicates that they do not recognise a quarter in the context of time, nor do they grasp proportion in the context of art or drawing. As a result, identifying the angle and explaining its significance (Mabotja et al., 2018) became a much more challenging task. Most of these pupils were uncertain as to whether the clock moves in a "clockwise" or "anti-clockwise" direction (refer to Figure 2), these are some of the learners falling under coder 0. Nevertheless, in the image below, a learner successfully rotated the quarter away from the noon position, even though it is in the wrong direction (anti-clockwise).

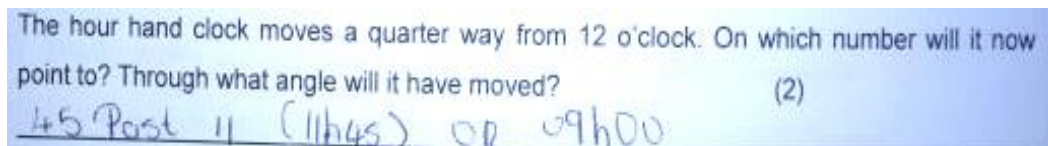


Figure 2: Learner NMT18 response

Students who fall under code 1 are those who answered question 1 partially correctly, with varying degrees of understanding. Some understood the concept of a "quarter" but were unable to identify (Jones, 2002) the correct angle, while others correctly identified the angle but had an incomplete understanding of how a clock moves in a "clockwise" direction (see figures 3 and 4 below). On the other hand, students who fall under code 2 correctly answered question 1, indicating a strong comprehension of the question (refer to Figure 5 below).

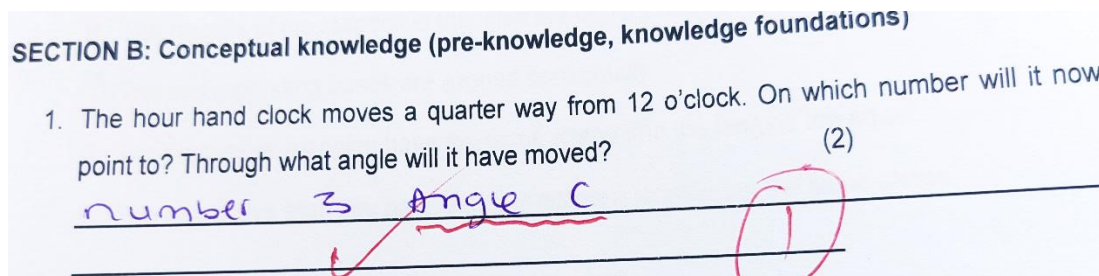


Figure 3: Learner NMT22 response

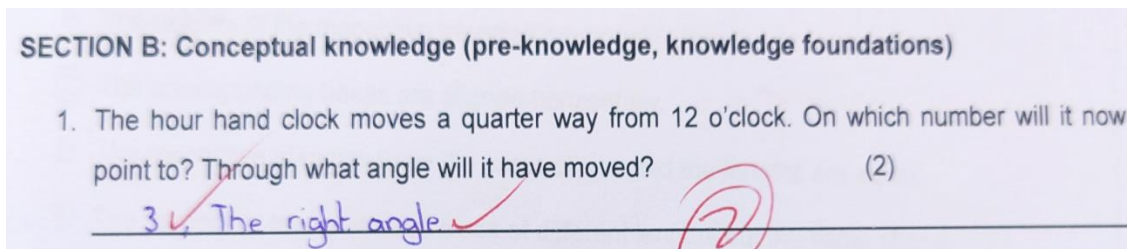


Figure 4: Learner NMT15 response

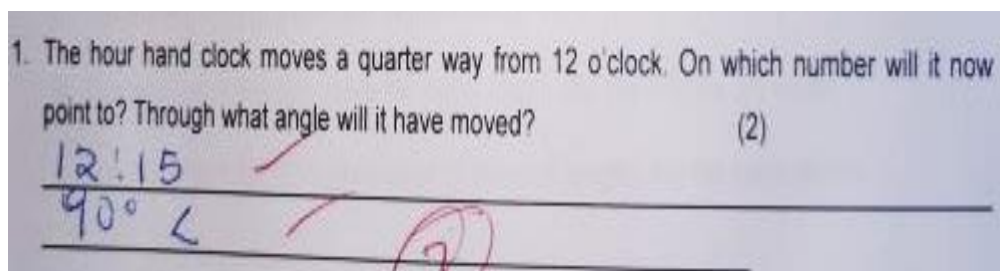


Figure 5: Learner NMT28 response

Both tests

The baseline and post-test were compared based on the following: a comparison of each item between the baseline and post-test, as well as a comparison of the same items between the baseline and post-test. The purpose is to evaluate whether learners have improved understanding or not. The learners who improved positively might mean that they took the intervention very seriously, while the learners who declined might mean that they were comfortable and relaxed during the intervention. Tables 2 and 3 below indicate those comparisons.

Same item comparison, looking at the order of improvement.

Table 2 below indicates the comparison of the same question items from the baseline test and post-test. The following are the items: B8, B9, B12, B15, and B16 from the baseline and P4, P6, P10, and P11 from the post-test. The above items were compared, and Table 2 below shows that 48.8% (21) of the learners' marks improved, 18.6% (8) of the learners' marks remained the same, and 32.6% (14) of the learners' marks declined. When looking at learners whose marks remained the same, they have scored the marks from different items, which shows that learners are not sure.

Table 2 Same items comparison

	No. of learners	Percentage (%)
Positive improvement	21	48,8
No changes	8	18,6
Negative improvement	14	32,6
TOTAL	43	100

Comparison per items from both tests

Table 3(a) below indicates the comparison per item from the baseline test and post-test looking at the order of improvement based on average. It shows that the average mark percentage from baseline and post-test items are 28.5 % for B8 and B9, 48.8 % for P4, 0.58 % for B12, and 3.5 % for P6, 46.5 % for B15, and 44.2 % for P10, and 20.9 % for B16, and 36.1 % for P11. It also indicates that B8, B9, and P4, B12, and P6, B16, and P11 are items that increased, and B15 and P10 are the only items that decreased slightly. It further shows that B8, B9, and P4, B12, and P6, B16, and P11, their average mark percentage increased positively, and B15 and P10 decreased slightly. Table 3(b) corroborates the findings on average mark percentages by demonstrating a positive increase in the average per item, as indicated in Table 3(a).

Table 3(a): The item-by-item comparison

Items (marks)	Baseline average mark performance (%)	Post-test average mark performance (%)
B8 (2), B9 (2) and P4 (4)	28,5	48,8
B12 (4) and P6 (4)	0,58	3,5
B15 (2) and P10 (2)	46,5	44,2
B16 (2) and P11 (2)	20,9	36,1

Table 3(b): The item-by-item comparison

	B8	B9	B12	B15	B16		P4	P6	P10	P11	
	1.1.1 (2)	1.1.2 (2)	2.1 (4)	4.1 (2)	4.2 (2)	Total (12)	1.4 (4)	2.1.1 (4)	3.1.1 (2)	3.1.2 (2)	Total (12)
Total scored	23	26	1	40	18	108	76	6	35	28	145
Average per item	0.5	0.6	0	0.9	0.4	2.5	1.8	0.14	0.8	0.7	3.4

Post-test additional items

The baseline assessment included challenging questions to gauge learners' knowledge and progress in Grade 10 Euclidean geometry and most of them were similarity questions. 20.9% of learners confidently answered these questions, indicating a strong conceptual understanding (Mabotja et al., 2018) of similarity. This group has the potential to excel in advanced geometry topics in Grade 12 and beyond. A Continuous challenge possibly assists learners to fully be prepared for academic success.

The study reveals that learners in the baseline test performed within the range of Grade 7 learners. The results show that the learners' prior knowledge or fundamental knowledge and skills are not solid according to their grade level. Thus, learners are not reasoning logically according to their level of maturity rather than their grades. For example, in the conceptual knowledge section B learning activity, **NMT18** and other learners were unable to understand how the hand of the clock moves, such as "*it will point to 11:45*" and how certain angles are formed based on Euclidean geometry. These were also highlighted by concepts like clock rotation, *forming a revolution in a circle*, which is equal to 360° . Furthermore, forms other angles; namely, acute, obtuse, right angle, etc. As the clock moves a quarter on the watch, it forms an angle which is a quarter of 360° is equal to 90° .

A portion of Grade 10 students understand that architectural drawings are scaled representations on paper, emphasising the significance of scale and proportion in construction. The accuracy and realism of the scale or ratio on paper are crucial. The findings indicate that students lacking fundamental understanding and the capacity (Mabotja et al., 2018) to connect geometric principles (such as proportion) to architectural concepts or building plans struggled to answer the questions. Only four students successfully comprehended and answered the questions. These mathematical concepts interconnect with rates, decimals, fractions, and percentages. According to the Grade 4 mathematics CAPS document, these topics form part of the curriculum for Grades 4 to 6 (DBE, 2011). Consequently, learners in Grade 10 should have a solid grasp of these concepts, given the progression of the syllabus from Grade 4. However, the study uncovered a concerning discrepancy: learners struggled with basic multiplication calculations, making this question one of the most challenging for them.

Finally, in the structured question, which entails grades 8 and 9 content, all those concepts are evident to learners' fundamental and present knowledge. The parallel line with a transversal

line is introduced in grades 8 and 9 and consists of the basics of geometry, for example, straight line, alternative, and vertically opposite angle, etc. This study indicates that learners can identify other shapes of geometric figures, but their reasoning does not justify the correct statement (Gunhan, 2014). Thus, learners use alphabet symbols to conceptualise and learn different types of angles. For example, learners recognise the symbolic alphabet such as “Z” on a geometric figure that is associated with an alternative angle, but they struggle to recognise it when the symbol is skewed or not straight.

The research findings suggest that certain students struggle to grasp enlargement concepts, which involve ratio and proportional thinking (Sanwidi & Swastika, 2018). They lack understanding of how to adjust or scale up a photo to a different size. Additionally, when tasked with depicting a two-dimensional (2D) figure based on given dimensions, students displayed difficulties. Some drew three-dimensional (3D) shapes, while others represented a figure without proper dimensions. This concept is applicable in real-life situations. Sharma and Bansal (2017) endorse the constructivist idea that observation and scientific study facilitate optimal learning. Consequently, students are encouraged to observe such concepts in their everyday experiences to enhance their understanding.

Question 2: *How does the learner-learner interaction approach influence the learning of similar triangles?*

Following the implementation of the designed intervention, learners underwent a post-test aimed at evaluating both the efficacy of the intervention and the extent of their understanding regarding parallel lines and similar triangles. The post-test results indicate observable changes in performance, as shown in the figures below:

1. Given the diagram below, $AB \parallel CD$.

1.1 Name two angles which are equal to x and give reason. (4)

STATEMENT	REASON
1.1.1 $\angle APJ$	Opposite to each other
1.1.2 $\angle OYD$	Parallel to each other

1.2 Name two angles which are equal to y and give reason. (4)

STATEMENT	REASON
1.2.1 $\angle OYD$	Parallel to each other
1.2.2 $\angle CJP$	Opposite angles

Figure 6 (a) : NMT08 baseline test

4. Given the diagram below, $AB \parallel CD$.

Name two angles which are equal to x and give reason.

STATEMENT	REASON
Q ✓	Opposite angles ✓
y ✓	Corresponding angles ✓

Figure 6 (b): NMT08 post-test

In the baseline, Figure 6 (a) illustrates a situation where a student struggled to produce an accurate statement (Bansilal and Ubah, 2019), such as “APJ and OYD”, and opted instead for an irrelevant reason like "opposite to each other" and "parallel to each other". In contrast, Figure 6 (b) demonstrates progress, as a student is now able to provide a correct statement with appropriate reasoning. However, it should be noted that in this case, while the statement itself is correct, one reasoning provided is not.

Name two angles which are equal to x and give reason. (4)

STATEMENT	REASON
1.1.1 Q ✓	Opposite angles ✓
1.1.2 y ✓	lo-interior angles ✓

Name two angles which are equal to y and give reason. (4)

STATEMENT	REASON
1.2.1 J ✓	opposite angles ✓
1.2.2 x ✓	lo-interior angles ✓

Figure 7 (a): NMT22 baseline test

Name two angles which are equal to x and give reason.

STATEMENT	REASON
$x = Q$ ✓	Vert. opp $\angle$ s ✓
$x = y$ ✓	lo-interior $\angle$ s ✓

Figure 7 (b): NMT22 post-test

Moreover, the data depicted in Figure 7 (a) highlights the learner's proficient grasp of fundamental geometry skills and knowledge. This is further corroborated by the learner's grade level, indicating that they possess the necessary mathematics knowledge relevant to their academic level. Additionally, Figure 7 (b) demonstrates a marked improvement in the learner's

reasoning ability and knowledge (Chiphambo, 2011), evidenced by their ability to use accepted geometry abbreviations when providing reasons, such as "angle ($\angle$)" and "vertically opposite angles (Vert. opp. $\angle$).". During their interaction, the learner expressed satisfaction with the strategies employed, citing their ease of use, and likening them to "FUN" and "Happy Face emojis." Furthermore, the learner explained how the "eye for an eye" strategy helped them understand the concept of similarity and "FUN" for alternative angles, providing further proof of their progress.

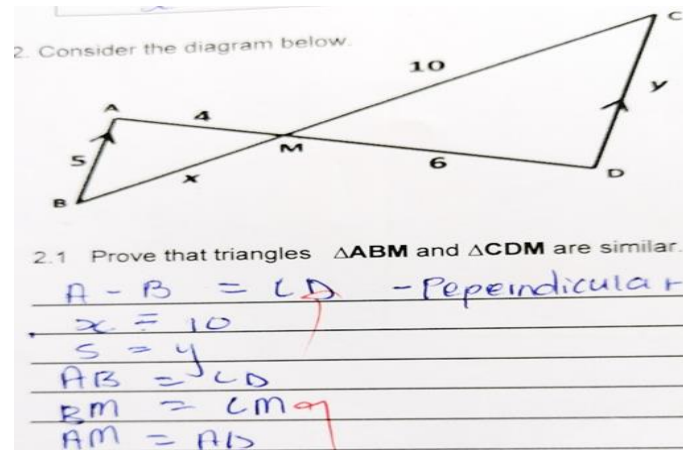


Figure 8 (a): NMT22 baseline test

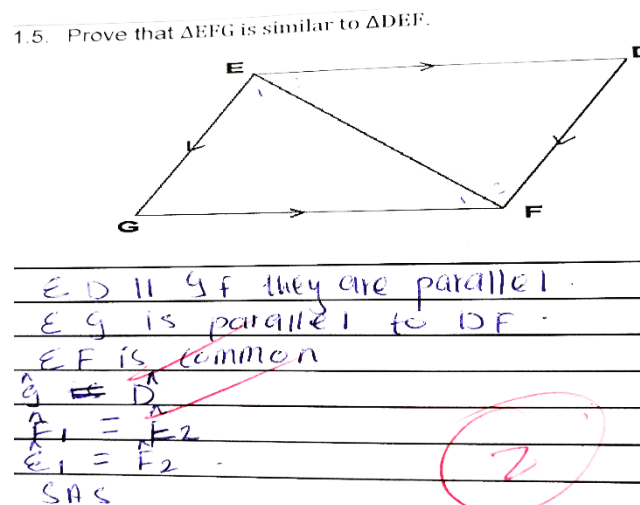


Figure 8 (b): NMT22 post-test

In the first diagram, marked as Figure 8 (a), the student couldn't apply their previous knowledge from Figure 7 (a) and (b) to solve similar triangles due to the differences in the diagrams. Additionally, the student failed to use the conditions required to prove similarity. However, in the post-tests Figure 8 (b), significant improvement was evident in the student's ability to write geometric symbols (Chauraya & Brodie, 2018), name angles differently, and use similarity conditions when proving similar triangles. While the student was able to identify $EG \parallel DF$, they did not provide reasons for their statements. The student successfully proved that triangle EFG and triangle DEF are similar by identifying three equal angles. Unfortunately, the student

incorrectly concluded that the triangles matched the (AAA) rule instead of (SAS) due to the unnecessary side.

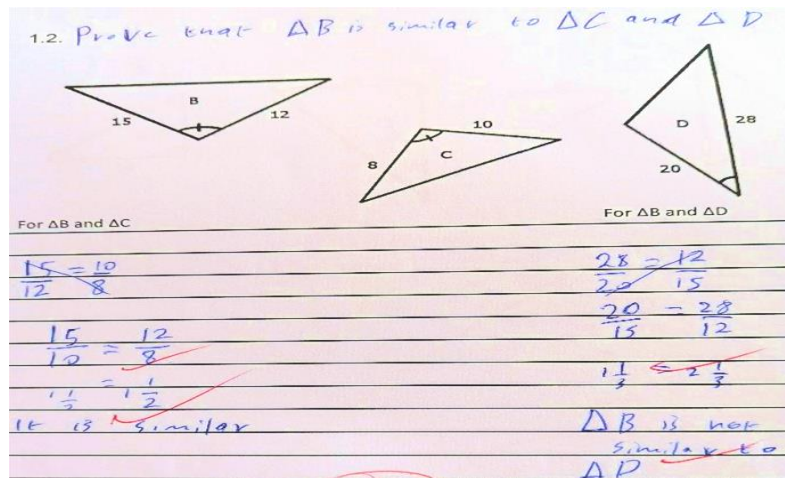


Figure 9: NMT36 post-test

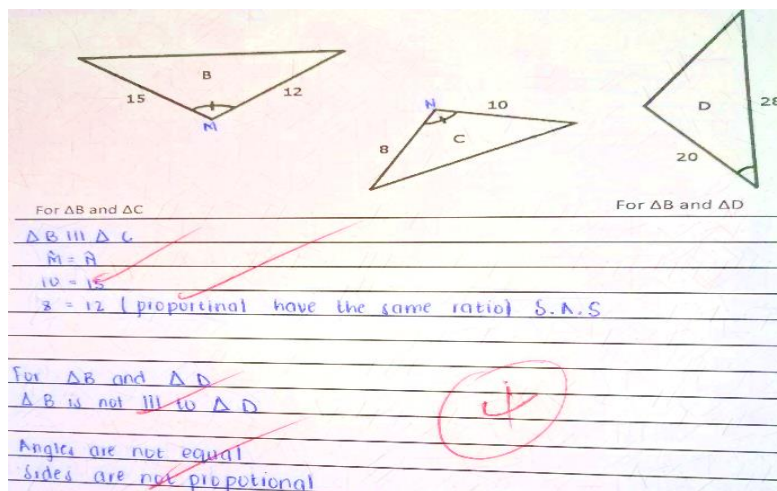


Figure 10: NMT01 post-test

It became apparent that the questions in the post were more challenging than those on the baseline (See figure 8 (a)), as the task involved proving similarity for three triangles in Figures 9 and 10. Despite using different techniques, both **NMT01** and **NMT36** were able to determine that triangles B and C were similar, but B and D were not. Both learners carefully analysed the diagram and correctly identified the corresponding sides (Alex & Mammen, 2014). **NMT36** calculated the ratio to confirm proportionality, while **NMT01** equated the angles and compared them to the size of their corresponding sides, utilising geometric symbols and applying the similarity conditions "S.A.S." Both learners arrived at the correct conclusion (Lin et al., 2020), yet **NMT01**'s method was more thorough and precise.

The study shows that the designed intervention was effective, and the results of the same items from both the baseline and post-test reveal that the mean mark (average) and percentage of tests are 2.5, with 20.9% and 3.4, with 28.1%, respectively, with 21 (48.8%), which means half of the learners in this group improved.

Conclusions and Recommendations

The significant research results regarding the authors' intervention approach were successful, showing that targeted methodologies can greatly improve learners' comprehension of similar triangles. Positive interactions between educators and learners significantly boosted learners' motivation and positively influenced their attitudes and perspectives toward learning. Learners who actively engage in a learning setting experience improved reasoning and grasp of similar triangle geometry. Recent studies indicate that social interaction is pivotal in enhancing student engagement, particularly in innovative educational approaches like flipped civic education. To enhance student engagement, educators should actively encourage collaborative activities like debates and discussions, fostering interactive learning experiences that promote critical thinking and active participation. These activities not only encourage enjoyable social interaction but also contribute to knowledge creation (Chen et al., 2023). To improve students' understanding of geometric concepts, educators should utilize teaching methods that emphasize self- and peer engagement. The findings indicate that certain aspects of Euclidean geometry especially similarity, including its vocabulary and concepts, continue to pose challenges for many students, and employing code-switching can help enhance engagement and comprehension (Mabele & Pule, 2023). It has been noted that a significant proportion of Grade 10 mathematics students exhibit knowledge levels comparable to or below those of Grade 7 students, highlighting a gap in their understanding relative to their expected academic level (Makhubele, 2014). This is in alignment with van Hiele's geometric thinking model which suggests that many students are operating below their potential level. This disparity is alarming and emphasizes the critical need for an immediate re-evaluation of current teaching methodologies to address the identified knowledge gaps effectively. As educators, it is crucial to implement collaborative learning approaches that actively engage students in the educational process, fostering a supportive and interactive environment conducive to enhanced learning outcomes. This necessitates a collaborative endeavour to enhance the calibre of mathematics education across all tiers.

To maintain uniformity in teaching similar triangles, it's advisable to introduce the topic gradually throughout the Further Education and Training (FET) band, with preceding Grades focusing on essential skills like proportionality and comprehension of angles and sides. Further research is needed to explore the efficacy of this customised intervention in mathematics education, providing insights into its potential effects. Lastly, when teaching Euclidean geometry, educators should prioritise instilling a thorough understanding of concepts rather than simply conveying information. This approach fosters deeper comprehension among learners, resulting in more meaningful and successful learning outcomes.

Ethical Consideration

The ethical considerations addressed in this research are briefly discussed in this section. The researcher sought and received permission to conduct the research from the Tshwane University of Technology through the Ethical Clearance Office. Permission for the research was also obtained from the provincial Department of Education, the district office, and the school. The participants were provided with a consent form. Signing the consent form indicated their agreement to take part in the study. They were informed that they could terminate their participation at any stage of the study without negative consequences. The researcher informed the participants that their identities would be kept confidential; therefore, their personal information would not appear in the research documents. Pseudonyms were used in the research document instead of the participants' real names.

Declarations

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ADAPTING MATHEMATICS TEACHING TO ARTIFICIAL INTELLIGENCE (AI): INSIGHTS FROM NAMIBIAN TEACHERS

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This study explores the perspectives of mathematics teachers in Namibia on the integration of Artificial Intelligence (AI) into teaching mathematics. The study employed a qualitative research design, using semi-structured interviews to collect data from 21 mathematics teachers. Thematic analysis was used to analyze the data. Findings revealed that the participating teachers exhibited varying levels of familiarity with AI, with a common understanding of AI as the ability of computers or machines to perform tasks that typically require human thinking. While some teachers recognized the potential of AI to support various domains of mathematics, others expressed concerns about its integration into teaching. These concerns include worries about plagiarism, over-reliance on automated solutions leading to rote learning, reduced human interaction in the learning process, and decreased academic integrity. The study emphasized the importance of a careful balance between technology and human interaction in the deployment of AI in teaching mathematics. It highlighted the need for mathematics teachers to acquire technological knowledge and skills, secure resources, align AI activities with curriculum goals, adapt teaching methods, and seek professional development opportunities. The study concluded that the deployment of AI in teaching mathematics demands a careful balance between technology and human interaction and emphasized the importance of teacher training and professional development in the successful integration of technology into teaching practices. These findings contribute to the understanding of the challenges and opportunities associated with the integration of AI in mathematics education, particularly from the perspective of mathematics teachers in Namibia.

Keywords: AI, Integration, Mathematics Education, Teaching, Technology

Introduction

The 21st century presents unique challenges that demand the acquisition of distinct skills compared to previous centuries. This has led several developed and developing countries to invest significantly in pedagogical reforms. A crucial part of these reforms is the integration of emerging technologies, such as Artificial Intelligence (AI) into the educational sector. Artificial Intelligence (AI) is defined as the branch of computer science that concentrates on developing machines or systems capable of carrying out tasks that usually necessitate human intelligence, including learning, problem-solving, and decision-making (Zulhilmi, Hidayat & Nabilah, 2022). AI is associated with developing machines or systems that can replicate tasks that require human intelligence (Benbya, Davenport & Pachidi, 2020). These tasks include learning, problem-solving, and decision-making (Alam, 2021; Zulhilmi, Hidayat and Nabilah, 2022). The integration of Artificial Intelligence as educational technology is affecting significant changes in 21st-century education (Pedro, Subosa, Rivas, & Valverde; Pham & Sampson,

2022). It transforms education tools and the future outlook of education. Furthermore, AI necessitates significant changes in the methods of teaching, even in classrooms where teacher presence remains pivotal. A particularly interesting area of study is the adaptation of mathematics teaching to Artificial Intelligence. In recent years AI educational technologies have seamlessly become a vital tool in teaching, and linking it to the teaching of mathematics could enhance the learning experience (Kooli, 2023).

From teachers' perspectives, leveraging AI for teaching mathematics promises a more interactive and engaging learning experience for students. This further underscores the inevitability of AI in shaping 21st-century education, particularly in subjects such as mathematics where AI's problem-solving attribute can be fully exploited. Given the ongoing evolution in the field of education due to the rapid change in new technology, teaching, and learning have also transformed. The rapid advancement of technology in the world prompted the Namibian government to adopt an ICT Policy for Education in 2003, which was later revised in 2005. The ICT policy aimed at "articulating the relevance, responsibility, and effectiveness of integrating ICT in education to meet the challenges of the 21st century" (Namibia. Ministry of Basic Education Arts and Culture , 2005 p.9). Even with such policies in place, there is still a noticeable gap in the incorporation of AI technologies in teaching mathematics, as most mathematics teachers lag in integrating these technologies into their practice (Van Vaerenbergh & Pérez-Suay, 2022). Despite the adoption of several educational technologies, disparities exist between Namibian government expectations, teachers' practices, and the actual adoption of these technologies in teaching.

Literature suggests that even teachers with access to technology are not integrating it into their practice (Nelson, Voithofer, & Cheng et al., 2019; Ifinedo & Kankaanranta, 2021). In some regions in Namibia, the Ministry of Education recognizes the need to integrate educational technologies in mathematics, spurring a massive rollout initiative. However, this success will depend not only on the availability of these technologies but, more importantly, on teachers' perceptions of educational technologies. Studies have suggested that teachers' perceptions significantly impact the acceptability and integration of ICT tools (Nelson, Voithofer, & Cheng, 2019; Pham & Sampson, 2022). Given this, understanding how mathematics teachers perceive AI educational technologies becomes vital to the success of ICT integration in Namibia. It is through understanding these perceptions that we can ensure that AI tools in mathematics teaching become properly utilized rather than underutilized or abandoned. Therefore, investigating the perceptions of mathematics teachers toward the use of AI educational technologies is the main focus of the case study in this paper.

This paper begins by reviewing literature related to the topic, followed by presenting the conceptual framework that guides the study. This is succeeded by the research methodology and data collection. Subsequently, the findings of the research and the discussion of the results are presented. The paper concludes with the findings and recommendations.

Review of literature

With the use of artificial intelligence (AI) growing every day, there is a fear that there will be excessive reliance on AI and technology in education which may lead to a lack of critical thinking (UNESCO, 2019; Preston, 2021; Njogu, 2023) and creativity among learners as well as potential loss of important social and emotional skills (Berendt et al., 2020). In addition, *the adoption of Artificial Intelligence in education requires a significant shift in traditional teaching and learning practices, which can be met with resistance from teachers and learners.* Adapting mathematics teaching to AI into 21st-century mathematics teaching presents challenges for teachers. This is consistent with Bingimlas (2009) who outlined that although most teachers have strong desires to incorporate technology into their teaching, they encounter many challenges. These challenges include “developing technical competence, accessing necessary resources, aligning AI integration with the curriculum, adapting pedagogical strategies, and accessing relevant professional development (Bingimlas, 2009, p. 1). With the emerging use of technology in education, teaching mathematics can be further enhanced through the integration of new emerging technologies, such as AI educational technologies which can provide dynamic visualizations, interactive activities, and personalized learning experiences that promote the discovery of connections between different mathematical concepts (Kamalov et al., 2023).

In a school context, many learners find mathematics to be one of the subjects with significant challenges (Langoban, 2020; Mamolo, 2022). The advancement of computer technologies, in particular, Artificial Intelligence (AI), could provide an opportunity to tackle this problem by diagnosing learners’ learning problems individually and providing personalized support to maximize their learning performance in mathematics (Berendt et al., 2020; Orhani, 2021). In line with Orhani, (2021), Pedro et al. (2019) argue that one of the domains with a promising future is the integration of Artificial Intelligence in education where it has the potential to be applied as a solution to some of education’s major challenges. Its capabilities for personalizing learning provide unique opportunities for educational growth. Notably, AI technologies can adapt to learner’s individual learning rates and styles promoting a grasp of complex mathematical concepts. For instance, Benbya et al. (2020) found that AI-based tutoring in algebra significantly improved learners’ grasp of complex concepts. AI also aids in aligning learning resources to align with learner’s knowledge gaps. Using data analytics, AI can offer insightful feedback to both learners and educators. This presents opportunities for continuous learning and development. Despite the advantages, ethical, privacy, and equity issues also arise with AI use. While AI tools facilitate learning, the role, and perspectives of teachers for mentoring and holistic mathematics education remain crucial. The deployment of AI in teaching mathematics demands a careful balance between technology and human interaction.

However, the claimed benefits could be questionable without addressing the issue of teachers’ roles and teaching competencies. Therefore, it is important to consider how teachers adapt to teaching using modern technologies. Research further indicates that the integration of Artificial Intelligence in 21st-century mathematics education provides concepts, methods, and tools for the design of flexible and relevant computer-based systems for teaching and learning purposes (Hwang et al., 2020; Kaushik et al., 2021). Several researchers have shown that Artificial

Intelligence technologies can be used to simulate the pedagogical and content knowledge of teachers, providing personalized guidance, feedback, or support to learners during the 21st-century learning process. To adapt mathematics teaching to the use of Artificial Intelligence mathematics teachers need to acquire technical skills, secure resources, align AI activities with curriculum goals, adapt teaching methods, and seek professional development opportunities (Novita & Herman, 2021; Trgalová et al., 2018; Weng, 2017).

Methodological and Theoretical Aspects

This research applies a case study design, a methodology that is recognized as being particularly valuable when the objective is gaining a thorough comprehension of an issue, event, or phenomena within its original, true-to-life context (Beaudry & Miller, 2016). Given the study objective, the research design aligns with the instrumental case study style coined by (Creswell & Miller, 2000). The data for this research was gathered by conducting semi-structured interviews with twenty-one (21) mathematics teachers in the Kharas educational region who were selected purposively. The aim was to gain a comprehensive understanding of the teachers' perspectives, and insights related to the integration of Artificial Intelligence (AI) technologies in teaching mathematics education. The interviews provided valuable insights into the teachers' engagement with Artificial Intelligence and its implications for teaching mathematics in the 21st century, thereby enriching our understanding of this topic. The teachers interviewed were all from schools with technological infrastructures for pedagogical use, such as computer laboratories, laptops, white board and internet connection.

The participants were asked (a) their understanding of AI and technological tools they use during the teaching of mathematics and how they use AI to develop their technological knowledge (b) their views on the use of Artificial Intelligence and how they open opportunities and challenges in their teaching and, (c) ethical concerns associated with adopting Artificial Intelligence to the teaching of mathematics in the 21st-century classroom.

Thematic analysis (Braun, Clarke & Weate, 2021) was employed to analyze the transcribed audiotaped data. The use of thematic analysis facilitated the identification and interpretation of shared perspectives, meanings, and experiences among the participating mathematics teachers, thereby enhancing the study's consistency. The teachers involved in the study were identified as Tr 1, Tr 2...Tr 21 to ensure participant's anonymity and confidentiality. Initially, themes were developed based on the interview questions, and additional themes emerged from the data itself during the analysis process.

Findings

This section of the study presents findings that emerged from the semi-structured interviews on the perspectives of teachers on adopting Artificial Intelligence into the teaching of mathematics. These findings specifically focus on the perspectives of mathematics teachers. All the teachers involved in the research utilized either technology in their teaching practices.

Understanding of the use of Artificial Intelligence (AI) in mathematics education

Participants in this study reported that they are familiar with the use of Artificial Intelligence in education. Their common understanding of AI is the ability of computers or machines to perform tasks that normally require human thinking. Some participants acknowledged the potential of AI to provide support across various domains of mathematics. In this study, teachers' understanding of Artificial Intelligence (AI) varies, but generally, it involves recognizing AI as a technology that empowers machines to perform tasks requiring human-like intelligence. Some teachers are aware of AI's broad applications, including machine learning and natural language processing. They understand its potential in education, such as personalized learning platforms and adaptive assessment tools. However, their perception leaned towards AI demonstrating enhanced efficacy in algorithmic calculations, particularly those involving procedural knowledge, conceptual understanding, computational mathematics, and topics that necessitate visualization.

In this study, participants revealed a diverse collection of Artificial Intelligence technologies employed for teaching mathematics, listed in Figure 1. The illustration visually captures the varied applications embraced by teachers, including PhotoMath, MathWay, Geogebra, Interactive Whiteboard Tools, Khan Academy and ChatGPT. These findings underscore the adaptability and innovation demonstrated by teachers in integrating Artificial Intelligence into their instructional approaches.

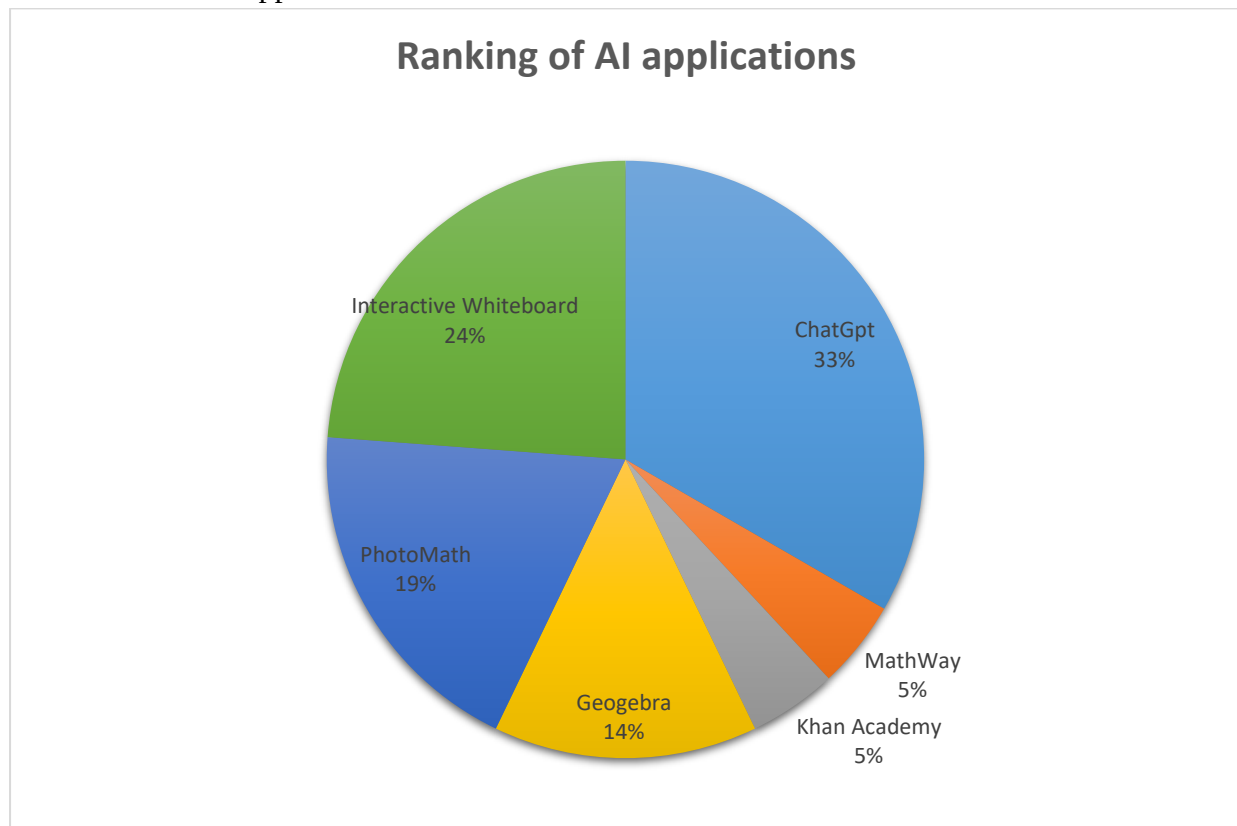


Figure 1: Ranking of AI applications used by teachers in the teaching of mathematics.

According to the data presented in Figure 1, ChatGPT emerged as the AI application most frequently utilized in mathematics teaching, with Interactive Whiteboard being the second most employed. PhotoMath ranked third in terms of frequency of use, while participants indicated that Geogebra, MathWay and Khan Academy were the least frequently utilized technologies for teaching mathematics. The participants expressed a high level of proficiency in acquiring knowledge and adapting to new AI technologies, highlighting their ease in learning how to utilize AI technologies.

Notably, some participants expressed a lack of awareness regarding any AI tools suitable for teaching mathematics. This finding underscores the need for comprehensive training programs aimed at equipping mathematics teachers with the knowledge and skills necessary for effectively integrating Artificial Intelligence into their instructional practices (Breddermann et al., 2017; Vásquez et al., 2022).

According to the participants, the evolving landscape of the world necessitates teachers' adaptation to emerging ICT trends and the enhancement of their technological knowledge. Some teachers acknowledged this paradigm shift as they reported developing their technological proficiency through specialized training on AI-driven platforms, which offer tailored resources to cater to their requirements. Additionally, these participants expressed an active interest in exploring the potential applications of AI in mathematics education. They also highlighted their utilization of AI-generated content and resources to facilitate their teaching and support their role as subject facilitators.

Benefits of using Artificial Intelligence in teaching mathematics

This study further sought insights into participants' regarding the integration of Artificial Intelligence (AI) in mathematics. Most of the participants seem confident in learning and using AI for teaching. A small percentage of participants (20%) feel less confident in solving AI-related technical problems and in learning how to use AI for teaching. The majority of the participants (80%) indicate that they play around with AI easily or keep up with important new AI technologies. Exploring the reasons behind the lower agreement percentages for being confident with AI technologies may provide insights into potential barriers or areas for improvement when adopting Artificial Intelligence in the teaching of mathematics.

Participants perceived the use of AI as beneficial when integrated into the teaching and learning of mathematics. The teachers indicated that adopting Artificial Intelligence to the teaching of mathematics has various benefits such as personalized learning, immediate feedback, adaptive learning, etc...

On AI benefits Tr 7 commented that:

“AI can positively contribute to the teaching of mathematics by providing teachers with relevant teaching strategies, solving complex problems, as well as serving as a data source for mathematics-related problems. AI can also help prepare lesson plans of difficult topics and even teach such topics in more relevant and effective ways.”

Further, Tr 11 added that:

“AI enables personalized learning by adapting to learners’ needs, offering tailored instruction and support. It enhances problem-solving experiences and receives real-time feedback. Additionally, AI gives teachers clues to solve certain problems if they are stuck, AI could solve many other mathematical problems”.

Tr 16 narrated that:

“AI enables teachers to recall concepts to enable them to conceptualize new concepts. AI helps teachers to teach some concepts confidently because they find algorithms or ways of presenting new concepts. Furthermore, AI provides immediate feedback to learners, solve math problems and identify and correct errors quickly, reinforcing their understanding of mathematical concepts”.

Tr 2 expressed less knowledge and experience of using of AI in teaching mathematics:

“Even though I have not explored Artificial Intelligence in detail in my teaching as a mathematics teacher, I believe it could be beneficial. I am saying so because if one enters a problem on ChatGpt and other AI applications for example, a solution and the approach to the problem can be generated”.

Participant 13 emphasized:

“Furthermore, reference books can be suggested, and worksheets can be generated using AI, which could be very helpful in the teaching and learning of mathematics in the 21st century classroom.”

In the same vein, Tr 6 and Tr17 explained that:

Integrating AI in the teaching of mathematics has the potential to be highly beneficial to learners. First, it will lead to personalized learning: AI can adapt to individual learners’ needs and provide personalized learning experiences”.

It can assess learners’ strengths and weaknesses, identify areas where they need additional support, and tailor instructional content accordingly. This personalized approach can enhance learners’ engagement and improve learning outcomes (Tr 17).

The findings from the teachers’ responses in the quotes above indicates a mostly positive stance towards integrating Artificial Intelligence into mathematics teaching, aligning with what other studies have suggested (Breddermann et al., 2017; Sayed Mnhrawi & Alreshidi, 2022). The notable benefits elucidated include immediate feedback, personalized learning, and assistance in teaching complex topics, which is in line with studies by Novita and Herman (2021) that highlighted AI’s potential in personalized learning. While the majority of the teachers in this study were confident about AI technology, some participants expressed concerns about lack of awareness and technical knowledge, indicating some potential barriers to broader AI adoption. This suggests the need for proper training or support systems for teachers. Specific experiences shared by participants echo the impactful role of AI in tailoring instruction to individual learners’ needs, enhancing problem-solving experiences, and facilitating real time feedback. This complements previous research by Benbya et al. (2020) suggesting AI’s potential in enhancing pupil engagement and learning outcomes.

Challenges experienced when adopting AI in the 21st century classrooms.

In the data collected from the participants' interviews, several noteworthy observations emerged. Firstly, ethical concerns were raised regarding the practice of directly providing learners with AI-generated texts without teachers fully grasping and comprehending the underlying concepts. Secondly, a sentiment was expressed suggesting that AI seems to be on a route to surpassing human intelligence, thereby evoking apprehension about the future role of human intelligence in the advancement of education, including the field of mathematics.

Tr 10 explained that:

“It appears to me that AI is on a mission to supersede human intelligence, something that instills fear in me regarding the future role of human intelligence in the evolution of education (including mathematics)”.

The responses from the participants underscore the significant emotional role that teachers hold within the classroom, as they exert influence on and are impacted by the emotions of their learners. The participants emphasize that an effective mathematics teacher is capable of encouraging and motivating their students to learn and possesses a genuine emotional connection with them, taking into consideration their weaknesses and individual needs. This stands in contrast to AI, as teachers can provide essential emotional support by demonstrating care, empathy, respect, and trust, attributes that cannot be replicated in machine-related technologies such as AI (Berendt et al., 2020; Chen & Chen, 2020; Sperling et al., 2022).

In the interview, Tr 13 conveyed,

“From my side, teachers can provide emotional support such as empathy, encouragement, and a personalized approach to learners, something that AI cannot do. The role of a teacher is irreplaceable in understanding and responding to the emotional needs of students”.

The participants highlighted key concerns in quotes regarding AI incorporation in modern classrooms. Ethical issues emerge when learners are supplied with AI-generated texts, a concern previously mentioned in the studies by Berendt et al. (2020) Hwang and Tu (2021). There also appears to be a growing belief that AI could potentially replace human intelligence within education, creating a sense of unease, notably with regards to mathematics education. The fears echo Tr 10's sentiment about the possible domination of AI over human intelligence. Another significant aspect underscored by responses is the crucial emotional element teachers bring to the classroom, which bolsters their ability to motivate students and cater to their unique requirements. The ability to show care, empathy, respect, and trust, qualities that are unattainable for AI, further underscores the irreplaceable role that teachers play in understanding and responding to learners' emotional needs, as articulated by Tr 13. This aligns with prior research (Sperling et al., 2022) indicating the limitations of AI in providing emotional support.

Ethical Concerns about Adopting AI in the Teaching of Mathematics

The integration of artificial intelligence (AI) in the teaching of mathematics brings forth several ethical concerns that necessitate scrutiny. During the interview, several concerns were highlighted regarding the use of AI in mathematics. One prominent concern is the potential for increased plagiarism in homework assignments facilitated by AI tools. The ease of access to AI-generated solutions may undermine the authenticity of learners' work and compromise the educational integrity of assignments. Additionally, there is apprehension about the impact of AI on critical thinking skills, as over-reliance on automated solutions may hinder students' ability to analyze problems independently.

During the interview, Tr1 responded that:

“First, there are ethical concerns of copying everything that appears on AI-generated texts and giving it to learners without conceptually and necessarily understanding such information as teachers. Secondly, AI has the potential to promote rote learning in mathematics, because both teachers and learners can now easily copy information without investing in any intellectual efforts”.

In addition, Tr 8 noted:

“Yes, there are some concerns associated with the use of AI in mathematics education. AI will reduce Human Interaction. We are scared, that AI might reduce teachers' roles and responsibilities in the classroom as learners will be using AI to solve problems without teachers' guidance”.

Tr 20 added that:

With AI, academic integrity will be overlooked as learners will depend on AI to solve activities or assignments”. I have not used it in mathematics yet. However, as I have used it in my research study, I concluded that it has the potential to contribute to academic laziness since students will be reluctant to do extensive readings.

The findings from the interviews indicate that educators have concerns about the integration of AI in teaching mathematics, similar to concerns highlighted in a previous study by Lo (2023). One significant worry is plagiarism, an issue potentially exacerbated by easy access to AI-generated solutions. The concern here pivots around the authenticity of students' work and the potential infringement of the educational value of assignments. Additionally, some teachers fear that AI could undermine students' critical thinking skills by fostering overreliance on automated solutions. Teachers fear that AI might promote rote learning rather than supporting the development of a deep understanding of mathematical concepts. All the teachers in this study expressed the risk of reduced human interaction in the learning process. Educators believe that the use of AI could lessen the roles and responsibilities of teachers and result in a lack of guidance for learners. Moreover, some teachers mention that their concern was over a decrease in academic integrity, as learners may overly depend on AI for solving assignments, leading to academic laziness.

Discussion of findings

The findings of the present study, which explored the perspectives of the selected Namibian mathematics teachers concerning the integration of Artificial Intelligence into teaching practices, mirror and extend the use of AI in mathematics education. The study highlights the importance of understanding teachers' perceptions of AI educational technologies to ensure their successful integration into teaching practices.

The study findings reveal that most mathematics teachers in Namibia have a varying understanding of AI, but generally, it involves recognizing AI as a technology that empowers machines to perform tasks requiring human-like intelligence. Some teachers are aware of AI's broad applications, including machine learning and natural language processing. This finding is consistent with previous studies that have highlighted the importance of teachers' technological skills in the successful integration of AI in education. Barao et al. (2022) suggest that mathematics teachers need to acquire technical skills, secure resources, align AI activities with curriculum goals, adapt teaching methods, and seek professional development opportunities to adapt mathematics teaching to the use of AI.

The study also found that mathematics teachers in Namibia recognize the potential of AI to provide support across various domains of mathematics. Several researchers have shown that AI technologies can be used to simulate the pedagogical and content knowledge of teachers, providing personalized guidance, feedback, or support to learners during the 21st-century learning process. Hwang et al. (2020) and Kaushik et al. (2021) concur with the findings that the integration of Artificial Intelligence in 21st-century mathematics education provides concepts, methods, and tools for the design of flexible and relevant computer-based systems for teaching and learning purposes.

Furthermore, while some mathematics teachers in Namibia are familiar with the use of AI in education, this study shows that they have concerns about its integration into teaching mathematics. One significant worry is about plagiarism, as easy access to AI-generated solutions could undermine the authenticity of students' work. Additionally, some teachers fear that AI could foster over-reliance on automated solutions, promoting rote learning rather than supporting the development of a deep understanding of mathematical concepts. Teachers also express concern about the potential for reduced human interaction in the learning process, leading to a lack of guidance for learners. Furthermore, some teachers worry about a decrease in academic integrity, as learners may depend overly on AI for solving assignments, leading to academic laziness.

These findings are consistent with previous studies that have highlighted the challenges of integrating technology into teaching practices. For example, Bingimlas (2009) found that teachers face several barriers to the successful integration of ICT in teaching and learning environments, including a lack of technical skills, inadequate resources, and resistance to change. Similarly, Nelson, Voithofer, and Cheng (2019) found that even teachers with access to technology are not integrating it into their practice, and Ifinedo and Kankaanranta (2021)

found that teachers' perceptions significantly impact the acceptability and integration of ICT tools.

The study also highlights the importance of teachers' roles and teaching competencies in the deployment of AI in teaching mathematics. While AI tools facilitate learning, the study emphasizes the need for a careful balance between technology and human interaction. The study suggests that to adapt mathematics teaching to the use of AI, mathematics teachers need to acquire technological knowledge and skills, secure resources, align AI activities with curriculum goals, adapt teaching methods, and seek professional development opportunities. These findings are consistent with previous studies that have emphasized the importance of teacher training and professional development in the successful integration of technology into teaching practices (Bingimlas, 2009; Trgalová et al., 2018).

Conclusion and Recommendation

The study concludes that the deployment of AI in teaching mathematics demands a careful balance between technology and human interaction. While AI tools facilitate learning, the role and perspectives of teachers for mentoring and holistic mathematics education remain crucial. The study highlights the importance of considering how teachers adapt to teaching using modern technologies. To adapt mathematics teaching to the use of Artificial Intelligence, mathematics teachers need to acquire technical skills, secure resources, align AI activities with curriculum goals, adapt teaching methods, and seek professional development opportunities.

The study's findings have several implications for mathematics education in Namibia and beyond. The study highlights the need for policymakers and educators to address teachers' concerns about the integration of AI in teaching mathematics. Policymakers and educators need to provide teachers with the necessary resources and training to effectively integrate AI into their teaching practices. Additionally, the study emphasizes the importance of balancing technology and human interaction in teaching mathematics. Policymakers and educators need to ensure that the deployment of AI in teaching mathematics does not undermine the role of teachers in mentoring and providing holistic mathematics education.

Limitations

The study involved a relatively small sample size of twenty-one mathematics teachers from Kharas educational region, which may limit the generalizability of the findings to a broader population. Additionally, the study did not explore the perspectives of students, administrators, or parents, which could provide a more comprehensive understanding of the implications of AI integration in mathematics education. Future studies could consider conducting a larger-scale investigation involving diverse educational settings and stakeholders to capture a more comprehensive and nuanced understanding of the challenges and opportunities associated with AI integration in mathematics education.

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GRADE 11 TEACHERS' PERCEPTIONS ON THE USE OF MATHEMATICAL MODELLING IN TEACHING LINEAR FUNCTIONS TO PROMOTE LEARNER ENGAGEMENT AND ACHIEVEMENT

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Effective teaching approaches are an appropriate way of engaging learners during teaching. This mixed-method study sought to explore Grade 11 teachers' use of mathematical modelling in teaching linear functions to promote learner engagement and achievement. The purpose of the study was to find out how teachers perceive the value of using mathematical modelling in promoting the more successful and engaging learning of linear functions. Data was collected from teacher questionnaires and learners' pre- and post-test results. The quantitative data was analysed statistically using ANOVA, in which measures of central tendency and dispersion were used to describe the findings, whereas the qualitative data was analysed using the Leuven scale of engagement. The study revealed that learners' engagement during linear functions lessons was moderate, showing dissatisfactory engagement. Furthermore, the study's results from post-tests revealed that teaching linear functions using modelling can enhance learner achievement. The study, therefore, recommends the use of mathematical modelling for teaching. It is further recommended that subject specialists assist teachers in selecting textbooks that include modelling to promote learner engagement even outside the school hours.

Key words: Mathematical Modelling, Linear functions, Learner engagement, Teaching, Learner achievement.

Introduction and background

In any classroom, teachers are required to teach in a way that learners can make sense of the learning content. The objective of many teaching activities is the achievement of learning outcomes and, consequently, learner achievement in the subject. In mathematics, it is difficult to achieve learning outcomes without bringing understanding into the teaching and learning process. The quality of learners' conceptual understanding that teachers develop in every section of mathematics builds on learners' prior knowledge, which will be used to develop and connect to new knowledge as they progress from one grade to the next. This calls for the use of teaching strategies that focus on learner engagement to enhance conceptual understanding. Modelling is one of the teaching approaches that took centre stage in research (Khan et al., 2021; Machado et al., 2020) and is regarded by many as the most effective teaching and learning approach. It is a process in which mathematics-based incidents are converted into an actual and realistic scenario (Ziatdinov & Valles, 2022). This implies that in modelling, learners' involvement in tasks is crucial. In this paper, we refer to learning tasks that involve modelling as modelling tasks. Using modelling simplifies learning content and assists in enhancing comprehension (Ahmad & Junaini, 2020). The comprehended content will form basis of their previous experiences and knowledge and will be used to make connections and make sense of the new content, which is called meaningful learning. In doing this, learners move away from memorizing content merely to achieve in summative assessments. Functions

are critical in mathematics, and they are used in different fields such as Economics, Engineering, sciences, and so on. They are used in Statistics to show relationships. In secondary school mathematics, functions are included in different sections such as Algebra, trigonometry, and Calculus. It is essential that learners have a thorough understanding of basic functions like the linear functions to prepare them for more complex functions like those in Trigonometry and Calculus.

During linear function lessons, learners are required to represent functions in different ways and identify and explain the relationships between variables. This requires learners to be engaged during the lesson. Halverson and Graham (2019) regard learner engagement as “the involvement of students’ cognitive and emotional energy to accomplish a learning task” (p. 155). Yu, Wang, and Xia (2022) regard learner engagement as taking part in assignments and activities, and an important element in improving scholastic performance. Shen and Chong (2023) discussed three types of learner engagement namely, behavioural, cognitive, and affective. According to Shen and Chong (2023), behavioural engagement is demonstrated when students interact with external sources, use different learning strategies, and revise to improve their work; cognitive engagement occurs when students succeed in understanding and interpreting content they are exposed to; and affective engagement is when students react emotionally and psychologically to the learning content. Keeping learners engaged throughout the lesson is a process that requires a lot of proper planning and preparation from the teacher's side, however, there is still a struggle among some teachers in this regard (Du Plessis, 2020). It is against this background that this study seeks to explore teachers’ perceptions regarding learner engagement when teaching linear functions through modelling to Grade 11 learners, and to determine the effect of using mathematical modelling on learner achievement in promoting the understanding of functions. To achieve this objective, the study focuses on the following research questions:

1. How do Grade 11 mathematics teachers perceive their learners’ level of engagement when using mathematical modelling approach in learning linear functions?
2. How does the use of mathematical modelling in teaching linear functions affect learner achievement?

Literature review

Modelling linear functions

Modelling is a means that teachers use to engage learners in learning activities to promote understanding; it is also used to account for learners’ progress within a series of lessons (Hidayat et al., 2022). In mathematics, the idea is to bring realistic situations to class to help learners understand concepts better, however, it is not always possible due to the abstract nature of the discipline. As a result, models are used to provide learners with experiences of realistic situations. In this paper, modelling refers to the use of models to provide learners with realistic experiences of mathematical concepts, for example, different representational forms of a mathematical concept, for example, graphing linear functions and representing them in a table form, and so on, see Figure 2. For example, the coordinates (1;3), (2;4), (3;5), (4;6), and (5;7) of an increasing linear function graph given in words as: Five girls save their pocket money

every month. Sanna, Mary, Boniswa, Ayanda, and Nhlanhla save R3, R4, R5, R6, and R7 respectively. This information can be modelled out as follows:

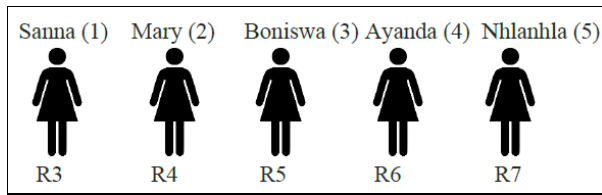


Figure 1: Modelling coordinates of a line graph

There are many different mathematical models identified, for example, calculators, pictures, real objects, algebraic analytical models (using formulae), graphical models, and so on. An example of an algebraic analytical model is the general formula of the straight-line graph, whereas a graphical model can be any graph, such as a line graph. We will focus on graphical and algebraic analytical models as they are more relevant to the teaching of linear functions. Hidayat et al. (2022) further assert that the success of a model depends on its simplicity when used as well as forecast accuracy. Modelling can be used in different mathematics topics and in other subjects, and if used appropriately, it has the potential to enhance conceptual understanding and knowledge retention, leading to an increase in learner achievement.

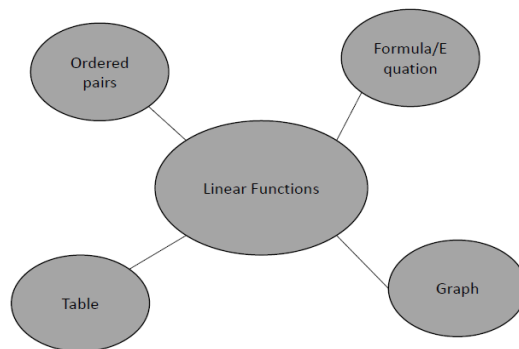


Figure 2: Modelling of functions

Mathematics curricula are permeated by the idea of a function, which presents obstacles for conceptualization, teaching, and learning due to its diversified, multifaceted, and complicated nature. (Niss, 2020). Furthermore, Niss points out that functions can be shown as a process, graph, or formula that uses the independent and dependent variables. In mathematical modelling, learners are first required to engage with existing ideas, skills, and reasoning abilities around the modelling activities, using representations to draw mathematical inferences (Hidayat et al., 2022). All these types of representations use variables, x and y to show the relationship between the values. A function can be given in one form and then changed to different equivalent forms; for example, a linear function can be given as an equation, and then learners can be asked to sketch the graph using the given equation. However, sketching the graph requires learners to perform some procedures that they need to understand, for example, calculating the gradient and intercepts of the graph or substituting in the given equation using a defined domain. Without this procedural knowledge, it will not be easy for learners to sketch the graph of the function. This demonstrates the interdependence between algebraic analytical

models and graphical models, in which knowledge of one leads to success in executing the other.

Mathematical engagement

Mathematics is regarded by many as an abstract subject (Muchyidin, et al., 2022; Yohannes & Chen, 2023), and most learners find it difficult to understand (Mahlaba & Mudaly, 2022). As a result, teachers need to engage learners meaningfully during their lessons to promote conceptual understanding. To do this, it requires teachers to do proper lesson planning and have sufficient knowledge of different instructional strategies that will promote learner engagement during lessons. Whether it is engagement, participation, or involvement, all these are important in any classroom. Teachers need to know if the learners they teach have understood the content, and it is through meaningful engagement that this can be measured. According to Padgett et al. (2019), engagement is described according to how it is measured in a particular situation; it can be an "outcome" or "predictor". Furthermore, Padgett et al. (2019) define engagement as an interactive process that is influenced by the individual's context. Halverson and Graham (2019) define learner engagement as "the involvement of the student's cognitive and emotional energy to accomplish a learning task" (p. 1). In the context of this paper, learner engagement is when learners try cognitively and emotionally to respond to the teacher's stimulus (learning content presented). Learners' actions of engaging with the learning content are referred to as indicators of learner engagement, and they depend on the ability of the chosen model and the teacher to arouse learner interest and sustain it throughout the lesson. When learners are curious, they will want to know what the model is all about, provoking their critical skills; hence, their concentration during the lesson will be sustained. The teacher has a critical role to play in allowing learners to debate and present their views. To do this, knowledge of the learners and their prior knowledge is crucial (Jacob, 2020) in helping teachers choose the appropriate model and modelling activities. Therefore, teachers need to provide a conducive environment for learners to be cognitively engaged in the lesson. Learner characteristics, prior knowledge, and the learning environment are regarded as facilitators of learner engagement (Halverson & Graham, 2019).

Conceptual framework

The conceptual framework of this study has been drawn from Halverson and Graham's (2019) explanation of the relationship between facilitators and indicators of learner engagement. Halverson and Graham (2019) regard indicators of learner engagement as those features that belong inside the constructs of engagement and facilitators as the causal factors emanating from outside of the construct. Learner engagement facilitators such as personal characteristics and thoughtful learning experience design are essential and have the potential to increase learner engagement (Halverson & Graham, 2019). The choice of modelling tasks should be guided by learners' characteristics and prior knowledge that is linked to their learning experiences.

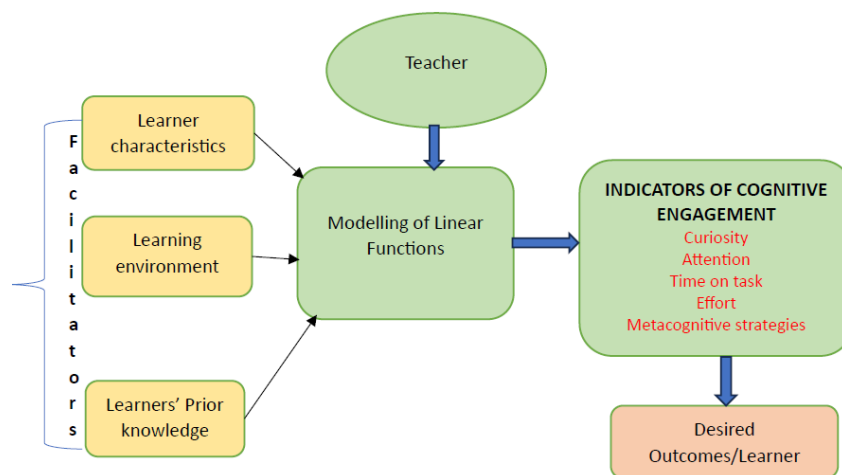


Figure 3: The relationship between facilitators and indicators of learner engagement during modelling. (Adapted from Halverson & Graham 2019).

The conceptual framework in Figure 3 explains that teachers can use modelling during mathematics lessons; however, learner characteristics, prior knowledge, and a conducive classroom environment are all necessary (Li & Wong, 2021) to facilitate effective modelling of functions. This implies that teachers need to use models that learners are familiar with when they teach. Furthermore, teachers need to have thorough knowledge of their learners (Garcia-Martinez et al. 2019), how they will engage them during the lesson, the choice of modelling activities depending on their learners' abilities, and whether learners will be able to cope with the teaching strategies they intend to use. Teachers need to consider the complexity and kind of models they use (Marques et al. 2020) during lessons in their struggle to arouse learner interest. This is crucial in capturing learners' attention and sustaining learner engagement throughout the lesson. When learners are interested in the lesson, they will use their metacognitive processes, such as goal setting, planning, and organizing, to put more effort into their work. It is believed that if these processes are followed, learners will be fully engaged during the lesson and the desired learning outcomes will be achieved.

Methodology

Research approach and design

The study followed a mixed-method research approach in which three case studies were used. The mixed method approach was preferred because the strengths of one research method complement the shortcomings of another, and a research question cannot always be answered solely by a qualitative or quantitative approach (Almeida, 2018). Teacher questionnaires and pre- and post-tests were used as data collection instruments.

Population and sampling

Three schools, namely, A, B, and C, were sampled from the Tshinane Circuit in 2021 using convenience sampling. From each of the three schools, one teacher and twenty Grade 11 learners were then sampled using purposive sampling. Purposive sampling was appropriate to use in this study as it is the type of sampling in which the researcher can intentionally select

participants because of the ample knowledge they have about the phenomenon being studied (Gill, 2020). The total participants in the study were three teachers and sixty Grade 11 learners. Teachers were selected based on their teaching experience in Grade 11 mathematics, and all three participants had a minimum of twenty years of teaching experience in Grade 11 mathematics.

Data collection

Data were collected using pre- and post-tests for learners, and teacher questionnaires. Pre-tests were given before learners were taught through modelling. Researcher 1 conducted an intervention program to assist teachers in using the modelling approach, and post-tests were then given after a week of teaching using modelling. Pre- and post-tests were administered to enable the researchers to compare learners' achievement before and after being taught using the modelling approach. With the assistance of the teachers, researcher 1 sampled only learners who were taught by their teacher and did not receive any extra tuition elsewhere. This was necessary because learner achievement is associated with active learning (Hartikainen et al. 2019) and receiving extra tuition on the same concepts might influence learner achievement in tests. Questionnaires assisted in determining the degree of learner engagement with functions content as perceived by teachers. Questionnaires were administered to get the results that either support or not support those from pre- and post-tests about using modelling in enhancing learner engagement.

Data analysis

Data collected from teachers' questionnaire was analysed qualitatively using the Leuven scale of engagement in learning (see Table 1), while that from learners' pre- and post-tests was analyzed quantitatively using measures of central tendency (mean, mode, and median) and ANOVA for measures of dispersion (variance and standard deviation). The pre-test was used as the independent variable, whereas the post-test was the dependent variable. This was done to check if learners' achievement would be affected by the intervention of teaching by using the modelling approach. The hypothesis was "Mathematical modelling can lead to better learner achievement in graphing of linear functions, consequently leading to better achievement", and the null hypothesis was "There is no effect in test results if the mathematical modelling approach is applied in teaching graphing of linear functions."

Table 2: The Leuven scale of engagement in learning.

Level	Engagement	Examples
1	Extremely low: Learner shows hardly any activity.	No concentration: Staring, daydreaming, An absent, passive attitude; No goal-oriented activity, aimless actions, not producing anything; No signs of exploration and interest; Not taking anything in, no mental activity.
2	Low: Learner shows some degree of activity, often interrupted.	Limited concentration: Looks away during activity, fiddles, dreams; Is easily distracted; Action only leads to limited results.
3	Moderate: The learner is busy the whole time, but without real concentration.	Routine actions, attention is superficial; Is not absorbed in the activity; activities are short lived; Limited motivation, no real dedication, does not feel challenged; The learner does not gain deep-level experiences; Does not use his/her capabilities to full extent; The activity does not address the learner's imagination.
4	High: There are clear signs of engagement, but these are not always present to their full extent.	The learner is engaged in the activity without interruption; Most of the time there is real concentration, but during some brief moments the attention is more superficial; The child feels challenged, there is a certain degree of motivation; The learner's capabilities and imagination are addressed in the activity.
5	Extremely High: During the observation of learning, the learner is continually engaged in the activity and totally absorbed in it.	Is absolutely focused, concentrates without interruption; Is highly motivated, finds the activity very appealing; Even strong stimuli cannot distract him/her; Is alert, has attention for details, shows precision; His/her mental activity and experience are intense; The learner constantly addresses all his/her capabilities; Imagination and mental capacity are in top gear; Obviously enjoys being engrossed in the activity.

Results

Research question 1

In answering the research question, **How do Grade 11 mathematics teachers perceive their learners' level of engagement when using mathematical modelling approach in learning linear functions?**

Data obtained from teachers' questionnaires were analysed. The results from the teachers' questionnaires were rated against the Leuven scale of engagement in learning given in Table 1. The teachers' responses to the questionnaire are given in Table 2 below.

Table 2: Teachers' responses from questionnaire

Cognitive engagement Indicators	Always	Usually	Sometimes	Not usually	Never
My learners listen attentively.		ABC			
Learners practise mathematics outside the classroom.		AC	B		
Some textbook questions involve modelling and can motivate learners to study function.	A	BC			
Textbook questions involve real-life situations.	A	BC			
Learners participate effectively in class.	C	AB			
My learners are inspired to study graphs of functions.		ABC			
Learners have the fundamental knowledge to tackle questions involving graphs of linear functions.		A	BC		
Learners use a variety of tools to solve problems.	A		BC		
Learners consult each other on different problems.	A	BC			
Learners use all the time they have available to study mathematics.			ABC		

The questionnaire contained statements about facilitators and indicators of learner engagement. In the facilitators of learner engagement, it is worrying that only Teacher A indicated that

learners demonstrated some knowledge about graphing linear functions, while Teacher B and C indicated that their learners sometimes lack this knowledge. Teacher A indicated that the textbooks used always contain real-life problems and use modelling activities, whereas for Teachers B and C, this usually happens. Indicators of learner engagement were listening attentively, participating effectively, finding inspiration to study, using a variety of tools in problem-solving, consulting, and using the available time to study. The questionnaire results show that according to teachers, learners are usually inspired to study and pay attention during lessons. Both indicators have all three teachers falling under the category of "usually", implying that there are instances when learners are not inspired to listen and pay attention. Only Teacher C indicated that learners always participate effectively during the lesson, with Teachers A and B saying their learners usually participate. For the indicator of using a variety of tools in problem-solving, only Teacher A indicated that learners always use a variety of tools, and Teacher B and C's learners sometimes use the tools. All teachers indicated that their learners sometimes use their available time to study. Teacher A indicated that learners always consult one another when they encounter problems in their studies, whereas for Teachers B and C, this happens sometimes. When it comes to practicing on their own, Teacher B's learners do this sometimes, whereas Teacher A and C's learners do it often.

The key learner engagement indicators (listening attentively, participating effectively, and being inspired to study graphs) are all rated by teachers as "usually" happening, and learners using time available to study as happening "sometimes". When comparing these ratings with the Leuven scale of engagement in learning, we concluded that the engagement level of Grade 11 learners who participated in this study was Level 3 (moderate).

Research question 2

In answering the research question, **How does the use of mathematical modelling in teaching linear functions affect learner achievement?** The researchers analyzed the data collected from the learners' pre- and post-test results using measures of central tendency (mean, mode, and median) and dispersion (standard deviation).

School A test analysis

Test	Mean	Standard deviation	Standard error of measurement (SEM)	Number of participants	Mode
Pre-test	15,4	3,5101	0,78270787	20	12 & 18
Post-test	18,25	2,8333	0,6402097	20	17, 19 & 21

H_0 (Null hypothesis): No difference in test means.

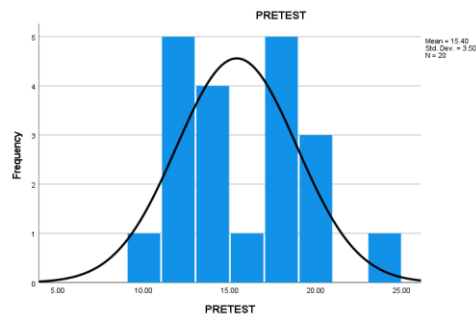
H_a (Alternative hypothesis): At least one difference in test means.

$F_{cal} = 7,943765$

$P = 0,007619$

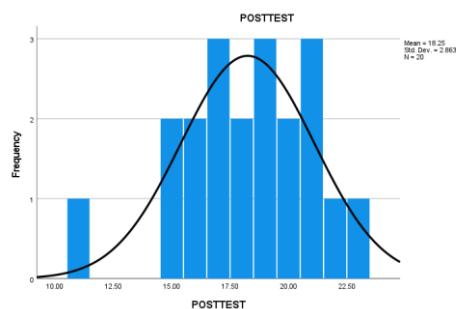
$\rightarrow H_0$ was rejected $\rightarrow H_a$ is accepted.

School A pre-test analysis



In School A 68% of learners scored between 11.89 and 18.91 in the pre-test, while 95% scored between 8.38 and 22.42, and 99.7% scored between 4.87 and 25.93.

School A post-test analysis



In School A, 68% of learners scored between 15.42 and 21.08 in the pre-test, while 95% scored between 12.59 and 23.91, and 99.7% scored between 9.76 and 26.74.

School B test analysis

Test	Mean	Standard deviation	Standard error of measurement	Number of participants	Mode
Pre-test	13,9	3,7403	0,85808843	20	14
Post-test	16,65	2,8333	0,65	20	17

H_0 (Null hypothesis): No difference in test means.

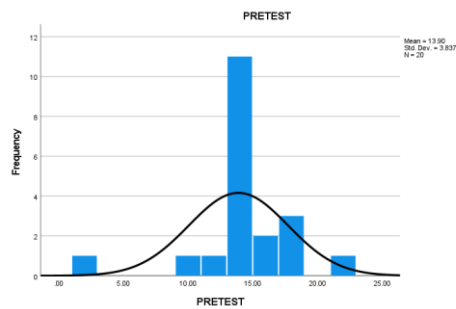
H_a (Alternative hypothesis): At least one difference in test means.

$F_{cal} = 6,5260587$

$P = 0,01460$

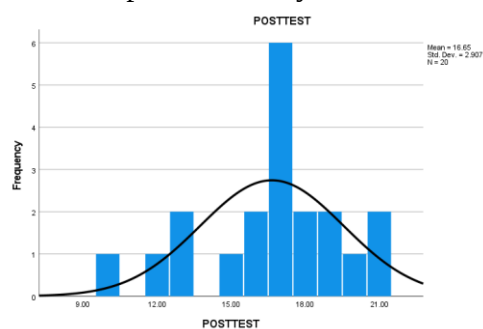
$\rightarrow H_0$ was rejected $\rightarrow H_a$ is accepted.

School B pre-test analysis



In School B, 68% of learners scored between 10.16 and 17.64 in the pre-test, while 95% scored between 6.42 and 21.38, and 99.7% scored between 2.68 and 25.12.

School B post-test analysis



In School B, 68% of learners scored between 13.82 and 19.48 in the pre-test, while 95% scored between 10.99 and 22.31, and 99.7% scored between 8.16 and 25.14.

School C tests analysis

Test	Mean	Standard deviation	SEM	Number of participants	Mode
Pre-test	14,6	3,2619	0,7483315	20	14
Post-test	17,05	2,7106	0,62186056	20	17

H_0 (Null hypothesis): No difference in test means.

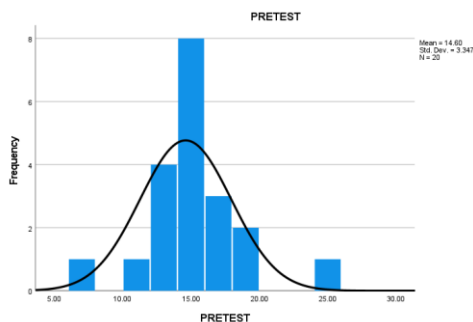
H_a (Alternative hypothesis): At least one difference in test means.

$F_{cal} = 6,3403754$

$P = 0,016135$

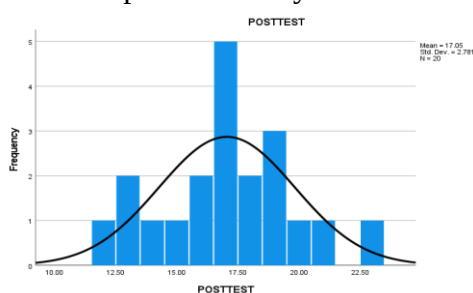
$\rightarrow H_0$ was rejected $\rightarrow H_a$ is accepted.

School C pre-test analysis



In School C, 68% of learners scored between 11.34 and 17.86 in the pre-test, while 95% scored between 8.08 and 21.12, and 99.7% scored between 4.82 and 24.38.

School C post-test analysis



School C improved in the post-test, with 68% of learners scoring between 14.34 and 19.76, 95% of learners scoring between 11.63 and 22.47, and 99.7% of learners scoring between 8.92 and 25.18.

Discussion

The results from the questionnaire reveal that only Teacher A was able to secure the "always" category in two learner engagement indicators of using a variety of tools and consultation in problem-solving. Furthermore, Teacher A secured the "always" category in the engagement indicator of using textbooks containing modelling and modelling activities. This serves as an indication that Teacher A might be using modelling regularly. Teacher A's learners also performed better in both pre- and post-test. We expected the learner engagement indicators of listening attentively and being inspired to fall into the category of "always" because these are crucial for learners to comprehend the content taught; however, this was not the case. It is the teachers' responsibility to keep learners engaged during lessons (Subramanian & Mahmoud, 2020). Essentially, when learners are not engaged during the lesson, it could imply that the learning environment does not provide opportunities for them to engage actively in the lesson. Teachers struggle to get learners' attention and concentration during their lessons (Du Plessis, 2020), and learners can be easily distracted. This also suggests that learners may not complete the modelling activities given due to distractions. The other concern is that of the learner engagement indicator, which uses the available time to study functions.

All three teachers indicated that learners sometimes use their available time to study, which is a concern for learners in Grade 11. However, this does not come as a surprise because when learners are not inspired to study, they will not put effort into their studies (Van Gog et al. 2020). This implies that learning content is not appealing enough to motivate learners to study

on their own, which may be the result of textbooks that do not always contain modelling activities.

It is worth noting that learners' prior knowledge is crucial and forms the basis of learning new content. Two teachers (B and C) indicated that learners sometimes lack fundamental knowledge of modelling activities, however, their categories are in "usually", indicating that sometimes they might not use modelling in their teaching; leading to learners not having thorough knowledge of modelling activities. This calls for teachers to have knowledge of what their learners know and use this as a basis of teaching new content. Moreover, regular use of effective teaching approaches is necessary for learner engagement with content. The fact that learner engagement level stands at moderate according to the Leuven scale is a concern because it is through learner engagement that teachers can determine whether learners understand or detect misconceptions and be able to address them at a very early stage.

The pre- and post-test results show that despite a drop in achievement for some learners, there was generally a considerable improvement in learner achievement in each class when teachers used modelling in their teaching. The lowest scores improved in all the classes, and the improvement is seen also in the mean scores for each class. Class A from 15.4 to 18.25, Class B moved from 13.9 to 16.65, and Class C from 14.6 to 17.05.

Improvement in the mean scores affected the standard deviations (SDs) in each class. The SDs in the post-test results of the three classes dropped as compared to the pre-test results. For Class A, the standard deviations dropped from 3.5101 to 2.8333; from 3.7403 to 2.833; and from 3.2619 to 2.7106 in Classes A, B, and C respectively.

The ANOVA tests of the three classes also show similar results; that in all the three classes, the H_0 (Null hypothesis) was rejected, and hence the H_a (Alternative hypothesis) was accepted. In answering the research question, **How does the use of mathematical modelling in teaching linear functions affect learner achievement?** It can be concluded that there is a considerable effect on learner achievement when mathematical modelling approach was applied in the teaching of graphing linear functions, which resulted in an average improvement of 8.9% in learner achievement.

Conclusion and recommendations

Learner engagement during teaching should be a priority for every teacher as it assists in managing classroom behaviours and increasing achievement (Taylor & Parsons, 2011). When learners engage cognitively during the lesson, this provides an opportunity for teachers to understand how they think (Aliabadi & Weisi, 2023). The use of mathematical modelling approach has the potential to increase learner engagement during teaching, and if used correctly, it can also improve learner achievement in mathematics because active learning is associated with the achievement of learning outcomes (Hartikainen et al. 2019). Teachers need to use models to develop a sense of curiosity that will keep them engaged throughout the lesson. This can be possible if teachers know the kind of learners they teach and the level of knowledge they possess about the content being taught, and the type of models to include in teaching. The

study, therefore, recommends firstly, that teachers strive to use different types of models to explain mathematical concepts as an attempt to keep them engaged in lessons. Secondly, schools need to provide teachers with resources that support the use of models and modelling approaches during their teaching.

The study relied on data collected from teacher questionnaires and learners' pre- and post-tests. Further studies may include lesson observations to explore how teachers use modelling and incorporate it in the teaching of linear functions to enhance understanding.

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USING INDIGENOUS KNOWLEDGE TO ENGAGE LEARNERS IN MEANINGFUL MATHEMATICAL EXPLORATION AND DISCOVERY IN PRIMARY SCHOOLS: A CASE OF SELECTED SCHOOLS IN ALFRED-NZO EAST DISTRICT IN THE EASTERN CAPE PROVINCE

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To foster meaningful mathematical exploration and discovery among learners in primary school settings, this article delves into the integration of indigenous knowledge to engage them in authentic learning experiences. The 2021 National Booklet Report analysis highlights a concerning trend in mathematics: a pervasive lack of understanding of geometric concepts spanning from primary school to high school levels. This deficiency correlates with subpar performance in Grade 12 mathematics results. In response, this approach seeks to remedy the issue by fostering a deeper comprehension of mathematical topics. It achieves this by embracing diverse perspectives and drawing upon the rich cultural heritage and mathematical practices of indigenous communities. Through this method, learners are encouraged to explore various ways of knowing, ultimately enhancing their mathematical proficiency and understanding. To address the question: In what ways does the integration of indigenous knowledge contribute to meaningful mathematical exploration and discovery for primary school learners, the researchers used a phenomenological design and qualitative research methods, underpinned by the Teaching for Robust Understanding in Mathematics (TRU) framework. Three teachers took part in semi-structured interviews based on the observed indigenous approaches used in teaching mathematics lessons. The researchers extracted some form of explanations and interpretations from the qualitative data that were generated by the participants and coded and categorised the data into several themes and sub-themes. The study found that teachers become more adept at recognizing cultural variations and assisting learners in connecting mathematical ideas to real-world situations. When teachers engaged with their learners, they observed productive struggles that enhanced learners' higher-order thinking abilities. It is recommended that the cultural practices relevant to some mathematics topics should be included in the curriculum to protect the social, economic, and environmental aspects of sustainable development.

Keywords: Indigenous knowledge, Exploration, discovery, engagement, meaningful mathematics

Introduction

The study of space and the connections between points, lines, surfaces, and their higher dimensional equivalents constitutes the focus of mathematical geometry (Bhagat and Chang, 2015). Meantime, Moloi (2013) emphasizes the imperative for African mathematics professionals to spearhead the integration of mathematics teaching with indigenous cultural values. Despite South Africa's democratic transition over two decades ago, it has been observed that colonial ideals persist in educational institutions and curriculum development (De Beer

and Petersen, 2016; Fataar, 2018). But, Mudaly (2018) contends that there remains a significant deficit of culturally relevant mathematics education within South Africa's educational system. Notably, all South African learners, regardless of their background, are required to master mathematics within a uniform National Curriculum, which predominantly reflects Western pedagogical approaches, side-lining indigenous knowledge systems (Howard and Kern, 2019).

Learners who rely on indigenous knowledge in their everyday lives or who are not exposed to English as their home language experience language obstacles due to the National Curriculum and Assessment Policy Statements (CAPS). Because of two centuries of continuous colonization, indigenous knowledge systems related to mathematics are no longer apparent in schools (Allen and Trinick, 2021; Hunter, 2021). It has been observed that indigenous knowledge and ways of being are undervalued, and efforts to give it more weight in educational contexts have been sluggish and troublesome (Allen and Trinick, 2021; Govender and Mutendera, 2020). The cause of the undervaluation and slow integration of indigenous knowledge and ways of being in educational contexts can be attributed to various factors such as institutional inertia, resistance to change, and a lack of cultural prioritization. However, in this paper, we encourage teachers to provide opportunities for learners to gain a deeper knowledge of mathematical principles while respecting different ways of knowing by utilizing the social, cultural, and mathematical traditions of indigenous communities.

Nicol et al., (2020) and Nutti, (2013) suggest that teachers must gain a thorough awareness of the sociocultural background and language of their learners to model and incorporate indigenous knowledge systems into the curriculum. Furthermore, Smith (2022) emphasizes the importance of holistic learning, which involves relating mathematical ideas to learners' cultural identities and real-world experiences. The author also notes that it would be best to incorporate indigenous knowledge in primary school mathematics education. Through the integration of indigenous ideas, teachers can cultivate a feeling of cultural pride and belonging among learners while also improving their mathematical comprehension. Nicol et al., (2020) emphasize that it is crucial that teachers provide mathematical challenges that demonstrate respect for indigenous knowledge instead of appropriating it. It is clear from a number of research studies (Garcia-Olp et al., 2019, 2022; Nicol et al., 2020; Nutti, 2013; Ruef et al., 2020) that indigenous knowledge is critical to fostering the expansion of learners' mathematical understanding. Through the use of indigenous knowledge inclusive learning environments that enable learners to make meaningful connections between mathematics and their cultural settings as well as to engage in meaningful mathematical exploration and discovery can be enhanced.

In some of the schools where this study was conducted, mathematics teaching resources are usually scarce due to socio-economic issues. Sometimes the Department of Basic Education (DBE, 2012) supplies some equipment relevant to the teaching of some mathematics concepts, however often teachers are not trained in using them. Thus, creating educational techniques that are consistent with cultural values and beliefs as a specific focus could make indigenous knowledge useful in mathematics classrooms. Similarly, Nicol et al. (2020) argue that real

connections are created when mathematics is set within culturally relevant practices and values. It is anticipated that with the integration of indigenous ideas into the mathematics curriculum, teachers can enhance learners' mathematical proficiency while also cultivating a more profound understanding of cultural variety.

Literature review

This study contests the Eurocentric approach to mathematics education, which bases its understanding of the subject on the idea that Europeans invented it and have higher intellectual capacity. For Conrad (2019) and Williams and Roth (2019), the Eurocentric approach denotes the perspective that prioritizes European culture over other world cultures, such as African culture. It follows that colonized Africans could not develop their goals and futures in rural areas where African indigenous knowledge, games, skills, and languages are respected and prevalent. Instead, they could only do so in contemporary urban area settings. It is from that background that, this study sought to engage indigenous knowledge in meaningful teaching and learning of mathematical exploration and discovery in primary schools.

Mathematical education in Western classrooms conveys the idea that the subject is individual and frequently competitive, requiring separate aims as opposed to shared interconnected goals (Garcia-Olp et al., 2022; Nicol et al., 2020). This means that in Western classrooms, mathematical education often promotes the perception of mathematics as an individualistic and competitive subject. It implies that learners are encouraged to pursue their own goals rather than working collaboratively towards shared objectives. Indigenous knowledge should be used to teach learners in decolonized classrooms where they can share their experiences and knowledge (Furo, 2018). Teachers in primary schools might encourage learners to apply the riches of their indigenous and cultural knowledge to improve their comprehension of mathematical ideas. Hunter and Hunter (2023) provide examples of the cognitive dissonance that learners felt when forced to work in the competitive setting of ability grouping in mathematics classes, as opposed to their sense of justice when given the opportunity to collaborate. Using indigenous knowledge to teach mathematics fosters cultural values by giving learners an opportunity to make connections between their learning and experiences.

According to Sefa Dei and Jaimunga (2018, p.1), one of the main goals of decolonization is subverting the taken-for-granted, everyday assumptions deeply held by society that work to routinely reinforce white power and privilege them. Opportunities to challenge prevailing practices, such as the Western conception of the normative self and whose knowledge is prioritized in mathematics classes, are presented by the decolonization of mathematics education (Ruef et al., 2020; Stavrou and Miller, 2017). Education for primary school mathematics that incorporates indigenous knowledge is in line with the larger educational objectives of social justice, diversity, and equity (DBE, 2012). Koya (2017) claims that decolonization of education, more especially decolonization of mathematics education is necessary given the diversity of perspectives and interactions that people have with the world. In this paper, we emphasize indigenous knowledge of collectivist cultural groupings as a crucial point of reference rather than colonization, building on an anti-colonialism approach. This is not to ignore colonial histories but instead to view how people resist and engage in

education in spite of, rather than in response to, colonial histories (Koya, 2017). We examine the ways in which learners might be positioned as collective communities that produce and disseminate information in order to foster agency and autonomy. Furthermore, by putting other ways of knowing and questioning Eurocentric perspectives, incorporating indigenous knowledge into the curriculum can aid in the decolonization of educational systems. In that way, indigenous knowledge could be best understood and transmitted within its cultural context and places a strong emphasis on community involvement, collaboration, and reciprocity.

Understandings of reality, values, and knowledge creation are positioned as the foundation of a research paradigm by drawing on decolonization theory, eliminating the need to connect and defend this in connection to Western perspectives (Sanga and Reynolds, 2017). This implies that indigenous knowledge can be included in primary school mathematics education to create more inclusive, culturally sensitive, and relevant learning opportunities for all learners. This can ensure that teachers promote tolerance for different cultures and methods of knowing while also assisting learners in developing a better understanding of mathematics by integrating indigenous ideas and practices.

Moleko (2018) complains that the degree of difficulty in the mathematical geometry curriculum combined with the abstract teaching methods used in schools is out of step with the educational level of the learners' underperformance. For this reason, the study attempts to close this gap by applying the indigenous knowledge approach. Nkopodi and Mosimege (2017), who claim that Western values have traditionally dominated South African education, attest to this. As a result, these authors claim, learners from underprivileged backgrounds are unable to make the connection between their education at school and their real-world experiences because mathematics education in schools continues to be dominated by a teacher-centered, textbook-oriented approach to teaching (Özerem, 2012).

According to Bhagat and Chang (2015), learners struggle to grasp geometric concepts, which leads to poor performance and a decrease in their interest in geometry. However, concerning geometric concept issues, the knowledge that supports learners in those areas is frequently ignored. Given that this research considers human activities in our culture and surroundings, it recommends using the indigenous knowledge approach to teach and learn mathematics.

Theoretical framework

Since the participants valued games, realistic, real-world contexts, indigenous knowledge, and cultural artifacts to help learners acquire mathematics, the TRU framework was important in this study. d'Entremont (2015) asserts that learning mathematics can be aided by the use of familiar real-world objects and contexts. This study was supported by the Teaching for Robust Understanding (TRU) framework, which identifies effective learning environments and offers a clear and understandable language for talking about what occurs (and should occur) in classrooms, in teacher preparation programs, and in teacher development (Prediger, 2021). This framework, as highlighted by d'Entremont (2015) and supported by Prediger (2021), emphasizes the use of familiar real-world objects, contexts, indigenous knowledge, and cultural artifacts to facilitate mathematics learning. By delineating effective learning environments and

providing comprehensible language for discussing classroom practices, teacher preparation, and professional development, TRU offers a structured approach to enhancing mathematical acquisition. TRU offers five dimensions that, when mastered, result in learners who are highly capable of thought (Schoenfeld, 2016). According to Schoenfeld (2016), these include the following: agency, ownership and identity, equal access to content, cognitive demands, content, and formative evaluation. Thus, it became crucial to examine how applying indigenous knowledge enhances understanding of geometric concepts through the lens of the TRU framework.

TRU underscores the significance of teacher activities in facilitating learners' interaction with the material and ensuring they are well-positioned to benefit from the wealth of knowledge within the field, highlighting the crucial role of effective training for educators. The term cognitive demand describes the degree of difficulty. Teachers must strike a balance so that learners can push their present understandings and build on their prior knowledge (Schoenfeld, 2022). The issue of whether there is unequal access to the material being taught in the classroom is the focus of equitable access to content (Schoenfeld, 2016). According to Zacharian and Silverstone (2020), learners exhibit intellectual agency when they express their opinions on the issue at hand rather than feeling compelled to provide a solution that they may or may not find convincing. The utilization of indigenous knowledge may be more convincing to learners than the Western approach to the understanding of mathematics concepts.

Methodology

According to the interpretivist paradigm, which underpinned this study, people interpret their experiences to gain knowledge about the planet they inhabit and work on (Cohen et al., 2017). The researchers examined the utilization of indigenous knowledge to foster meaningful exploration and discovery of mathematical concepts in primary schools situated in rural areas, employing a phenomenological case study research design. The questionnaire was administered to a diverse sample of primary school teachers from rural areas, comprising teachers with varying levels of experience and backgrounds in mathematics education. Based on the observed indigenous approaches used in teaching mathematics lessons, a sample of three intermediate-phase mathematics teachers in the Eastern Cape in Alfred Nzo East District, took part in the research. Three teachers were observed and selected to respond to semi-structured interviews after the lessons were presented. The two researchers visited schools on several days to gain insight into how indigenous knowledge was organized and prioritized when the teachers taught mathematics in Grade 6 classrooms. One researcher was asking questions while the other one was taking records and make follow up to some of the responses. The Kotoka (2012) observation tool was employed as a tool for documenting classroom activities.

To gather first-hand information, the researchers also used semi-structured interviews with participants, who were allowed to reply in any language of their choice. Three Grade 6 primary school mathematics teachers from three different schools in the Alfred-Nzo East district of the Eastern Cape, South Africa, were selected to participate in this study using a qualitative method called purposeful sampling. Observation tools, video clips, and notes were changed, and all observations were classified, transcribed, and kept on file. Thus, this study is addressed the

research question: In what ways does the integration of indigenous knowledge contribute to meaningful mathematical exploration and discovery for primary school learners?

Results

Segments of the data were transcribed, coded, categorized, and patterns were created. To maintain confidentiality, the teachers in the three chosen schools were assigned codes during the entire study: TA, TB, and TC. An observation schedule that attempted to examine (i) the various forms of teaching techniques, (ii) the language teachers used to elucidate geometric concepts, (iii) their connection with learners, and (iv) the teaching tactics utilized was implemented using an adopted tool (Kokota, 2012). This was consistent with the study's theoretical foundations. Two themes emerged from the data analysis that have to do with teaching geometry in primary schools using indigenous knowledge. These included using indigenous games to teach mathematics and having learners sketch geometric representations and projections. Each of these themes included categories that showed how the participants' use of indigenous knowledge facilitated the performance of mathematical relationships, interactions, and activities.

Learners' geometric representations and projections through drawing

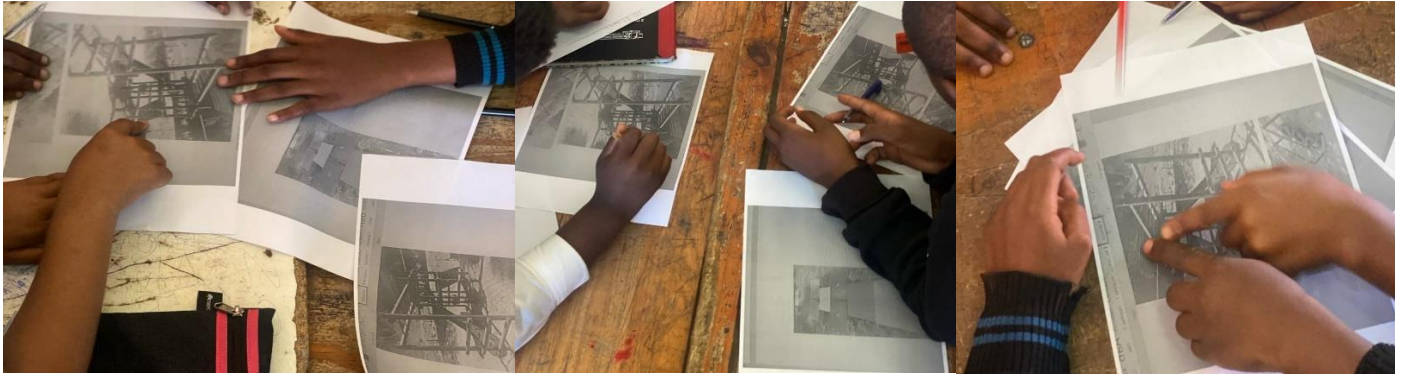
With 104 learners in Grade 6, the first teacher (TA) had a crowded classroom as she taught mathematics. Teachers and learners found it challenging to participate in the mathematics lessons since the classroom was too small and crammed with too many chairs. To encourage effective learning, the teacher needed to be creative, kind, and empathetic. She used the example of outdoor games found in the grade R field for learners to play.



Extract 1: Outdoor games to teach geometric concepts

The teacher came with printed pictures on A4 paper and put them on the chalkboard where learners could be able to see. The learners looked at the pictures seriously and some were pointing while others were so quiet as if they were waiting for the teacher to say something. The teacher asked the learners to name the different shapes they see from the pictures. The

questions the teacher asked to assess learners' prior knowledge appeared to be significant as the learners were asked about what they already knew. The teacher asked the learners to draw the different shapes they see from the pictures and name them. The learners created different drawings similar to the ones on the figures below.



Extract 2: Learners discussing different shapes from pictures

The learners were then grouped to discuss the different types of shapes from the pictures they were given. The teacher asked for the different shapes the learners could see in the pictures. After discussions and oral questions asked by the teacher, the learners were given a chance to draw the different shapes they could see from the pictures again.



Extract 3: Learners drawing different shapes from photos

The teacher asked the learners again to name the different shapes they drew. The learners who responded only mentioned the 2-D shapes and did not mention the 3-D as they were in the drawings. The teacher began to talk about the relationship between 2-D and 3-D shapes.



Extract 4: Different types of shapes drawn by learners

TA: There are 2-Ds and 3-Ds on the drawings but you mention only the 2-D shapes. What is the difference between the shapes on the drawings? Which one do they represent 2-Ds and 3-Ds?

Learner: In 2-Ds there is only one part yet in 3-Ds there are many parts.

The teacher explained the parts the learner mentioned using the correct terminology.

TA: The parts you are talking about are closed-shaped and they are called faces. In a 2-D shape there is one face yet in 3-D there is more than one face. In a 3-D shape, one can be able to see the different views or sides of a shape yet in 2-Ds, one can only see one view or face.

The teacher referred back to the pictures on the chalkboard and the ones given to the groups. In the pictures, there were wheels, shelter, and hopscotch (uchiki).

The teacher asked:

TA: How many faces in a wheel and what is the name of that shape?

Learner: A wheel is an example of a cylinder although it has an open space.

The teacher further asked:

TA: Which geometric shapes compose the cylinder? What types of lines are utilized to depict a wheel in the example?

Through the explanation of the teacher about the faces, the learners were able to explain that the faces that formed the cylinder were circles. They were also able to mention the types of lines used to draw the examples of wheels that were curved lines.

The teacher then used the example of a shelter from the outdoor games. The shelter was drawn in both 2-D and 3-D.

The teacher asked the learners to compare the two different shapes drawn. She numbered them, numbers 1 and 2.

TA: Look at the two drawings and compare them. Which one is a 2D and which one is a 3D? Please also state a reason to support your answer.

The learners seemed to be quiet after the teacher asked them to give their reasons. Very few learners lifted their hands as a sign of readiness to share the answers.

Learner: Number 1 represents 3-D because I can be able to see three views whereas number 2 is a 2-D shape as I see one face.

The teacher asked the learners to point to the faces the learner one talking about in the drawings.

TA: Please come and point at the faces or the views the learner was talking about.

One learner went to point out the views but she confused the faces as the shelter is made with different shapes. The learner saw the different shapes that formed the 2-D and she concluded that the shape falls under 3-D.

The teacher had to address the misconception of the faces that are formed with different shapes.

TA: The building we are using is an example of a 3-D object and when you are standing in front of it, you can be able to see even the roof. When you are asked to draw it, you will only draw the front part of the building. The front part you draw is the one that makes us conclude and say the shape is in 2-D.

TA further explained:

TA: The shapes that formed the faces are not used to determine to differentiate between 2-D and 3-D. 2-D of the shelter is formed with different forms of shapes but that does not make it to be 3-D.

The teacher also referred to 6 or 8 corner houses formed with different shapes.

The teacher introduced the vertices and edges of the shapes.

TA: Vertices are the corners is a shape or object whereas edges are the lines where the sides meet. Look at the chalkboard and let us count the number of edges, vertices, and faces. Also, look at the desk and count the number of edges, vertices, and edges.

The teacher had a commutative dialogue and discourse with the learners from the use of familiar examples to teach geometry in her classroom. The teacher engaged the learners to investigate the relationship between the edges, vertices, and faces in groups. The learners were asked to use the objects in the classroom or the ones they had such as lunch boxes or bottles they had. They had to pick four objects they would use to do the investigation. They were also given a chance to share their findings with the class.

When the researchers asked the teacher during the semi-structured interviews about the reasons of grouping the learners and giving them a choice to choose the objects they used for the activity, the TA said:

TA: I wanted to give learners freedom around their peers and a chance to learn to share ideas with others. I wanted them to learn to be exposed to debates and to be able to defend themselves by supporting their points or ideas. Also, group work gives learners a chance to use the language of their choice and the misconceptions might be addressed in the groups. The reason for asking them to choose the object of their choice was to give them a chance to use the object they would be able to name and find the properties of them. I also wanted them to know that the objects are around us and that would make learners understand that geometry is part of real-life context.

Use of beadwork to teach geometry

The second teacher, TB, was observed teaching mathematics in a rural under-resourced school. The teacher posted a picture of a lady wearing bead attire that all learners could be able to see. When the teacher posted the pictures, the learners started to talk.

Teacher asked:

TB: When do people usually wear like this and why are they not wearing like that every day?

Learner: People usually wear the beads when it there are traditional ceremonies to represent their culture. For example, on cultural days, people usually wear traditional attire.



Extract 5: Beadwork to relate geometric concepts with indigenous knowledge

TB: Look at the picture carefully and name the different types of shapes you can see from the beadwork.

The learners mentioned the shapes.

Learner 1: I see quadrilaterals from the apron and triangles from what is on the head and neck.

Learner 2: I see circles from what is on the head and neck.

TB: Why are you shouting at the one who is giving the answer. She did not say, “We see circles” but she said, “I see circles”. Give her a chance to show you the circles or explain where are they. No one has a right to say the person is wrong but give the person a chance to defend himself or herself.

Learner 3: The artefacts on the head and neck give a round shape when they are won. It by that reason that I said I see circles.

The teacher looked surprised and she confessed that she did not see in that way and she was convinced that there were circles.

TB: The neck and head look round although the shape of the head is not round. The artefact adopts the curved shape of the head or a neck when it is won.

The teacher also introduced the curved shape using the one artefact that is tied to the chest.

TB: There are also curved shapes in the artefact that are tied around the chest that forms semi-circles of different sizes. The semi-circles also form a pattern there.

The teacher explained semi-circle and circles.

TB: Draw a circle and a semi-circle.

The drawings were not accurate as the learners were using free-hand drawing.

TB: You see that your drawings are too accurate although you used curved lines you say they are circles. Same thing to your heads, they are not spherical as soccer balls but we say round.

The learners have begun to laugh.

Learner: Other heads are like ruby balls, some have corners, Mem.

The teacher got back the response of quadrilaterals.

TB: What are quadrilaterals?

Learner 4: Quadrilaterals are shapes with four sides and four angles.

When the learner mentioned the angles, the teacher drew an example of the rhombus from the artefact and asked the learners to explain angles. She also asked them to mark the angles.

TB: What is an angle? One learner must come and mark the angles in the shape.

Learner: Angles are corners, Mem.

One learner went on the chalkboard and marked the interior angles.

TB: There are eight angles in a quadrilateral and six angles in a triangle. All the shapes have interior angles and exterior angles. When you talk about angles in a shape, you need to be specific about the ones you are referring to.

The teacher explained while she was marking the angles on the drawing.

The teacher explained the properties of shapes using the new geometric terms such as vertices, edges, and faces.

When the teacher mentioned the term face, some of the learners touched their faces.

TB: As you have one face, also the 2-D shapes have one face despite the number of edges or vertices. Some of the shapes have no vertices or edges but they have faces whereas some have no faces and no edges.

One learner asked:

Learner: Why do some shapes have no edges or faces?

TB: When the shape is completely round or has no flat sides has nor faces. For example, the ball has no faces which is why it rolls nor slides.

TB: Work as teams to draw the shapes you see in the artefacts and discuss the properties of each shape.

During semi-structured interviews, the teacher was asked about the significance of beadwork in teaching shapes.

TB: I decided to find something that is related to the culture and familiar to the learners to teach 2-D shapes. I wanted to take mathematics from the textbook and relate it to real-life context. I also wanted the learners to be aware that in any form of education, whether formal or informal, people practice mathematics, meaning that mathematics is part of daily lives.

Researchers: How can you reflect on your teaching in the classroom?

TB: I have realized that through the innovation of what learners know, they become interested in the lesson and participate actively. Even the learners who are shy in the classroom, get a chance to participate actively in the lesson as they talk with what they know or things that seem to be general or out of the topic of mathematics. The learners do not feel the stigma of mathematics as it seems that they are talking about something else generally in reality not for mathematics education. These kinds of activities engage learners to participate actively and to grasp mathematics concepts.

Researchers: How would you advise on how to choose relevant indigenous artefact, games, or play for mathematics lessons?

TB: As a teacher, I need to watch how learners play and think of the things they are familiar with from the environment and use that to enhance their understanding of geometric concepts. You can then use the artefact of a game to twist the questions that could lead learners to the understanding of mathematics concepts. It is the duty of the teacher to think creatively to find the questions that could lead learners to an understanding of the topic.

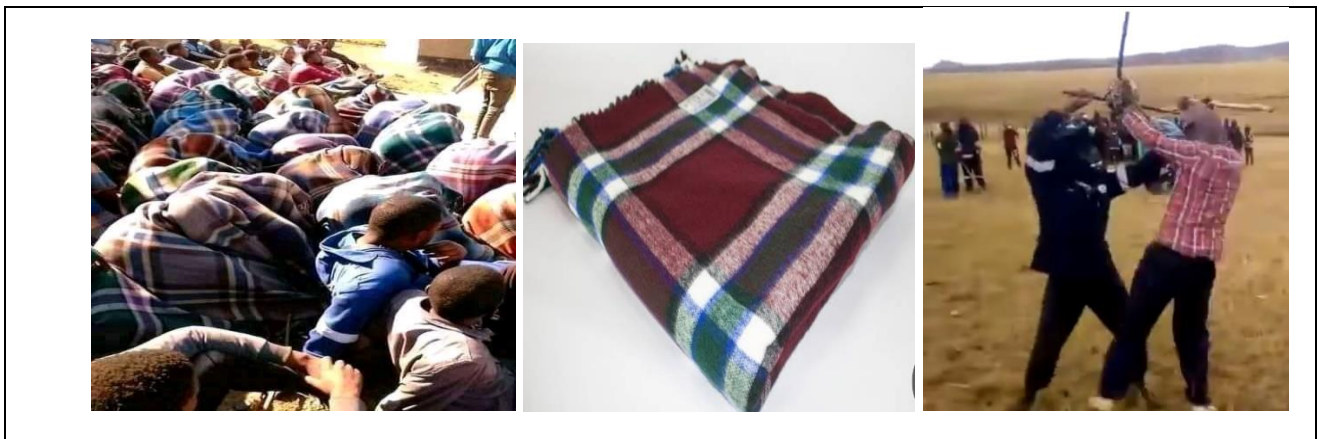
Researchers: How can you motivate the other teachers to use indigenous knowledge to teach mathematics?

TB: It is not easy to convince other teachers but I can share how the use of indigenous knowledge works in the classroom and how it enhances the understanding of mathematics in

learners. I can invite some of the teachers to observe how I use IK in the classroom so that they can try it in their classrooms. By doing that I think I can motivate many teachers as the others will motivate other teachers.

This shows that the teacher is aware that using indigenous knowledge could be one way to improve learners' comprehension of geometry in primary school. The learners were able to recognize those plays. While some learners were busy pointing to others to form groups so they could dance, others inquired as to whether they would be sent outside to dance. The learners were familiar with the context that was created.

Rug blanket to teach geometry in the classroom



Extract 6: Rug blanket and iintonga (stick) game to relate geometry with indigenous knowledge system

The teacher came with a printed picture of the initiation ceremony where boys covered themselves with blankets. He posted the picture on the chalkboard where learners could be able to see it. All the learners looked at the picture while others were whispering to their peers. Some learners were so excited to see the pictures, especially the boys. The teacher asked:

TC: Why do you all look excited when you see the picture?

Learner: One day Sir, I will go for initiation but not in this group as my siblings did not go in this group.

Learner: My brothers were initiated in this group so I have to go to it when I am old.

The teacher asked,

TC: Why they are covering themselves with the blankets, heads are covered with doeks, and have rods in their hands? Why are they fighting with iintonga?

The learners gave different answers and there was a debate in the class.

Learners shouted: They are not fighting, Sir, they are playing.

The teacher had to control the class as learners were overwhelmed. She asked the questions:

TC: *Look at the blankets and name the shapes you can see from the blankets and doeks they usually cover their heads.*

Learner 1: *I can see squares and rectangles.*

TC: *What is the difference between a square and a rectangle?*

Learners were able to tell that a square and a rectangle are the same. The other learners were saying that rectangles and squares are one thing different.

Learner: A rectangle has two opposite sides equal and in a square, all sides are equal.

The teacher explained:

TC: *A square is a special kind of rectangle with all sides equal in length. It is a quadrilateral with all four angles and right angles and that is why it is called a special kind of rectangle. Rectangles are not squares as all sides and angles in a square should be equal.*

The learners were able to mention that doeks are in the form of triangles and some in a square of rectangles.

TC: *What is a rectangle?*

Learner 2: *A rectangle is a shape with three sides and three interior or exterior angles. It has got three corners.*

The teacher began to introduce the different types of 2D shapes starting with the quadrilaterals. He also asked learners to relate the shapes with what they know from the environment or at home.

TC: You can use the sticks they use for playing during the initiation ceremony for different types of shapes. It is when you can see you can see the different types of angles from different shapes practically. For example, as the two rods in the picture are crossed, the formed angles.

The researchers asked the teacher why he used the blanket to introduce the different types of shapes in the classroom.

TC: *I used the initiation ceremony picture as I wanted something that would draw learners' attention from real-life experience. I know the learners are familiar with the kinds of ceremonies and they enjoy initiation ceremonies. I also wanted to relate mathematics to real-life context.*

The researchers asked: *Why do you think real-life examples would be relevant for teaching geometry in the classroom?*

TC: *We apply mathematics in our everyday situation and we do not consider that we are doing mathematics. I wanted my learners to understand that indigenous knowledge they have or used at home is also relevant for the understanding of mathematics in formal education. This could also assist learners to be observant at home when they are doing things and associate cultural activities with formal education. This could assist learners in understanding mathematics knowing that is something relevant to real-life contexts.*

Researchers: *Do you think indigenous knowledge is important to enhance understanding of geometry in learners?*

TC: *Indigenous knowledge is important for teaching and learning mathematics as it relates what learners learn in the classroom with what they experience or relevant to real-life contexts. The learners use different shapes at home as objects they use at home are having shapes. They also experience the use of shapes when they are playing. For example, cards are in the form of shapes and they have different shapes drawn in them. I can say that life is about mathematics and is more about geometry. also, money has shapes. I can conclude and say life is about mathematics. The teacher has to build on what the learners already know to enhance their understanding of formal mathematics.*

Researchers: *When you decided to take the picture of initiation, were you not afraid that the class might lose control as some learners became more excited when they saw the picture?*

TC: *I decided to use it to draw learners' attention as I know that most learners enjoy these kinds of ceremonies. That was the strategy to make learners free and pay more attention with interest on what I was going to say in the classroom about the picture. Also, that was something that would make learners interact with one another and think creatively.*

Probing further we asked: *Why did you decide to choose an initiation ceremony as communities are facing challenges of young boys who go for initiation without getting permission from the parents or guardians?*

TC: *I was not promoting that children should go for initiation at younger ages but I wanted something that would make a lesson more interesting to learners and make them participate actively. If we are afraid of the things that are happening in the environment, it means we cannot relate mathematics with indigenous knowledge, and some of them have negative and positive sides.*

The teacher further explained:

TC: *'It is important for learners to be exposed to learning opportunities that link mathematical concepts and principles with their applications encountered in everyday experience.*

Researchers: *How could you justify yourself by the kind of example you chose to enhance understanding of geometry in your classroom?*

TC: *These kinds of activities are used in Art and Culture as one of the subjects taught at schools. I usually see the learners practicing ceremonies for their formal activities and initiation is one of them. I usually observe how happy they become when acting during Art and Culture and how dedicated they become in their groups. I also wanted to promote that group work is so important for an understanding of mathematics. They also practice Ubuntu values in their groups as they learn to respect others. They go to investigate about some of the things about initiation as they do not know them since they are still young. This is one of the activities that make learners think and give them a chance to debate.*

Discussions

Teaching for Robust Understanding was observed in most lessons observed. The three teachers were observed fostering the understanding of mathematical concepts by using and referring to indigenous processes like initiation customs. This is one of annual events that all young boys look to accomplish when they are of age. Part of that custom is a special blanket called 'iragi' that the teacher referred to in his classroom. This indicates that the teacher was aware of the potential for using indigenous artefacts to improve primary school learners' understanding of geometry. This concurs with Prediger (2021) who emphasizes the use of familiar real-world objects, contexts, indigenous knowledge, and cultural artifacts to facilitate mathematics learning. The participant expanded on the geometry teaching that the learners received. The introduction and identification of shapes and lines displayed in those blankets opened the learners' understanding of geometry. This supports the claims made by Ball and Ladson-Billings (2020) who assert that every learner in a classroom in the twenty-first century brings valuable, individual cultural experiences and viewpoints to the table that should be valued and included in education. In addition, this indicates how the teacher was striking a balance for cognitive demand such that learners could push their present understandings and build on their prior knowledge (Schoenfeld, 2022). This is also connected and achieves the CAPS document's objective, which states that learners should have a critical understanding of how mathematical relationships are applied in social, environmental, cultural, and economic contexts while also developing a love and curiosity for the subject (DBE, 2012 p. 13). The teacher also introduced different kinds of lines with examples from their drawings. The image helped the teacher explain 2-D and 3-D concepts as well. For instance, the learners were able to comprehend the round and conical forms present in the circular wheels. This is consistent with the findings of Alex and Mammen (2018), who contend that mathematics fosters more original thought and reasoning in the context of real-world issues that may be resolved through the use of geometry.

The teacher demonstrates an awareness of the significance of IK in the teaching and learning of mathematics. The teacher is aware that Ubuntu encourages the exchange of ideas and viewpoints, which can be utilized in the classroom for educational objectives. Ubuntu may encourage learners' commitment to learning by encouraging group projects. Through the use of indigenous artefacts in the schools, learners were introduced to the informal deduction of van Hiele's level as they started to relate the numerous figures' features to one another and to

order and apply logical reasoning and mathematics ideas abstractions (van der Walle, et al., 2017). The learners were able to recognize the various forms from outdoor game fields, initiation ceremonies, and bead-work. When the learners drew different shapes, they needed to understand which shape they used to make the outdoor game field or bead-work use. For example, when the learners drew shapes, they needed to know the shapes they were used in each artefact and estimate the size of those shapes. Also, the learners should know the angles that were formed in those shapes and their sizes. The sharing of indigenous knowledge promotes critical thinking and ubuntu values where people pay respect, care for one another, and discuss ideas.

Conclusion

The results of the data collection showed that learners' prior knowledge necessary for understanding geometry might be activated by the pre-lesson assessment activities. The purpose of the study was to investigate how the use of indigenous knowledge could engage learners in meaningful mathematical exploration and discovery in rural situated primary schools. The findings showed that teachers' use of games, activities, and indigenous knowledge enhanced learners' comprehension of geometry. It is advised that mathematics teachers receive training on how to contextualize mathematical space and shape learning, as well as how to use native teaching aids that differ based on the learning style of the learners. Mathematical discourse, active involvement, and learner motivation were all improved by the integration of games and indigenous knowledge. This method of teaching geometry that incorporates games and indigenous knowledge also enhances learner collaboration and teamwork, attitudes toward mathematics and social skills, and learning and comprehension of complicated problem-solving.

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FOUNDATIONAL KNOWLEDGE FOR GRADE 12 ALGEBRA ASSESSMENT

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Algebra is an important part of the mathematics curriculum in South Africa and the rest of the world. However, there are always issues in the way Algebra is taught and this affects learner performance in algebra and other mathematics content where algebra plays a key role. In this paper some Algebra questions of Mathematics Paper for 2023 are worked out, analysed and the background knowledge needed to answer these questions traced to earlier grades. The spiral nature of the mathematics curriculum and the knowledge required for mathematics teaching feature prominently in these analyses. The findings reveal there are key algebraic concepts that are regularly assessed in the grade 12 examinations and these concepts are introduced to learners as early as grade 7 or 8. This means that schools should strengthen the teaching and learning of mathematics in the earlier grades if performance in Algebra and other topics, and consequently Mathematics, is to improve in grade 12.

Introduction

Algebra is one of the main topics for FET Mathematics in South Africa (DBE, 2011). The mark allocation for Algebra (which includes equations and inequalities) in FET year-end assessment is shown in table 1:

Table 1: The assessment of Algebra in Grade 12 Mathematics

Grade	Mark allocation
10	30 ± 3
11	45 ± 3
12	25 ± 3

The mark allocation for the Algebra portion of the Grade 12 Mathematics Paper remained the same when the Mathematics Examination Guidelines was revised in 2021 (DBE, 2021a). We observe that there is an increase from grade 10 to grade 11, and then a drop in grade 12. However, it should be noted that the work taught in grade 11 is assessed in both grade 11 and grade 12.

Algebra is also one of the key content areas in the Senior Phase where it is combined with Patterns and Functions (DBE, 2011a). The mark allocation for Patterns, Functions and Algebra in the Senior Phase is shown in table 2 (as percentages).

Table 2: Patterns, Functions and Algebra in the Senior Phase

Content Area	Grade 7	Grade 8	Grade 9
Patterns, Functions and Algebra	25%	30%	35%

We observe in table 2 that there is a progressive increase in percentage allocation from grade 7 through to grade 9. This is, obviously, in preparation for the more “demanding” work in grades 10 to 12.

Problem Statement

There is no doubt from the discussion in the introduction of this paper that algebra is an integral part of the mathematics curriculum, from grade 7 through to grade 12. Thus, learners should be exposed to a variety of classroom activities where their algebraic knowledge and skills are nurtured and developed.

The various activities for Algebra in both the Senior Phase and FET is clearly spelt out in the CAPS Document for Senior Phase Mathematics (DBE,2011a) and FET Mathematics (DBE, 2011).

For example, in the Senior Phase, the key focus of Algebra content is

- Investigation of numerical and geometric patterns to establish the relationships between variables
- Expressing rules governing patterns in algebraic language or symbols
- Developing algebraic manipulative skills that recognize the equivalence
- between different representations of the same relationship
- Analysis of situations in a variety of contexts to make sense of
- them
- Representation and description of situations in algebraic language
- formulae, expressions, equations, and graphs

The CAPS document is very specific about what should be done in grade 7, grade 8 and 9. It appears to be developmental, going from “easier” work in grade 7 to an increased number of topics, with a bit more “difficulty” in grades 8 and 9. It also recommends that each year starts off with a revision of key concepts of the previous years. This is clearly in line with the spiral approach to learning (Bruner, 1960).

The CAPS curriculum for Mathematics is very specific about what is included in Algebra in grades 10 – 12. A summary of the Algebra content for grades 10 to 12 are:

Grade 10:

- Work with real numbers (rational and irrational)

- Simplify expressions using the laws of exponents
- Identify surds
- Manipulate algebraic expressions by multiplying a binomial by trinomial
- Manipulate algebraic expressions through factorisation
- Sum and difference of cubes
- Solve equations (linear, quadratic, literal), linear inequalities, systems of linear equations, word problems

Grade 11:

- Work with non-real numbers and the nature of roots
- Apply exponential laws
- Add, subtract, multiply and divide simple surds
- Revise factorisation
- Solve quadratic equations and quadratic inequalities
- Solve equations in two unknowns (one linear and one quadratic) algebraically and graphically

Grade 12:

- Definition of a logarithm and laws needed to solve real-life problems
- Understand the remainder and factor theorem; factorise third-degree polynomials

A common term used to describe these activities in research, teaching, learning, and assessment is “Algebraic Manipulation”. There appears to be no formal definition of algebraic manipulation, but rather a description of what it entails. In this regard, the writer attempts to describe it as “ the manipulation of algebraic expressions, using appropriate mathematical techniques or procedures, into a simpler or recognisable forms in order to solve given problems”.

One area where algebraic manipulation features strongly is in the National Diagnostic Report for Grade 12 Mathematics where it is stated that:

“The algebraic skills of the candidates are poor. Most candidates lacked fundamental and basic mathematical competencies which could have been acquired in the lower grades. This becomes an impediment to candidates answering complex questions correctly”

This statement appears in the National Diagnostic Report for Grade 12 mathematics for 2019 to 2022. (DBE, 2019; DBE, 2020; DBE, 2021; DBE; 2022). Thus, it would be reasonable to argue that South African mathematics learners tend to struggle with algebra, and this is likely caused by inadequate preparation or learning in earlier grades.

Research question

In the light of the discussion on algebra and algebraic manipulation in the previous section and the National Diagnostic Report for Mathematics, the following research question was posed for this paper:

What background knowledge and skills do grade 12 learners need to answer questions involving algebraic manipulation?

The following sub-questions were formulated in line with the research question.

- What type of algebraic manipulation techniques are needed to solve some questions in grade 12 mathematics papers?
- When are learners likely to be taught or encounter, for the first time, these algebraic manipulation techniques?

Literature Review

This short literature review examines some issues on algebraic manipulation as highlighted by researchers. Angle (2007) contents that most mathematical concepts can only be understood fully if procedural skills of the concept are acquired by the learners' themselves. Sarwadi and Shahrill (2014) state that students tend to struggle with mathematical concepts late in high school due to a number of misconceptions learnt by them and this appears to result in poor performance in tests and examinations.

Matzin and Shahrill (2015) describe algebraic manipulation as the rearrangement of variables in an equation to obtain the desired outcome. They refer to the use of these basic skills in manipulating an equation and in other mathematics content such as functions, geometry, and trigonometry. They conclude that students' inability to manipulate algebraic equation properly has led them to have difficulties in solving word problems where knowledge of algebraic techniques is key when translating the problem into an equation.

In this regard, Kani and Shahrill (2015) say that students often misinterpret the statement or directly translate the words into an equation, making errors in expansion and factorization of algebraic expressions or equation and errors in rearranging or manipulating algebraic equations. Other factors may include low English proficiency in interpreting word problems. When students find a mathematics question difficult to understand, they are not likely to attempt the question or may use an inappropriate approach in solving the problem (Kani and Shahrill, 2015)

In a large-scale study in Italy (Ferretti, 2017), grade 10 mathematics students had to choose which expression was equal to $\left(a - \frac{1}{a}\right)^2$. Four choices were given.

A. $a^2 + \frac{1}{a^2} - 2$

B. $\frac{a^4 + 1}{a^2}$

C. $a^2 - \left(\frac{1}{a}\right)^2$

D. $\frac{a^2 - 2a + 1}{a^2}$

The correct option was A, $a^2 + \frac{1}{a^2} - 2$

The number of correct answers was lower than 35%, the number of wrong answers was about 63%, and the number of missing or invalid answers was very low (2.5%). The fact that the correct answer was option A may indicate that many students did in fact read it, as it was the first option but decided to reject it. Among the wrong answers, the most popular was option C which involved the difference of squares of monomials from the algebraic expression in the task text. It also emerged that the top achievers who chose the wrong answer were likely to choose option D, rather than option C. Another interesting fact is that as regards high ability levels (the last 3 deciles), the students who chose the wrong answer were more likely to choose option D than option C.

This study by Ferretti confirmed other studies about the challenges associated with the learning of algebra. Ferretti, further, states that the difficulties are of a semiotic nature (related to signs and symbols) and that students should be given adequate preparation to understand the meaning of algebraic terms. This may allow them to resolve algebraic expressions without using algorithmic procedures that may often lead to mistakes (Ferretti, 2017).

It is earlier stated that the algebraic skills of grade 12 learners are poor with a number of them lacking fundamental and basic mathematical competencies which should have been learnt in earlier grades. The National Diagnostic reports (for 2019 to 2022) provide suggestions on how learners' algebra difficulties may be resolve. These are shown in table 3:

Table 3: Suggestions on addressing issues with algebra:

Suggestions	Years applicable
The skills learnt in earlier grades should be revised from time to time. These skills are essential to solving many questions in Grade 12. Learners must be reminded to use the information sheet for the correct formula	2019 - 2022
A more thorough teaching of factorisation in Grades 9 and 10 is needed. There should an emphasis on how to identify the type of factorisation that is applicable to the	2019, 2020

given expression. Also, writing a quadratic equation in standard form means to make the RHS of the equation equal to 0	
In teaching inequalities, integrate algebra with functions so that learners have a visual understanding of inequalities. The meaning of the inequality signs in the teaching of both algebra and functions, should be emphasised. Teachers should demonstrate different methods to solve inequality problems so that learners can choose the method they understand best	2019 - 2022
When solving surd equations, learners should be reminded that both sides of the equation should be squared in order to maintain the balance; not only the radical parts of the equation. Teachers must emphasise that implicit restrictions are placed on surd equations and learners should continue to test whether their answers satisfy the original equation	2019 - 2022
When teaching quadratic number patterns, it is essential to show learners how the formulae: $T_1 = a + b + c$, the first term of the first differences $= 3a + b$ and the second difference $= 2a$, are deduced	2020 - 2022

Despite these suggestions given on addressing algebra issues in examination of grade 12 Mathematics Paper 1, these issues remain, based on the National Diagnostic reports for Mathematics.

Research Design and Methodology

Algebra is a challenge to learners, internationally and in South Africa. Despite measures to address these challenges in South Africa through suggestions in the National Diagnostic Reports, these challenges remain.

In this paper, the writer traces the foundational or background knowledge that learners need to respond to questions involving Algebra in grade 12 examinations in South Africa. The structure of the examination papers has remained the same over the years (DBE, 2021). Thus, for purposes of this study, the writer examines the questions of one paper, Mathematics Paper 1 from November 2023.

For each question, a possible solution is given and the background knowledge to attempt the question is outlined. The writer uses the Mathematics Paper 1 review completed by AMESA

at the end of 2023 to indicate the level of difficulty of the questions. Thus, this research could be classified as descriptive research (Manjunatha, 2019) as it describes the “what” of the algebra assessment such as the solutions and background knowledge rather than the “why”.

Theoretical framework

There are two key elements which are considered in locating this paper within a theoretical framework, the spiral curriculum, as applied to Mathematics, and mathematical knowledge for teaching. The spiral curriculum is a form of learning that encourages the revising of topics and key concepts, building on previous knowledge in a cyclical and spiralling manner. This approach

enables students to gain a deeper understanding of fundamental principles and ensuring that they are regularly exposed to subject matter at different levels of complexity. This is very effective when teaching abstract mathematics concepts within algebra and other topics (Main, 2022).

In the Senior Phase in South Africa (DBE, 2011), the allocation of topics in grades 7, 8 and 9 for Algebra is done according to the spiral curriculum approach with fewer topics, mostly at the basic level in grade 7. This increases both in number and complexity in grades 8 and 9 and allows for revision of topics from the previous grades. Table 4 shows how this approach is used in the solution of equations. The table is self-explanatory.

Table 4: Equations in grades 7 to 9

Grade 7	Grade 8	Grade 9
Solve and complete number sentences by inspection; trial and improvement (no mention of equation; number sentence is used instead)	- Solve equations by inspection - Extend solving equations to include using additive and multiplicative inverses; using laws of exponents	- Solve equations by inspection using additive and multiplicative inverses; using laws of exponents - Extend solving equations to include: using factorisation; equations of the form: product of factors = 0

The outline of content in the FET is mentioned earlier in this paper. It also follows a spiral format. In grade 10, there is consolidation of grade 9 content with additional topics introduced. Grade 11 builds on this with more topics introduced. Since the assessment in grade 12 is based on grades 11 and 12 work, just two topics are allocated in grade 12. However, while not specified, it is important for teachers to allocate time for the revision and consolidation of grade 11 topics

Mathematical Knowledge for Teaching, commonly referred to as MKT is defined by Ball, Thames and Phelps (2008) as a practice-based theory that encompasses the mathematical

knowledge needed to teach mathematics. This theory allows for the perspective, habits of mind, and sensibilities that are important in the pursuance of effective teaching of the content.

Mathematical Knowledge for teaching should be a priority in the training of pre-service teachers. Superfine and Li (2014) posit that mathematics teacher educators need to understand mathematical knowledge for teaching for themselves and should be knowledgeable about ways to connect preservice teachers' mathematical learning to the practice of teaching in schools. They propose a model for the professional development of teachers that provide opportunities for teacher educators to develop and refine their mathematical knowledge.

A study by Mosvold and Fauskanger (2016) of 15 teachers that investigated teachers' beliefs about mathematical knowledge for teaching students the definitions of mathematical terms, using content analysis, referred to these beliefs as epistemological. The majority of the teachers viewed knowledge of mathematical definitions to be an important aspect of MKT; but did not see why this is important for themselves. This led Mosvold and Fauskanger to suggest an extension of Beswick's (2012) three categories of teacher beliefs: beliefs about the nature of mathematics; beliefs about mathematics teaching; and beliefs about mathematics learning. They called this fourth category "beliefs about MKT" which consisted of three subcategories: remembering content, understanding content, and adjusting and differentiating.

Building on from the spiral nature of the mathematics curriculum, there is no doubt that mathematics teachers should be well-versed in the ideas of Mathematical Knowledge for teaching (MKT). MKT starts when teachers are being trained and should continue when they start teaching. In the context of this paper, teachers should be fully aware of the importance of having appropriate mathematical knowledge to teach algebra topics in grades 8 -12 using the spiral approach.

Thus, an appropriate framework in which to locate this paper is:

Mathematical knowledge for teaching Algebra using a spiral approach

Grade 12 Mathematics Paper 1 2023

Due to space constraints, only questions 1 and 2 of this paper are worked out. The mathematical techniques and approaches used are discussed and the foundational knowledge for such questions are outlined. There is also an indication of the cognitive level of the question.

Question 1

1.1 Solve for x:

1.1.1 $x^2 + x - 12 = 0$

Solution

In this question the left-hand-side is a trinomial right-hand side is 0; so learners need to factorise the left-hand side to form a product of 2 binomials. Since the right-hand side is zero, each binomial is equated to 0. This is foundational knowledge from grade 9 and 10.

$$\begin{aligned}x^2 + x - 12 &= 0 \\ \Rightarrow (x + 4)(x - 3) &= 0 \\ \Rightarrow x = -4 \text{ or } x &= 3\end{aligned}$$

Due to its basic requirements, this question was classified as a knowledge question (AMESA, 2023)

1.1.2 $3x^2 - 2x = 6$ (correct to TWO decimal places)

Solution

This question involves finding the solution to a quadratic equation. The stimulus to the question about two decimal places may indicate that, once in standard form, the quadratic portion, on the left-hand side does not factorise. The quadratic formula, introduced in grade 11, should be used. The writing of the quadratic equation in standard form, where the 0 is on the right-hand is first done in grade 9. Learners are usually introduced to decimals in the intermediate phase.

$$\begin{aligned}3x^2 - 2x &= 6 \\ \Rightarrow 3x^2 - 2x - 6 &= 0 \\ \Rightarrow x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} \\ \Rightarrow x &= \frac{2 \pm \sqrt{76}}{6} \\ \Rightarrow x &= 5,36 \text{ or } -3,36\end{aligned}$$

This question was classified as a routine procedures question (AMESA, 2023)

1.1.3 $\sqrt{2x+1} = x-1$

Solution

In this question, both sides of the equation are squared (see table 3). The terms are then arranged in standard form. The left-hand side is then factorised, using a common factor x , to yield two solutions. However, the solutions should be checked as the squaring of both sides is likely to increase the number of solutions. For $x = 0$, the left-hand side is 1 and the right-hand side is -1. So, we reject this solution. Only $x = 4$ is valid.

$$\begin{aligned}
\sqrt{2x+1} &= x-1 \\
\Rightarrow (\sqrt{2x+1})^2 &= (x-1)^2 \\
\Rightarrow 2x+1 &= x^2 - 2x + 1 \\
\Rightarrow x^2 - 4x &= 0 \\
\Rightarrow x(x-4) &= 0 \\
\Rightarrow x = 0 \text{ or } x = 4 \\
\Rightarrow x = 4 \text{ only}
\end{aligned}$$

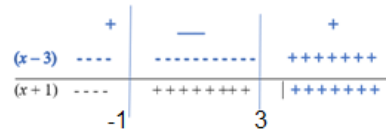
This question was classified as a routine procedures question (AMESA, 2023).

$$1.1.4 \quad x^2 - 3 > 2x$$

Solution

We start off in a similar manner when working with equations by moving all the terms to the left-hand side with 0 on the right-hand side. The left-hand side factorises, a type seen in grade 9 for the first time. When we equate these terms to 0, we get the critical values to the inequality, -1 and 3. We use these values on a number line to create 3 regions, the left of -1, between -1 and 3 and to the right of 3. We take values within each of these regions and substitute into the factors $(x + 1)$ and $(x - 3)$. We note to the left of -1, both are negative, and the product is positive; between -1 and 3 the product is negative and to the right of 3, the product is positive. The solution to this question is written in a particular way as shown below.

$$\begin{aligned}
x^2 - 3 &> 2x \\
\Rightarrow x^2 - 2x - 3 &> 0 \\
\Rightarrow (x+1)(x-3) &> 0 \\
\Rightarrow x < -1 \text{ or } x > 3 \\
\text{Solution set } \{x: x < -1\} \cup \{x: x > 3\}
\end{aligned}$$



This question was classified as one involving routine procedures, where basic factorisation was needed, and complex procedures where the critical values are used to divide the number line into 3 regions and the sign of the product checked in each region (AMESA, 2023).

Question 1.2 Solve x and y simultaneously

$$x + 2 = 2y \quad \text{and} \quad \frac{1}{x} + \frac{1}{y} = 1$$

Solution

In a question of this nature, the linear equation is expressed in terms of one variable; in this one x is expressed in terms of y . This involves changing the subject of the formula which is

introduced to learners in grade 9. For the other expression, we find the LCM and then multiply both sides by the LCM to get a form of the equation which is easier to work with. We substitute for x in this equation, to get a quadratic equation in y , which factorises. This type of factorisation is done in grade 9 and 10. In this question, first solve for y , and then get the corresponding x values. The final solution is written in a particular way.

$$x + 2 = 2y \text{ (A)} \quad \frac{1}{x} + \frac{1}{y} = 1 \text{ (B)}$$

From (A) we get $x = 2y - 2$

$$\text{From (B) we get } \frac{1}{x} + \frac{1}{y} = 1 \Rightarrow \frac{y + x}{xy} = 1$$

So $y + x = xy$ (C)

Now substitute for x in (C)

We have:

$$y + 2y - 2 = (2y - 1)(y)$$

$$\Rightarrow 3y - 2 = 2y^2 - 2y$$

$$\Rightarrow 2y^2 - 5y + 2 = 0$$

$$\Rightarrow (2y - 1)(y - 2) = 0$$

$$\Rightarrow y = \frac{1}{2} \text{ or } y = 2 \text{ and } x = -1 \text{ or } 2$$

$$\text{Soulution set } \left\{ (x; y) : \left(-1; \frac{1}{2} \right); (2; 2) \right\}$$

This question was classified as a combination of routine and complex procedures, due to the unusual nature of the second equation. It also required a bit of algebraic manipulation (AMESA, 2023).

1.3 Given $2^{m+1} + 2^m = 3^{n+2} - 3^n$ where m and n are integers. Determine the value of $m + n$

Solution

This was an unusual question which required common factorisation, involving exponents, on both sides of the equation. The question is then rewritten with fractions on both sides, with the unknowns on the left- hand side. This involves some algebraic manipulation, using division. The numerator of the right-hand side is expressed as a power of 2; the denominator of the right-hand side is expressed as a power of 2. We are now able to solve for m and n as shown to eventually obtain the sum of m and n .

$$\begin{aligned}
2^{m+1} + 2^m &= 3^{n+2} - 3^n \\
\Rightarrow 2^m(2+1) &= 3^n(3^2-1) \\
\Rightarrow 2^m(3) &= 3^n(8) \\
\Rightarrow \frac{2^m}{3^n} &= \frac{8}{3} = \frac{2^3}{3^1} \\
\Rightarrow 2^m &= 2^3 \text{ and } 3^n = 3^1 \\
\Rightarrow m &= 3 \text{ or } n = 1 \\
\Rightarrow m+n &= 4
\end{aligned}$$

This question was classified as problem solving as it was an exponential equation in two unknowns and involved quite a bit of algebraic manipulation (AMESA, 2023).

Question 2

2.1 Given the arithmetic series: $7 + 12 + 17 + \dots$

2.1.1 Determine the value of T_{91}

Solution

In this question, the first term and common difference are known. To find the 91st term, we just need to substitute these values in the n th term of an arithmetic series as shown below. As stated earlier, learners can substitute in formulas in grade 8 and 9. They should know that $T_n = a + (n-1)d$ so they can do the necessary substitution as shown.

$$\begin{aligned}
T_n &= 7 + (n-1)5 = 5n + 2 \\
T_{91} &= 5(91) + 2 = 457
\end{aligned}$$

This question was classified as a knowledge question as it was just substitution in a known formula (AMESA, 2023).

2.1.2 Calculate S_{91}

Solution

This question can be done in two ways. In both cases it involved substitution in a known formula. In the first case, we know the number of terms, the first term and the last term. In the second cases, we know the number of terms, first term and common difference.

$$\begin{aligned}
S_n &= \frac{n}{2}[a + l] \\
S_{91} &= \frac{91}{2}[7 + 457] = 21112
\end{aligned}$$

OR

$$\begin{aligned}
S_n &= \frac{n}{2} [2a + (n-1)d] \\
S_{91} &= \frac{91}{2} [2(7) + 90(5)] \\
&= \frac{91}{2} [14 + 450] \\
&= 21112
\end{aligned}$$

This question was classified as a knowledge question as it was just substitution in a known formula (AMESA, 2023)

2.1.3 Calculate the value of n for which $T_n = 517$

Solution

Here we are given a specific term of the arithmetic series. We must determine which term it is. To solve this question, we substitute the values of a , d and T_n in the general term $T_n = a + (n-1)d$. To determine the value of n , we do a bit of algebraic manipulation so that n is on the left-hand side, and we solve for n :

$$\begin{aligned}
T_n &= 517 \\
\text{We know that } T_n &= a + (n-1)d \\
\text{Now } a &= 7; d = 5 \\
\Rightarrow 7 + (n-1)5 &= 517 \\
\Rightarrow (n-1)5 &= 510 \\
\Rightarrow n-1 &= 102 \\
\Rightarrow n &= 103
\end{aligned}$$

This question involved substitution into a known formula and using some algebraic processes to obtain the solution. This question was classified as a routine procedures question (AMESA, 2023).

2.2 The following information is given about a quadratic number pattern:

$$T_1 = 3; T_2 - T_1 = 9; T_3 - T_2 = 21$$

2.2.1 Show that $T_5 = 111$

Solution

This question involved working with the given information to find the first three terms of the series. Then, using these three terms, we use the second difference, which is common, and the first differences, which are different, to obtain the fourth term as 66 and the fifth term as 111. The mark allocation for this question was 2 marks which may indicate that the question did not require as many steps as shown here. However, it would be fair to say that the question involves a bit of analytical thinking and algebraic manipulation.

$T_1 = 3; T_2 - T_1 = 9; T_3 - T_2 = 21$
 $T_2 - T_1 = 9 \Rightarrow T_2 - 3 = 9 \Rightarrow T_2 = 12$
 $T_3 - T_2 = 21 \Rightarrow T_3 - 12 = 21 \Rightarrow T_3 = 33$
 The sequence is: 3; 12; 33;....

So, we have: 3; 12; 33; T_4 ; T_5

First difference: 9; 21; a; b

Second difference: 12; 12; 12; 12;

So, $a = 21 + 12 = 33$ and $b = 33 + 12 = 45$

Thus, $T_4 = 33 + 33 = 66$ and $T_5 = 45 + 66 = 111$

The question was classified as a routine procedures question as teachers involved in the AMESA review probably knew this shorter method and could have explained this method to their learners (AMESA, 2023).

2.2.2 Show that the general term of the quadratic pattern is $T_n = 6n^2 - 9n + 6$

Solution

This question requires knowledge of the n th term of the quadratic equation and how to use the first and second differences to obtain the values of a , b and c in the general form of the equation. It involves some algebraic manipulation to get three key terms

The second common difference which is $2a$; the first common difference ($T_2 - T_1$) which is $3a + b$; and the first term which is $a + b + c$.

These three key terms are also shown in table 3.

General form of quadratic sequence: $T_n = an^2 + bn + c$

The first three terms are:

$$a + b + c; 4a + 2b + c; 9a + 3b + c$$

1st diff: $3a + b; 5a + b$

2nd diff: $2a$

We apply this to the given sequence:

We have $2a = 12$ so $a = 6$

$$3a + b = 9 \Rightarrow b = 9 - 3(6) = -9$$

Now $a + b + c = 3$

$$\Rightarrow c = 3 - a - b = 3 - 6 + 9 = 6$$

Thus, $T_n = 6n^2 - 9n + 6$

This type of question has become commonplace in grade 12 examinations. As such it was classified as a routine procedures question (AMESA, 2023).

2.2.3 Show that the pattern is increasing for all $n \in N$

Solution

This question can be worked in two ways, in the first way we list the terms of the sequence and observe that it is increasing:

3; 12; 33; 66; 111;

We note that the minimum term of the sequence is 3 and the sequence is increasing for all $n \in N$

In the second way, we differentiate the general term and substitute $n = 1; 2; 3; \dots$

$$T_n = 6n^2 - 9n + 6$$

$$\Rightarrow (T_n)' = 12n - 9$$

Now $12n - 9 > 0$, for $n = 1; 2; 3; \dots$

Thus, the quadratic pattern is always increasing

This question was classified as a complex procedures question (AMESA, 2023) as, in the first solution, it depended on the given information and the calculation of the fifth term to establish, via inductive reasoning, that the sequence was increasing. In the second solution, learners had to bring in knowledge from another section of the syllabus, that of differential calculus. Since the derivative is positive for all $n \in N$, it can be deduced that the function is increasing.

Findings

The interrogation of question 1 and question 2 of the Mathematics Paper 1 from November 2023 has resulted in some interesting findings which are now discussed within the context of the research sub-questions for this paper:

- **Substitution in formulae:** Learners are first introduced to substitution in the primary school where they must identify the number or value that satisfies certain number sentences. In the senior phase they encounter substitution in expressions as early as grade 7 and this continues in various forms until grade 12. In grade 12 learners have to substitute in given formulae which are relevant for the given question. In question 2.1 they had to select the appropriate formulae from the Information Sheet for the n th term and the sum to n terms of an arithmetic series and use these to solve all three parts of question 2.1.
- **Like and unlike terms:** Once learners are introduced to algebra in grade 7 and 8, they work with like and unlike terms on a continuous basis. In the mathematics paper they worked with like and unlike terms in questions 1.1.3 and 1.2 where the like terms were added to ensure there was a trinomial on the left-hand side of the quadratic equation.

- **Factorisation:** In the questions analysed, factorisation occurred in questions 1.1.1, 1.1.3 and 1.2. It was factorisation of a trinomial in questions 1.1.1 and 1.2 and using a common factor in question 1.1.3. Learners are exposed to factorising using a common factor in grade 8 and factorisation of trinomials in grade 9. In grade 9, they are also shown that if a product of different factors is equal to 0, then each factor is equal to 0.
- **Surd equations:** Surd equations are usually done for the first time in grade 10 or 11. Learners must rearrange the terms such that the surd part is on one side and the non-surd part is on the other side. In this way, both sides can be squared, the terms then rearranged to be on one side to form a trinomial with 0 on the other side. In question 1.1.3 of the paper analysed, two solutions emerged; but one was rejected after substitution.
- **Quadratic inequalities:** In the primary school learners are introduced to “greater than”; “less than” and “equal to” in a number context when using the four operations. This is taken further in an algebraic context with linear inequalities in grade 10 and quadratic inequalities in grade 11. As observed in the solution for 1.1.4, the key to solving a quadratic inequality is to write the left-hand side in standard form, then equate the left-hand side to 0 to get the critical values of the solution. These values are placed on the number line to form three regions; numbers are chosen in each region and substituted into the factors; these are then multiplied to check whether the region is positive or negative. In question 1.1.4, the solution to the inequality was the two extreme regions which were > 0 . This solution is then written in a unique way.
- **Simultaneous equations:** Simultaneous equations, one linear and one quadratic, are an integral part of Mathematics Paper 1. This builds on what is learnt in grade 10 when learners work simultaneously with two linear equations which may be solved algebraically or graphically. In question 1.2 of the mathematics paper analysed, candidates had to express one variable in terms of the other and substitute in a simplified version of the other equation. This resulted in a quadratic equation in y which was solved; and x values were obtained via substitution in the linear equation.
- **Quadratic number patterns:** When working number patterns which are quadratic in nature, it is important to note that the first differences are not the same; there is a common second difference. In question 2.2.1 there had to be some manipulation of the given terms to determine the first 3 terms of the quadratic sequence. This involved techniques which learners may encounter in grade 8 and 9. To determine the fourth and fifth term, one had to use the common second difference to work out the next two first differences and then T_4 and T_5 . Then the second common difference a , the first difference $3a + b$ and the first term of the quadratic sequence of $a + b + c$ are used with the given information to arrive at the general term of the quadratic pattern $T_n = 6n^2 - 9n + 6$. This is done using simple simultaneous linear equations in a , b and c . These are usually done in grade 10.

Conclusion

This paper examined the type of algebra and algebra-related questions which appear in the grade 12 Mathematics Paper 1 and traced the foundational knowledge required to attempt these questions. This foundational knowledge is acquired by learners in the earlier grades with some work being done in primary school level.

This paper has shown that teaching of algebra and other related topics across the school grades should be according to the spiral curriculum approach where learners acquire basic knowledge first before going on to more demanding work in the grades that follow. We are fortunate, in South Africa, that the mathematics curriculum is structured in this manner and that teachers should use the curriculum rather than textbooks when planning their lessons. However, textbooks and ICT resources are useful in aiding lesson planning and preparation.

Teachers should also be fully conversant with the mathematics content in the various to make choices about the activities that should take place in their classrooms to ensure that learners acquire the necessary mathematical skills and knowledge. In this regard, the importance of Mathematical Knowledge for Teaching (MKT) cannot be overemphasised. Finally, the approach used in this study may be applied to other paper 1 topics such as functions and graphs, differential calculus, finance and growth and decay. It may also be relevant to some paper 2 topics.

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PERCEPTIONS OF MATHEMATICS TEACHERS ON DEVELOPING TPACK THROUGH LESSON PLANNING INTEGRATING TECHNOLOGY IN OSHANA REGION, NAMIBIA.

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The Namibian current curriculum underscores the importance of digital technologies in improving students' grasp of mathematical concepts, reasoning, and communication skills. This requires teachers to understand technological pedagogical content knowledge (TPACK) to effectively integrate technology into their teaching, as technology alone does not guarantee better outcomes. Therefore, this intervention case study examines Mathematics teachers' perceptions of developing their TPACK while planning technology-integrated lessons. The study involved six purposively selected Grade 9 Mathematics teachers from three Secondary Schools in Oshana region, Namibia. Data collection methods included questionnaires, lesson plan analysis, and semi-structured interviews. Mishra and' Kochler's (2006) TPACK theory informed the study, using an interpretive and pragmatic paradigm. The study revealed that Mathematics teachers positively perceived TPACK development through planning algebra lessons that integrate technology. They gained a deepened understanding of TPACK and boosted their confidence, fostering greater technology integration in their future teaching. The study suggests further research on implementing planned lessons in mathematics classrooms.

Keywords: Curriculum, Technology integration, mathematics teachers and TPACK.

Introduction

The rapid advancement and increasing ubiquity of technology in society has led to growing expectations for its integration into public school classrooms. Scholars continue to advocate for using digital technologies in mathematics education, viewing them as vital resources (Henriques & da Silva, 2022). They aid students in understanding mathematical concepts, enhancing their reasoning skills, and effectively communicating their thoughts (Ziatdinov & Valles, 2022).

Consequently, the 21st-century classroom demands that teachers possess a profound understanding not only of the mathematical content they teach but also of the available technological tools and their potential for exploring mathematical ideas, as well as the pedagogical aspects related to teaching and learning mathematics with technology (Goos et al., 2020). In response to these demands, mathematics teachers must develop and acquire a specialised form of knowledge known as technological pedagogical content knowledge (TPACK) (Koehler & Mishra, 2006). TPACK represents the interplay of three foundational knowledge bases: content knowledge (CK), pedagogical knowledge (PK), and technological knowledge (TK). These knowledge domains must be synergistically combined to facilitate effective technology integration in mathematics instruction (Koehler & Mishra, 2009).

In today's technology-driven era, having a good understanding of mathematics is incredibly important. It's not just useful for everyday life, but it also helps in developing critical thinking skills (Maass, Geiger, Ariza, & Goos, 2019; Iithete & Hiloshi, 2021). However, in Namibia, many learners find topic like Algebra, a part of mathematics, quite tough to grasp (Hamukwaya & Haser, 2021; Chirimbana, Nzwala, & Martin, 2023). One reason for this is the increased pressure brought about by the recent changes in the curriculum. As noted by Bezuidenhout (2021, p. 2), "advanced algebra concepts are introduced quite early, in grade 9, which is a critical point before entering the secondary phase." In contrast to the previous curriculum, the new Namibian curriculum mandates students to study mathematics from grade 8 to 11.

As attempt to help teachers and learners cope with the demands of the new curriculum, the Ministry of Education organized workshops to introduce teachers to some of the more challenging topics in mathematics for grades 8 to 12 (Angula, 2019; Josua, 2022). Furthermore, similar workshops have been arranged to introduce advanced teaching methods in other subjects. Amidst the challenges faced by the current education system in most developing countries such as Namibia, Mathematics teachers acknowledges that the integration of modern technology in learning mathematics offers the potential to create engaging, relevant, and personalised learning experiences for all students, thereby meeting educational objectives more effectively (Kanandjebo, & Lampen, 2022). Nevertheless, to master complex mathematic concepts such as algebra, learners require engagement in effective teaching and learning activities (Nicol, 2021). Thus, integrating technology into mathematics education could offer promising avenues for designing exciting and engaging lessons (Freiman & Tassell, 2018) that help students overcome learning obstacles through practical activities rather than rote memorisation.

Consequently, the Ministry of Education in Namibia has implemented various initiatives to promote technology integration in education. These include implementing the ICT policy for Education across all subjects (Ngololo, 2010), the Digital Skills Foundation training (Tjijombo, 2022), and mathematics advisory teacher in the Oshana region has facilitated workshops focusing on building a technological knowledge base. However, despite initiatives by the Namibian government to equip schools with technology and provide teacher training workshops, significant challenges remain in leveraging technology to enhance mathematics instruction. Many mathematics teachers lack the knowledge to select appropriate technology and effectively integrate it into their teaching practices (Kanandjebo, 2022). Some teachers may use technology merely because it is available, resulting in superficial or inconsistent integration within their lessons (Miller, 2019). Developing teachers' Technological Pedagogical Content Knowledge (PACK) is imperative to address these challenges and design practical mathematics lessons that integrate technology (Kambeyo, 2018; Ashipala, 2021).

Literature suggests that technology provides pedagogical affordances that support practical mathematics learning and promote conceptual understanding and critical thinking (Bray & Tangney, 2016; Perienen, 2020). However, the successful integration of technology into mathematics instruction necessitates appropriate planning at the intersection of standard-based curriculum requirements, effective pedagogical practices, and technology's affordances and constraints (Harris & Hofer, 2011). Therefore, this paper investigates grade 9 teachers'

perceptions of their self-reported Technological Pedagogical Content Knowledge (PACK) development through an intervention study focusing on planning lessons integrating technology in mathematics.

This paper commenced with a literature review of the subject matter to address this study's research question and objective. Subsequently, the researcher introduced a conceptual framework to steer the study. The research approach and data-gathering procedures were then presented, followed by the exposition and analysis of the research findings. Lastly, the researcher concluded the paper by recapitulating the results and proposing suggestions for future research.

History of TPACK

Over the years, a significant emphasis has been placed on teachers' subject matter knowledge, as it is thought that well-prepared teachers can positively influence student learning. As a result, considerable research has concentrated on teachers' pedagogical knowledge. For example, Shulman (2015) underlined the significance of subject matter knowledge and introduced the notion of pedagogical content knowledge (PCK). Shulman's concept posits that teacher knowledge consists of at least seven categories: (a) content knowledge; (b) pedagogical knowledge; (c) curriculum knowledge; (d) knowledge of learners and learning; (e) knowledge of educational contexts; (f) pedagogical content knowledge; and (g) knowledge of educational philosophies, goals, and objectives (Shulman, 1987, p.8). In essence, pedagogical content knowledge is a combination of content knowledge and pedagogical knowledge, incorporating content and pedagogy and their interaction, including aspects such as representations, misconceptions, and teaching knowledge (Mishra and Koehler, 2006).

Although some teachers utilised traditional technologies in educational environments historically, Shulman (2015) did not include a technology knowledge component in the concept of PCK in the 21st century. Researchers have begun incorporating technology into the various aspects of teachers' knowledge. Mishra and Koehler (2008) and Koehler and Mishra (2006) emphasised the importance of technology knowledge in the context of teaching. As various digital technologies continue to increase, integrating technology into educational settings has become increasingly vital. Integrating technology requires teachers to utilise and incorporate technology into their teaching effectively. However, frameworks that emphasise technology have arisen, with technology becoming more integral to teaching and learning. Consequently, Mishra and Koehler (2006) proposed the TPACK framework (previously referred to as TPCK), which has since become a central area of research related to the use of technology in an educational context.

Mathematics Teachers TPACK Development.

Over a decade since its introduction, there has been a notable increase in methods to support in-service teachers in developing their Technological Pedagogical Content Knowledge (TPACK) (Doukakis et al., 2021; Bas & Senturk, 2018; Agyei & Voogt, 2012). Nonetheless, only a handful of studies have involved in-service teachers in design-based science and mathematics education activities using various approaches (Kafyulilo et al., 2015; Njiku,

Mutarutinya & Maniraho, 2021). In the realm of professional development within initial teacher education, lesson design activities have also been shown to improve preservice teachers' TPACK (Agyei & Voogt, 2015).

Moreover, Harris and Hofer (2011) introduced an approach to effectively assist teachers in integrating technology into their teaching methodologies. This strategy connects learners' learning needs with specific subject-related tasks, pinpointing suitable technological tools to enhance lesson delivery. While contemplating integrating technology into mathematics education, Collins, Brown, and Newman (2018) underscore the importance of instructors considering the subject matter, teaching techniques, and chosen technological resources. However, this study's central focus is developing the essential expertise necessary for seamlessly incorporating technology into mathematics instruction. It is worth noting that only a restricted number of investigations have involved active teachers, as evidenced by Alemdag et al. (2020), who worked with in-service educators, and Kafyulilo et al. (2015), who engaged preservice teachers in design-based activities for science and math educators. For instance, Alemdag et al. (2020) observed that collaborative, hands-on tasks significantly fostered teachers' technological and content-related knowledge (TPACK). Similarly, within the context of initial teacher training, lesson design exercises have proven effective in cultivating preservice teachers' TPACK (Lee & Kim, 2014; Kafyulilo et al., 2015; Paskins, 2023).

Several researchers propose specific strategies to support teachers in developing TPACK, including collaborative learning-by-design (Koehler & Mishra, 2006; Harris & Hofer, 2011), instructional systems design (Lee, & Kim, 2014), and collaborative reflection-upon-practice (Kearney, Pressick-Kilborn, & Maher, 2012). Likewise, Hernawati (2019) applied the TPACK framework to examine the specialised knowledge mathematics teachers need to enable effective learning. Similarly, Dalal, Archambault & Shelton (2017) used the TPACK framework to investigate factors influencing technology integration among teachers, discovering that teachers' technological knowledge and institutional backing significantly contributed to TPACK development. On the other hand, Voithofer et al. (2019) found that factors related to teaching, curriculum, and organisational support affect teachers' TPACK and technology integration more broadly. Despite these endeavours, most research was carried out in Western countries. However, there is evidence that teachers in many developing countries like Namibia are not effectively integrating technology into mathematics instruction (Ngeama, 2018). This evidence highlights the need for initiatives to enhance teachers' technological content knowledge, especially when planning lessons that include technology.

Theoretical Framework

Technological Pedagogical Content Knowledge (TPACK).

The TPACK framework comprises seven domains that represent teacher knowledge, including content knowledge (CK), pedagogical knowledge (PK), technological knowledge (TK), pedagogical content knowledge (PCK), technological content knowledge (TCK), technological pedagogical knowledge (TPK), and technological pedagogical content knowledge (TPACK). Koehler and Mishra (2008) explained each knowledge domain, as demonstrated in **Figure 1**.

Figure 1 demonstrates that the TPACK framework consists of three interconnected knowledge bases that influence each other reciprocally. To successfully integrate TPACK, teachers and practitioners must understand the knowledge domains and context and apply them appropriately. For example, the subject matter (content) may determine which technologies to use when teaching a specific discipline. Advanced computing technologies can benefit learning disciplines like mathematics, which rely heavily on visualisation and graphical representation (Mishra & Koehler, 2008). Education is an ill-structured phenomenon, making it challenging to adopt a single, universal educational technology integration model (Mishra & Koehler, 2006). Since different content areas necessitate distinct pedagogical approaches and technology utilisation, it is crucial to ensure that mathematics teachers develop TPACK while planning technology-integrated lessons. As the TPACK framework offers a clear description of technology in the learning of secondary school mathematics teachers, it is likely to produce the desired TPACK outcomes for them. Consequently, the TPACK framework served as the conceptual framework guiding this study.

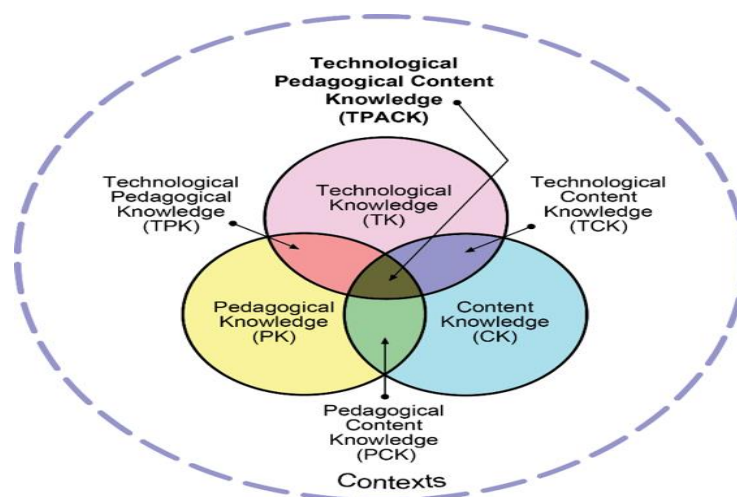


Figure 1 Components of technological pedagogical content knowledge

TPACK, as a conceptual framework, acknowledges the interplay between various teacher knowledge bases while highlighting the importance of context for secondary school Mathematics teachers (Lee & Kim, 2014; Hamukwaya & Haser, 2021; Kafyulilo et al., 2015). Context is a crucial yet often overlooked component of the TPACK framework. Researchers have employed different elements to describe context, such as subject matter, grade level, student background, and available technologies (Mishra & Koehler, 2006; p. 1032), the design of the learning environment and its characteristics (Papadakis, Kalogiannakis & Zaranis, 2021), and the classroom layout, available support elements in the school, and societal conditions like state and national standards (Hamukwaya & Haser, 2021). These various contextual factors may enable researchers or mathematics teachers to elucidate how and to what extent TPACK

develops. Depending on the unique context of diverse teaching and learning situations, these factors may help comprehend how TPACK evolves within those environments.

This study employed TPACK as a theoretical framework for integrating technology into the planning of teaching and learning process. This framework plays a crucial role in forming an essential part of teaching, specifically in lesson preparation. Subsequently, it becomes of significant interest to comprehend the cognitive process involved in instructional planning. However, this research is fundamental to understanding the TPACK development in planning instructions incorporating technology.

Methodology

The study adopted a qualitative case study. A case study is a research method used to understand and provide insights into a specific phenomenon in a real-world setting (Creswell, 2012). In this study, the case under examination was the participants' perceptions regarding developing TPACK before and after participating in the workshop based on planning mathematics lessons that integrate technology. Therefore, it was appropriate to adopt the interpretive paradigm (Thanh & Thanh, 2015), supported by the pragmatic paradigm (Clarke & Visser, 2019), as our goal was to practically describe and interpret the world's phenomena to obtain shared meaning with others.

The research study involved a purposive sample (Creswell, 2012) of six mathematics teachers chosen from three public secondary schools in the Oshana region in Namibia. Four participants (T3, T1, T4, and T6) had more than five years of experience teaching mathematics at the secondary school level and had attended one technology workshop in the circuit. The remaining two participants (T2 and T5) had less than five years of experience teaching mathematics and science at the secondary level and had attended three technology workshops. Table 1 offers an overview of the participants' characteristics.

Table 1. Participants characteristics

Participant	Gender	Years experience	of Technology access
T1	F	12	Yes
T2	M	3	Yes
T3	F	7	Yes
T4	F	9	Yes
T5	M	4	Yes
T6	F	13	Yes

The intervention's initial phase lasted four weeks. For this study, the researcher investigated secondary school Mathematics teachers' perceptions of their TPACK development in a single

case through multiple data sources such as semi-structured questionnaires, semi-structured interviews, and lesson plans. Moreover, during the study, the data were collected from a single case to deeply understand how the participants perceive their TPACK development when planning algebra lessons that integrate technology. The first author originally designed a semi-structured questionnaire separated into three sections. This tool was then reviewed and refined by a second author. The questionnaires were administered before the first workshop. It included questions regarding the participants' teaching experience, their previous attendance at training sessions, their access to technology, and an open-ended query prompting them to articulate how they integrated technology into their teaching before the workshop. This intervention study explored the secondary school Mathematics teachers' perception of developing their TPACK before and after planning algebra lessons that integrated technology. This data answers the following research question: How do secondary school Mathematics teachers perceive their TPACK development before and after participating in the workshop on planning algebra lessons that integrate technology?

Organisation of workshop

The workshop was planned in this study to offer learning of study activities through social interactions so that participants would be afforded an opportunity to develop their own interpretations, challenge their initial understanding and grow as critical thinkers. The study conducted three workshops. In the first workshop, Mathematic teachers share their views and technological knowledge regarding the educational technologies that could be integrated into algebra lessons.

All six Mathematics teachers, one ICT expert and one mathematics expert (subject advisory teacher) were invited to attend the face-to-face workshop. The first workshop took place on Friday, April 2023, after school hours at one school in town. The participants agreed upon the date and the venue. During the workshop, Mathematics teachers sensitised each other about the possible technologies that can be integrated into mathematics lessons. It is worth noting that the Mathematic teachers possessed technological knowledge at some point because they attended the digital skills training offered by the region. After introducing the constructs of TPACK and the training on planning algebra lessons that integrate technology, Mathematics teachers demonstrated their lesson planning in groups, which were discussed for necessary improvements. The second and third workshops took place online and served to give additional development and lesson planning support where Mathematics teachers did individual algebra lesson planning considering their own context; their lessons were discussed and commented on for necessary improvements.

After the workshop, Mathematics teachers were asked to provide three lesson plans within four weeks after the third workshop. Yet, semi-structured interviews were conducted to gain insight into Mathematic teachers' perception of TPACK development after participating in planning lessons that incorporate technology. These interviews, comprising six participants, were audio-recorded to ensure accuracy. The interview questions drew insights from Harris & Hofer's (2009) and Mishra & Kohler's (2006) TPACK framework. The researcher conducted

personal, face-to-face interviews to ensure the reliability of the data collected. Finally, an analysis of the three lesson plans provided by each participant was undertaken, the results of which are presented in this paper.

Data analysis Procedure

The researchers followed Maguire & Delahunt's (2017) thematic analysis guidelines. The first step involved becoming familiar with the data by reading and re-reading the interview transcripts and lesson plans. The second step involved generating initial codes and identifying and labelling different ideas and concepts from the data. These codes were then collated and organised into potential themes. In the third step, the researchers searched for themes by reviewing the collated codes and looking for patterns and connections among them. The fourth step involved reviewing and refining the themes by checking for internal consistency and coherence and ensuring they accurately reflected the data. The fifth step involved defining and naming the final themes and subthemes that emerged from the data. Finally, the researchers wrote up the themes and subthemes coherently and meaningfully, drawing on examples from the data to illustrate each one. The result was a set of themes that captured the essential findings and insights from the data related to the development of TPACK among secondary Mathematics teachers through planning lessons that integrate technology.

As a qualitative study, questionnaires, semi-structured interviews and lesson plan data sources were utilised to ensure the reliability and robustness of the findings, which served as a triangulation method by providing corroborating evidence. Furthermore, the participants performed a member check by examining their case descriptions and analyses to ensure the correctness of interpretations. The use of rich, descriptive quotes from the participants further added to the credibility of the findings by allowing the readers to engage with the participants' responses and experiences directly.

Findings from the Study

The primary data used for this research came from open-ended questionnaires and semi-structure interviews. Table 2 shows a breakdown of the participant's responses to the questions in the questionnaire. This step began with six participants' results describing their lesson planning before the intervention and competencies in the questionnaire aligned with TPACK framework. Items 1 was about Mathematics teachers' content knowledge (CK); items 2 was about technological knowledge (TK); items 3 & 5 were about pedagogical content knowledge (PCK); and item 4 was about technological content knowledge (TCK).

Table 2. Results of the teacher skills with frequencies of Mathematics teachers' responses.

Criteria	When choosing technology:	Strongly agree	Agree	Not sure	Strongly disagree
1. Curriculum goals Technologies use (CK)	I consider curriculum goals for my lesson.	0	1	3	2
	Sometimes, the content may affect my technology choice.	1	0	2	3
	Sometimes, the available technology will determine the content to teach.	0	2	2	2
2. Technology Selection versus classroom and school contexts (TK)	I consider my classroom context, school context and availability of tools.	0	1	2	3
	Sometimes, the availability of tools at school determines my technology integration.	0	1	3	2
	Sometimes, the learner's learning needs affect my technology integration.	0	0	2	3
3. Select activity types to combine and sequence. (PCK)	I consider taxonomy activity types to combine and sequence my lesson.	0	0	1	5
	I consider the taxonomy stipulated in the syllabus when choosing technology	0	1	3	2
	Knowing the taxonomy percentage for the grade I teach is essential.	0	0	3	3
4. Select assessment strategies (TCK)	I consider assessment policy requirements on assessment.	1	0	4	1
	The assessment strategy determines the technology to use.	0	1	1	4
	I may not integrate technology when it does not seem to afford	2	1	2	1

5. Selection of technology (PCK)	me to achieve the lesson objective.				
	My choice of technology is affected by the content to teach and the teaching method.	0	1	2	2

Note: One participant did not answer the last question of the questionnaire.

Most mathematics teachers described their planning of lessons integrating technology before participating in the first workshop by making choices as **Strongly agree, agree, not sure and strongly disagree**. Most statements in the table were correct; hence, a strongly agreed response was regarded as a strong component of the lesson description. While strongly disagreeing was regarded as a weak lesson description. However, most Mathematics teachers described their lesson planning as very weak, ranging from being not sure to strongly disagreeing, as seen in **Table 2**.

The results are presented as themes in response to the guide research question: How do the Mathematics teachers perceive their TPACK development before and after planning algebra lessons that integrate technology in the Oshana region? The themes are supported by participants' quotes from semi-structured interviews and our analysis of lesson plans. The data analysis revealed that secondary school Mathematics teachers perceived their TPACK differently as their technological pedagogical content knowledge (TPACK) developed while planning mathematics lessons that incorporate technology. The teachers revealed how they think about their TPACK development after participating in the integrated lesson planning workshop as follows:

Perceived improvements in the application and utilisation of specific technologies

All the participants in this study indicated that participating in planning mathematics lessons that integrate technology offered them an opportunity to acquire new knowledge about applying specific technologies in teaching mathematics. By incorporating technology into their lesson plans, the teachers learned how to use particular software or tools that are suitable for teaching mathematics. The teachers were able to improve their understanding of the potential benefits and limitations of different technologies and to make informed decisions about their use in the classroom.

Some participants reported that the workshop exposed them to various technological tools and how they can be incorporated into education. Despite having taught mathematics for over five years and having access to technology, this was the first time they had planned mathematics lessons integrating technology, limiting their technological knowledge and making it challenging for them to operate specific technological tools. T5 and T4 account provides a clear example of their development of technological knowledge,

"I learned how to use different technology, not just YouTube. I now know how to use Kahoot and Gamification in my lessons, which makes them more interesting. Before, I didn't have much experience with using technology in teaching because this was my first time trying to integrate it". [T5 Interviews].

"Engaging in planning and comments has erupted from planned lessons during the intervention has awakened my mind. I realised that there are several issues that I have to consider before integrating any technological tool. I can see that I have been using technology closed-minded". [T4 Interviews].

Shifted the perceived knowledge on how to improve teaching outcomes by effectively incorporating technology

Most Mathematics teachers expressed that the workshop experience provided valuable insights into improving teaching effectiveness using technology. For example, T2 mentioned that while planning lessons integrating technology, she better understood how to use technology effectively. He intends to use two computers, one for displaying PowerPoint presentations and another for opening the Kahoot application. Furthermore, he expressed his desire to learn how to integrate two technological tools in one lesson for instance, Kahoot for assessment and YouTube video during lesson presentations.

The workshops offered a platform for some Mathematics teachers to enhance their comprehension regarding how pedagogy and technology interrelate and impact one another. The teachers developed their technological content knowledge through discussions and feedback on their technological tool choices (TCK). Additionally, this intervention helped participants gain insights on how to use the tool beyond information transmission, thus improving their confidence to incorporate technology than it was perceived. For instance, it was observed in T3 lesson plans that she aimed to explore learner-centred approaches when using technology in the classroom. Therefore, for assessment purposes, she planned to use Kahoot, and her technological knowledge informed her choice of assessment method. Furthermore, T4 lesson plans suggest that she intends to encourage learners' use of mobile technologies in the classroom. For T2, T3 and T5:

"While I was preparing for my technology-integrated lesson and discussing our lessons, I spent a significant amount of time suggesting various technologies and considering their potential impact on my teaching, especially since we discussed our lesson plans during the workshop". [T2-interview]

"Participating in planning lessons that integrate technology has helped me become more aware of how to present my lessons in an interesting way to my learners. This involves considering activities for both slow and fast learners. I have also explored many technologies that could help me achieve my lesson objectives. After careful consideration, I have gained the confidence to integrate technology into my future lessons." [T3-interview]

"Attending a lesson planning workshop has encouraged me to carefully consider my teaching approach and the technology that best aligns with my pedagogy and assessment requirements.

For example, when assessing my students during or after the lesson, I decided to incorporate algebra video clips from YouTube as an introduction to the topic. This is because learners need to remember the algebra rules from the previous grade. Videos are easy to operate, allowing me to pause, stop or forward to facilitate questioning." [T5-interview]

During the interviews, emotional responses were a frequent occurrence, which was found to be associated with the mathematics teachers' reported limited pre-technological knowledge. The possession of limited technological knowledge while preparing a lesson incorporating technology could lead to emotionally intense reactions: T4 expressed

"But, but aaa.... I sometimes undoubtedly feel quite annoyed and highly frustrated when I lack knowledge about technologies that my learners ask me to incorporate into my teaching. This feeling tends to happen throughout the process, eventually reaching a point where I become unwilling to persist and ultimately refrain from using technology in my lessons. [T4-Interviews].

Data from the interviews revealed that before the workshop, Mathematics teachers seemed to be confused between technological knowledge (TK) and (TPACK), therefore finding it difficult to balance technology, content and pedagogy when planning for technology integrated lessons.

Discussions of the findings

This research explored how participants perceived their development of Technological Pedagogical Content Knowledge (PACK) before and after planning algebra lessons that integrated technology in the Oshakati education circuit. Previous studies by Harris et al. (2009) and Hofer et al. (2011) suggested that TPACK could be accessed more effectively through interviews, prepared activities, and lesson plans rather than just using PACK scales, which only measure teachers' thoughts. Therefore, this paper analysed data from semi-structured questionnaires, semi-structured interviews, and lesson plans to better understand how mathematics teachers perceive their PACK development before and after participating in lesson planning workshops.

Preparing technology-integrated lessons has positively impacted Mathematics teachers' perception as they planned algebra lessons incorporating technology. Preparing lessons that integrate technology led to improvements in Mathematics teachers' Technological Knowledge (TK) and a deeper comprehension of how to use technology more effectively in their teaching by applying TPACK (Technological Pedagogical Knowledge (TPK). These findings align with Njiku et al.'s (2021) research, suggesting that the impact of technology on teaching depends on how teachers engage with it. Additionally, planning technology-integrated lessons helped teachers develop the skills to achieve their objectives through technology integration and boosted their confidence by enhancing their overall TPACK. The participants' lesson plan analysis revealed that collaborative and hands-on activities were crucial in improving teachers' TACK and understanding of TACK constructs. This is because Mathematics teachers described their lessons with a deeper understanding of TPACK after participating in the workshop. These findings aligned with research by Lee & Kim (2014), demonstrating that

teachers who used the TACK framework in lesson design activities significantly improved their technological competencies more than those who did not participate.

The findings from the interview indicate that Mathematics teachers have improved the description of their lessons after the intervention than before. The results revealed that Mathematics teachers planning lessons integrating technology considered the relationship among content , pedagogy, and technology after the intervention . Yet, teachers indicated that participating in the workshop focusing on technology integration improved their TPACK development and positively affected their TPACK development perception. Similarly, Alemdag et al. (2020) found that active participation in collaborative, hands-on activities was crucial in developing teachers' technological pedagogical content knowledge.

Contextual factors, such as classroom behaviours, time, and resource availability, were also found to influence the planning and implementation of technology-integrated lessons. Throughout the interviews and analysis of lesson plans, the interplay between technology, pedagogy, and content emerged as a critical factor. This interplay refers to the relationship between technology use, the teaching methods employed, and the teaching content. The study highlighted that all types of teacher knowledge are contextually sensitive, and the way technology, pedagogy, and content interact is heavily influenced by the specific teaching context. Context can include various elements, such as the culture, learners' behaviours, the school environment, and the learning environment itself. Moreover, the teaching content also shapes how technology, pedagogy, and content come together. For instance, some subjects, such as mathematics, may lend themselves naturally to technology-based caring.

Implications of the study

The results from this study offer empirical evidence that consistent and targeted progression in technology integration is essential for Mathematics teachers. It suggests that an enhancement in a teacher's drive to improve their technological pedagogical content knowledge (TPACK) skills considerably influences their capacity to assimilate technology into their lesson design and activities. Furthermore, the more teachers comprehend the Technological, Pedagogical, and Content Knowledge (TPACK), the less they view technology as merely a technical tool. Moreover, the study identifies a distinct variance in TPACK comprehension among teachers who persistently refine their understanding of best practice methods in information technology and its practical usage in education. The evidence from this study indicates that Mathematics teachers exhibit enhanced TPACK after participating in intervention workshops compared to their initial perceptions.

Limitations and Recommendations for Future Research

While this study has provided valuable insights into integrating technology into lesson planning, several limitations require further investigation. One of the limitations is the small sample size of six in-service mathematics teachers. Although the thematic analysis employed in this study generated quality data, the results may not be generalisable to a larger population due to the limited number of participants. Therefore, future research with a larger sample size

is recommended. Additionally, the researcher suggests conducting a follow-up study that allows teachers to implement their planned lessons in teaching practice, which would provide a more comprehensive assessment of the impact of the intervention program on teachers' TPACK development.

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BRIDGING THE SCHOOL-TO-UNIVERSITY TRANSITION GAP IN MATHEMATICS: CHALLENGES AND STRATEGIES AT UNIVERSITY

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The transition from high school to university mathematics poses a persistent challenge for learners, and the gap between these two educational levels appears to be widening. In this study, we delve into the nature of this gap, exploring the perceptions of educators and students. We shed light on the unique challenges and opportunities in teaching and learning trigonometry and calculus. Our central question is: “What intervention strategies can be used to enhance and align the teaching of trigonometry and calculus at university with the high school curriculum”. To address this question, we draw upon the rite of passage theoretical model of Clark and Lovric (2008) to explore the school-university transition, and the Cultural-Historical Activity Theory (CHAT) as an analytic lens.

We employ a mixed-methods approach, combining quantitative data from a survey, and interviews with mathematics educators to get insights into the challenges faced and strategies to bridge the gap. Our findings highlight specific areas off trigonometry and calculus that students find particularly challenging. Moreover, these difficulties are influenced by prior educational experiences, socio-economic factors, and cultural contexts, all which impact students’ readiness for university-level mathematics. We also recommend several targeted interventions and support mechanisms to facilitate a successful crossing of the liminal space.

Keywords: School-university transitions, mathematics, trigonometry, calculation, intervention strategies

Introduction

The transition from high school to university is a critical juncture in a student’s academic journey. For mathematics learners, this transition can be especially daunting. Notwithstanding many efforts to address the gap, it remains unresolved and even seems to be widening (Machisi, 2018). In this paper, we explore the factors contributing to this gap and propose strategies to bridge it effectively. Despite the education policy reform in our country since to redress historical imbalances, however, the school-to-university transition problem remains unresolved, and the gap appears to be getting wider. The present study delineates and unravels the nature of the gap between high school and first-year university mathematics. The intervention strategies offered by universities to support students who come to university with learning deficits in mathematics i.e., extra tuition and allowing them to register for extended programmes are not efficient to close the school-university transition gap sufficiently. A concerted effort from both institutions, high school, and university, is required to navigate the gap and make the transition smoother. We, therefore, attempt to answer the following question: “What intervention strategies can be used to enhance and align the teaching of trigonometry and calculus at university with the high school curriculum”.

Literature Review

We reviewed existing literature on the school-to-university transition, focusing on mathematics education. Previous studies have identified challenges related to curriculum misalignment, teaching methodologies, and student preparedness. By synthesizing this literature, we gain insights into the complexities of the high school-university transition process.

From the university perspective, several studies reported on the difficulty that students face in transitioning from high school to university mathematics. Hourigan and O'Donoghue (2007) found that students' inability to cope with the transition is due to the big difference between the nature of mathematics experience at high school level and that at university in mathematics-intensive courses. Other researchers such as Craig (2007) and Hoyles, Newman, and Noss (2001) have reported on the differences between university expectations and the wide spectrum of mathematical abilities of new students. The changing school curricula have been identified as one of the reasons for these differences (De La Paz, 2005; Hoyles et al., 2001). Clark and Lovric (2008) highlighted the transition to mathematics in higher education as a 'rite of passage'. There is a clear need to investigate the deficiencies in the current high school mathematics curriculum, and the differences between high school and university mathematics curricula to improve the transition of students from high school to university mathematics. Di Martino et al. (2023) assert that the focus at university to address the transition gap shifted from purely cognitive aspects to a more holistic approach, including socio-cultural and affective issues. However, there remains an underrepresentation of research from certain geographical areas, such as South America and Africa. This study also follows up on the findings of George and Rzyankina (2023) in which they explore the school-university transition gap from the high school educator perspective in South Africa. They found that it lacks depth in critical content areas of mathematics, there is an overemphasis on procedural knowledge oppose to conceptual knowledge, lack of localised application of mathematics concepts, and misinterpretation of word problems by learners. Their study also highlighted the problematic questions in calculus and trigonometry over the past five years in national senior certificate examination (NSC) that might have contributed to the consistent low performance. These mentioned literature studies provided the background and set the basis of this study.

Theoretical Framework

Researchers used various theoretical frameworks to investigate the high school-university transition (also referred to as secondary-tertiary transition) gap. The most prominent frameworks used in this research field are found in the three worlds of mathematics (Tall, 2008), the Anthropological Theory of the Didactic (Hausberger, 2018), the theory of commognition by Sfard (2008), and Clark and Lovric's (2008) well-established anthropological theory - rites of passage. We purposefully used the rites of passage theoretical model proposed by Clark and Lovric (2008) which consists of three stages – separation, liminal and incorporation - explaining the transition experience of students (Fig. 1).

This anthropological approaches to rites of passage have been profoundly influenced by Van Gennep's classic proto-structuralist model. According to Van Gennep (1960) such rites

commence with the radical separation of the person or persons being transformed, often marked through special adornment, locale, or comportment.

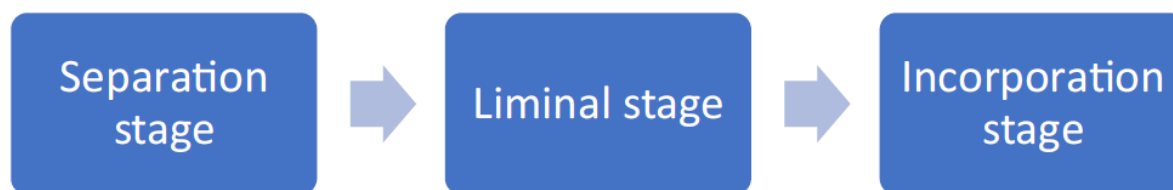


Fig. 1: Three stages of rite of passage (Di Martino et al., 2023)

The liminal stage includes the period from the last part of high school to the first part of university. It is characterised by an unavoidable crisis because known mathematical knowledge is challenged, and first-year students need to find their place in a new mathematics community. Arnold van Gennep (1960) describes and analyses certain events “life crises” (e.g., birth, betrothal, marriage, or death) and developed the theory of what he called rites of passage. When a person undergoes a “life crisis,” the usual and customary routines in her/his life are interrupted, changed, and distorted. This stage, in which the person experiencing a crisis gets “removed” or isolated from the rest of the community (family, social group, etc.) is called the phase of separation. Although the crisis seems to be an event related to that individual, it affects the life of the community that the individual belongs to and will belong to when the rite is completed. This way, the individual’s crisis becomes, in many ways, the crisis of the whole community, and the essential purpose of a rite of passage is to re-establish the balance and bring back the customary routines of daily life to the community. This model explains not only the cognitive shock of the passage from informal to formal language and reasoning in mathematics, but the affective reactions - the initial reaction of euphoria, the feelings of inadequacy during the crisis, and the recovery after the resolution of the crisis.

Additionally, this study leverages Cultural-Historical Activity Theory (CHAT) as a foundational framework to probe the socio-cultural contexts that significantly influence mathematics teaching and learning. By illuminating the historical and cultural forces at play, we aim to shed light on the nuanced dynamics that shape first-year students' engagement with emerging mathematics concepts. Central to our analysis is Vygotsky's (1978) concept of the Zone of Proximal Development (ZPD), introduced as a critique of the traditional psychometric assessments prevalent in Russian schools. Such assessments, focusing solely on learners' current level of achievement, neglect the potential for future developmental growth. The ZPD underscores the gap between what learners can accomplish independently and what they can achieve with the guidance of a more capable peer or adult, thus emphasizing the importance of prospective development in educational settings. In applying the ZPD to the context of mathematics education, this research specifically examines the Grade 12 curriculum content of trigonometry and calculus. These subject areas are used as mediating tools within the ZPD framework to explore how curriculum engagement can facilitate students' interaction with complex mathematical concepts and support their developmental progression, thereby aiming to improve concept understanding and application.

Echoing the significance of mediation in human action, as highlighted by Wertsch (1991), our study also emphasises the role of semiotic means (tools) in learning. According to Vygotsky (1978), semiotic tools, which include a diverse array of resources such as language, counting systems, mnemonic devices, and algebraic symbols, not only facilitate learning processes but also encapsulate the cultural and historical essence of mathematical knowledge. Our investigation focuses on the utilization of numerical and symbolic representations within the Grade 12 mathematics and the first-year mathematics curricula, examining their impact on students' comprehension and engagement with mathematical concepts. Building on Vygotsky's concept, mediated action is conceptualised as a dynamic interplay among the learner/student (subject), the educational tools such as mathematics curriculum, and the learning objectives (object). This triadic relationship is instrumental in understanding how students interact with mathematical concepts within their socio-cultural environments, highlighting the significance of this interaction in the high school-university transition process.

The second generation of the Cultural-Historical Activity Theory (CHAT) of Engeström (1999) introduce an activity system that shifts the focus from individual actions to collective activities (Fig. 2). This paradigmatic shift emphasises the intricate interrelations between individual subjects and their communities within educational settings. Our analysis utilises a triangular model to examine these interactions, focusing on the cultural and historical dimensions inherent in the educational process. In this model, learners/students (the subjects) engage with mathematical concepts through an intricate network of tools, objectives, and community interactions, aiming to achieve conceptual understanding.

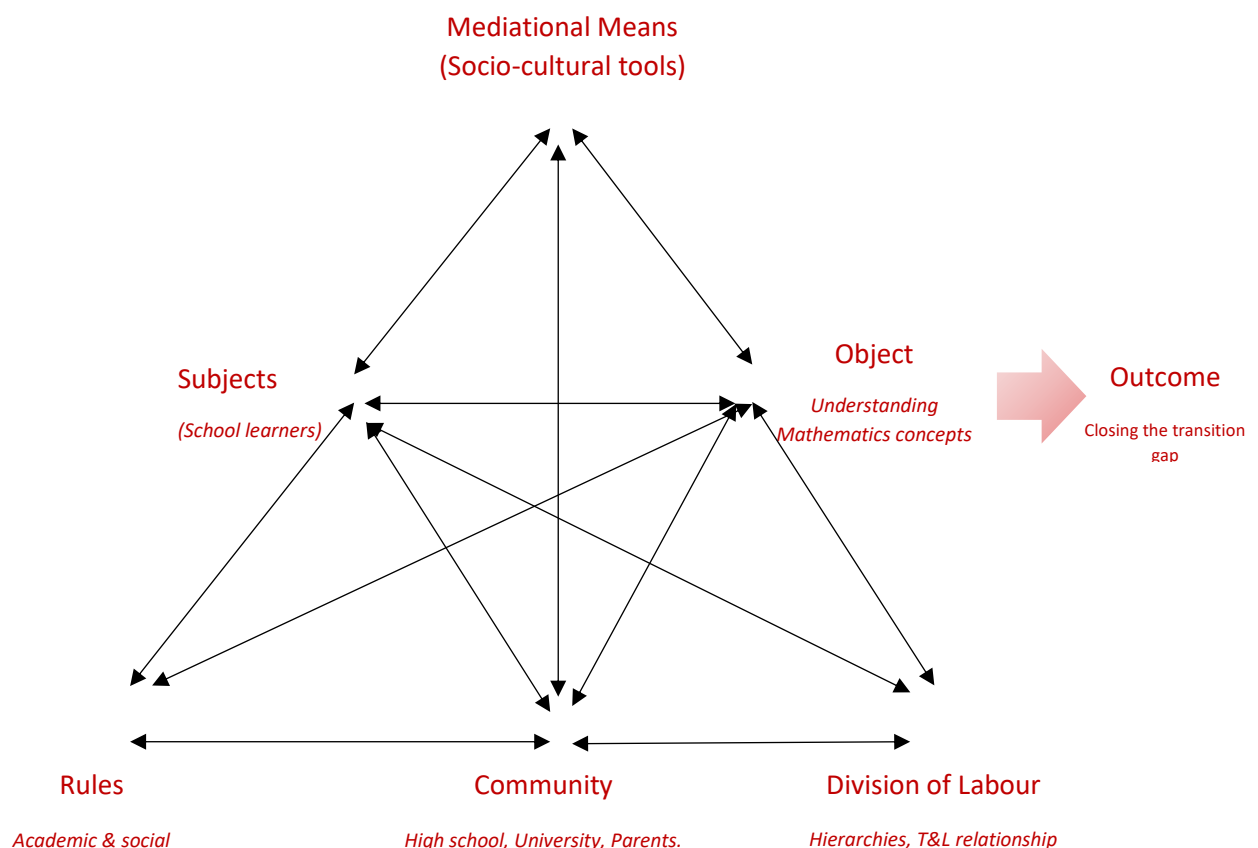


Fig. 2: Second generation CHAT (Engeström, 1999)

A cornerstone of CHAT is the examination of systemic contradictions within the activity system. These contradictions, far from being mere obstacles, are inherent challenges that stimulate change and development. By adopting CHAT as an analytic lens through which to view the complexities of the high school-university transition gap in mathematics, this study contributes to a deeper understanding of the socio-cultural influences on learning. Through a detailed examination of the mediated action, semiotic means, and systemic contradictions within the educational activity system, we offer insights into the potential for enhancing teaching and learning practices in mathematics, particularly at the critical juncture of the school-to-university transition.

Methodology

We made use of an exploratory case study with a mixed methods approach located within the interpretive research paradigm (Creswell, 2014). To investigate the high school-university transition gap in mathematics, we conducted a survey and interviewed 28 first-year mathematics lecturers. These lecturers are from eight South African universities in different provinces purposefully selected in terms of traditional, comprehensive, and universities of technology. The survey questionnaire consisted of 15 questions using the likert scale – Strongly Disagree (SD), Disagree (D), Unsure (U), Agree (A), and Strongly Agree (SA). We used five questions in the interview questionnaire to collect qualitative data. In both the survey and interview questionnaires we collect data about the lecturers' perceptions of the transitioning students' conceptual understanding of trigonometry and calculus topics with the following themes: mathematics notation and application skills, curriculum alignment, effectiveness of existing intervention strategies, cultural sensitivity and inclusivity, and high school-university collaboration in mathematics. We used the tripartite model of rites of passage to discuss the liminal space during the high school-university transition in mathematics. We also make use of Vygotsky's zone of proximal development (ZPD) and Engstrom's Cultura-Historical Activity Theory (CHAT) to analyse the data collected methodically.

Findings and discussion

The rite of passage – survey responses

The rites of passage theoretical model proposed by Clark and Lovric (2008) is very useful to explain the high school-university transition perceptions of lecturers. Table 1 reflects the summary of the responses to the survey.

Table 1: Frequency table of lecturers' perceptions

Survey: First-Year Lecturers' Perceptions of Transitioning Students (N = 28)						
Item #	Item	SD f (%)	D f (%)	U f (%)	A f (%)	SA f (%)
1	Your student had a good conceptual understanding of the trigonometry and calculus topics	19 70%	6 20%	- -	3 10%	- -
2	Your students were able to apply and effectively used the required mathematics notation.	1 5%	17 60%	3 10%	7 25%	- -
4	The high school curriculum (CAPS) is appropriately aligned with the first-year trigonometry and calculus topics.	8 30%	8 30%	6 20%	3 10%	3 10%
8	The existing intervention programs (tutorials, bridging courses, etc.) are effectively supporting transitioning students to be successful in first-year mathematics.	- -	8 30%	6 20%	13 45%	3 10%
13	The first-year mathematics lecture presentations at your university are culturally sensitive and inclusive to transitioning students.	1 5%	6 20%	1 5%	7 25%	13 45%
14	Your university have existing mathematics collaboration program with high schools.	4 15	14 50%	4 15%	6 20%	- -

The data in Table 1 suggest that most of the participating lecturers (90%) think that the transitioning students do not have the adequate conceptual understanding of the trigonometry and calculus topics, which support their view (65%) that these students do not apply, understand, and use mathematics notation correctly. The data also shows that the lecturers (60%) are of the opinion that the high school mathematics curriculum is not appropriately aligned with first-year mathematics topics, and there are mainly no existing mathematics collaboration programs with high school (65%). However, the most lecturers (55%) responded that the intervention programs are effectively supporting transitioning students; the majority of lecturers (70%) also agree that the first-year mathematics lecture presentations are culturally sensitive and inclusive.

The low frequency in the lecturers' responses to the students' conceptual understanding and mathematics notation could be attributed to what Gennep (1960) term as the unavoidable crisis

when the known mathematical knowledge is challenged, and first-year students need to find their place in a new mathematics community. This stage of separation where the high school learners experiencing a crisis and gets “removed” or isolated from the high school community affects the life of the community that the transitioning students belongs to and will belong to when the rite is completed. The transitioning student’s crisis becomes the crisis of the whole community (high school and university), and the essential purpose of a rite of passage is to re-establish the balance and to incorporate the student into the university community (Fig. 3). Although most of the responses of the lecturers confirm the challenges that widen the high school-university gap in mathematics i.e., lack of curriculum alignment, collaboration programs, teaching methodology, etc., they agree that there are positive initiatives to narrow the gap like intervention programs and creating a culturally sensitive and inclusive space for transitioning students during the liminal stage. The lack of high school-university collaboration programs in the responses of the lecturers have to be addressed to contribute positively to the separation stage.

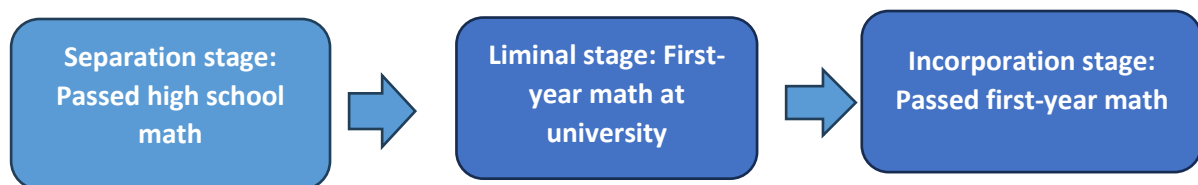


Fig. 3: Rite of passage for transitioning students

During the liminal stage the transitioning students might experience a cognitive shock of being exposed to emerging formal language and reasoning in trigonometry and calculus, and the initial reaction of euphoria and the feelings of inadequacy. According to the responses of the lecturers the university community do attempt to create the conducive environment with their intervention programs and culturally sensitive and inclusive practices. In order to resolve the crisis and enter the final stage of the rite of passage – incorporation – effective intervention strategies by the involved communities (high school and university) must be put in place.

Transition gap – the interview responses

We summarised the 15 survey questions into five interview questions in order to a deeper understanding of the data in the frequency table. The participating lecturers provided extensive responses which more light on their perceptions about first-years conceptions of trigonometry and calculus. We only highlight the responses which were expressed by the majority of the lecturers.

In terms of the conceptual understanding of transitioning students most lecturers mentioned share the expression of Lecturer 5.

L5: “One common struggle lies in conceptual understanding, particularly in grasping abstract concepts such as limits, derivatives, and integrals. Many students find it difficult to connect these abstract ideas to real-world problems, which are crucial for applying mathematical principles practically.”

In further discussions with the lecturers on this theme they mentioned the challenge of students to unlearn the tacit misconception they hold i.e., completeness of real numbers, inverse trigonometry graphs, applications of integral, etc.

In terms of application, and understanding mathematics notation:

L21: “Mathematical notation can be daunting for new students. I must spend a lot of lecture time on teaching new students on notations such as sigma notation, the derivative symbol, and trigonometric identities before I can proceed with the first-year mathematics.”

This item in the frequency table scored the highest (90%), which raise a real concern. It is also interesting to note that this observation is across the three categories of universities.

In terms of curriculum alignment of the high school curriculum with the foundational concepts taught in university mathematics courses.

L13: “The alignment between high school curriculum and university mathematics courses is another area of concern. High school mathematics only introduces foundational concepts, but it lacks the depth and rigor of university-level material for example limits, inverse trigonometric function, etc. As a result, students may struggle to bridge the gap between what they learned in high school and the more advanced concepts of calculus in university courses.”

Since 1994 the South African education system has reformed the curriculum to readdress equality and equity in terms of Outcome-based education (OBE). Curriculum 2005, and Curriculum Assessment Policy Statements (CAPS). However, these policy changes did not intentionally aim to close the high school-university gap (Department of Basic Education, 2011).

In terms of intervention and collaboration programs of high school and university levels.

L7: “We need to integrate university-level mathematical concepts like inverse trigonometric functions, hyperbolic functions, integration, etc. into high school curricula to better prepare students for higher education. Professional development opportunities for high school mathematics teachers is essential to enhance their knowledge and teaching methods. University lecturers must also be knowledgeable about the high school curriculum to better support students in overcoming the content gap and fostering a deeper understanding of trigonometry and calculus from the outset.”

Other responses expressed by the lecturers about this theme were - early exposure to university-level content, introduce learners to university-level mathematical concepts during their final years of high school, identify problematic topics, offer advanced elective courses or workshops that cover topics relevant to calculus and trigonometry at university level.

Intervention - ZPD and CHAT

The theoretical construct of the zone of proximal development (ZPD) is a useful tool to explain the transition of the high school-university mathematics gap (Fig. 4). In terms of ZPD high school learners at the separation stage find themselves at the potential level (below Step 1). When entering the liminal space, being confronted with emergent mathematics concepts and guided by more knowledgeable others the transitioning students move through the zone of proximal development (Steps 1, 2, and 3). In the zone of proximal development, the transitioning students are mediated and scaffolded with intervention strategies. The external stakeholders – high school management, university lecturers, parents – also play important roles in the mediation process. When the transitioning students mastered the mathematical concepts they reached the zone of acquired development (Step 4) and are able to solve complex mathematical concepts independently. Thus, they are successfully incorporated into the university community.



Fig. 4: The zone of proximal development (ZPD)

Transitioning from high school to university-level trigonometry and calculus involves a complex interplay of epistemological and cognitive challenges, mathematical notation, social identity, peer pressure, teaching methods, math anxiety, real-world application, and transition shock, all of which can be understood through the lens of the second generation Cultural-Historical Activity theory (Fig. 5). Epistemological and cognitive challenges arise as students encounter more abstract and rigorous mathematical concepts in university.

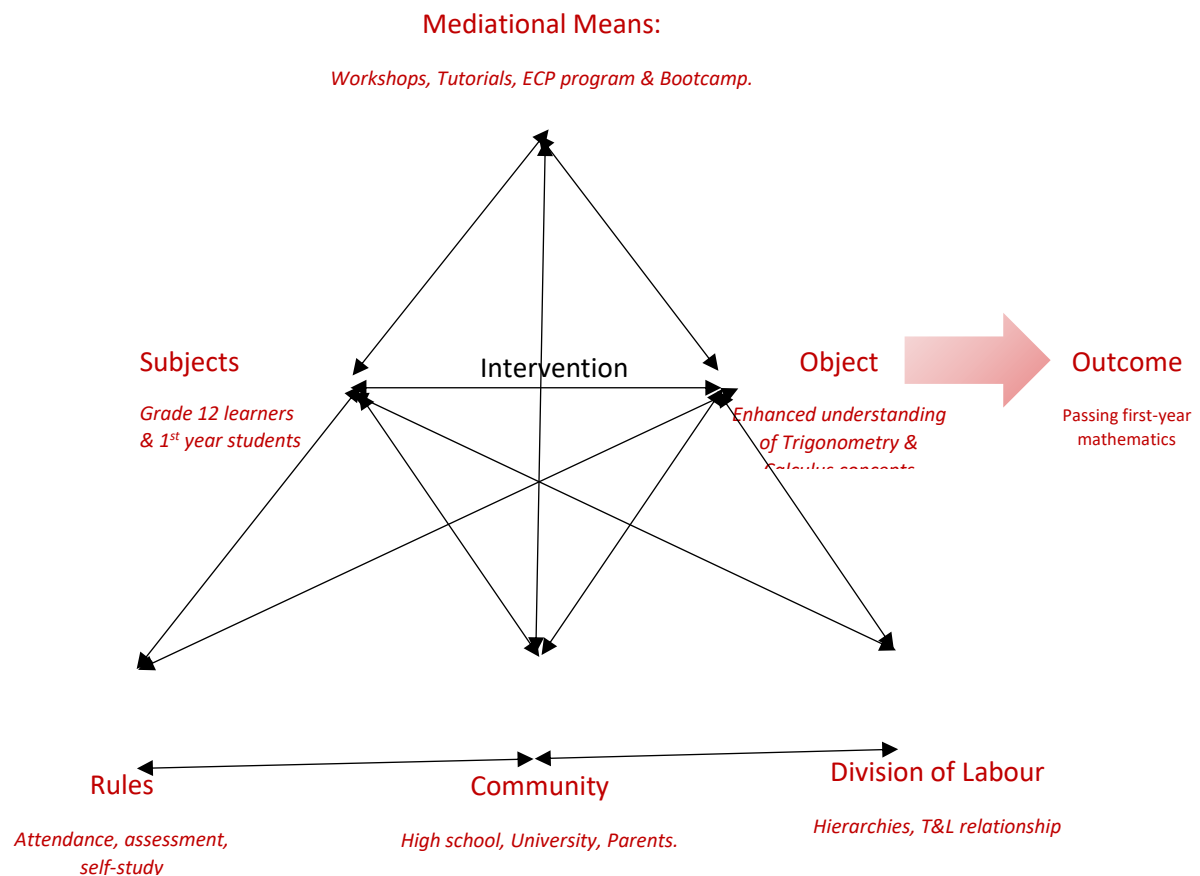


Fig. 5: School-university transition activity system (Engeström, 1999)

The shift from rote learning to deeper understanding requires learners to develop new ways of thinking and problem-solving, which can be daunting. Mathematical notation serves as a symbolic language in mathematics, but its complexity increases at the university level. Mastery of notation is crucial for communicating mathematical ideas effectively, yet it can pose a barrier to understanding for students transitioning from high school.

Some of the tension that is evident the transition activity system is, social identity and peer pressure which play significant roles in the transition process. Students may feel pressure to conform to academic norms and expectations, which can impact their confidence and performance in trigonometry and calculus. Social identity can influence students' perceptions of their mathematical abilities and their willingness to seek help when needed. Furthermore, mathematics anxiety, characterised by fear or apprehension about mathematics exacerbate these challenges. Transitioning to university-level trigonometry and calculus can intensify math anxiety due to increased academic pressure and the perceived difficulty of the material. The shift from high school to university trigonometry and calculus can trigger transition shock as students navigate academic and social changes while grappling with the aforementioned challenges.

The mediation tools in the activity system includes high school and university educators, tutorials, teaching methods also differ between high school and university, with university instruction often emphasizing independent learning and critical thinking (Jacobs, George, & Angaama, 2023). This shift can be challenging for students accustomed to more structured approaches in high school. Intervention strategies should also include real-world applications of trigonometry and calculus are emphasized more in university courses, requiring students to connect mathematical concepts to practical problems. This shift highlights the importance of understanding how mathematical principles are applied in various contexts. Understanding these dynamics through the Cultural-Historical Activity theory can inform interventions and support mechanisms to facilitate a smoother transition for students entering university-level mathematics courses. By addressing epistemological, social, cognitive, and affective factors, educators can help students build confidence, develop effective learning strategies, and thrive in their mathematical studies.

In summary, addressing these challenges involves a holistic approach that considers cognitive, affective, and socio-cultural factors. Successful strategies include curriculum alignment, early exposure to university-level content, and support mechanisms to ease the transition.

Conclusion

The transition from high school to university-level mathematics, particularly in trigonometry and calculus, is marked by various challenges and complexities. Despite efforts to bridge the gap, it remains a significant issue in many educational systems. Our exploration of this gap has highlighted the multifaceted nature of the challenges faced by transitioning students and the importance of addressing them effectively. The literature review underscores the discrepancies between high school and university mathematics curricula, as well as the cognitive, social, and affective factors influencing students' transition experiences. Theoretical frameworks such as rites of passage and Cultural-Historical Activity Theory provide valuable insights into understanding the transition process and the systemic contradictions within educational systems.

Findings from our survey and interviews with first-year mathematics lecturers further illuminate the challenges faced by transitioning students, including conceptual understanding, mathematical notation, curriculum alignment, and the effectiveness of intervention strategies. These findings underscore the need for collaborative efforts between high schools and universities to support students through the transition process. Drawing on theories such as the Zone of Proximal Development and Cultural-Historical Activity Theory, intervention strategies can be developed to address the diverse needs of transitioning students. By integrating university-level concepts into high school curricula, providing professional development opportunities for educators, and fostering a culturally sensitive and inclusive learning environment, the transition gap can be narrowed.

In essence, addressing the transition from high school to university mathematics requires a holistic approach that considers cognitive, social, and cultural factors. Through concerted

efforts from both institutions and educators, we can strive to make the transition smoother and empower students to succeed in their mathematical studies at the university level.

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REFLECTIONS ON THE ONLINE USE OF CONCEPT CARTOONS TO FACILITATE PROBLEM-SOLVING BEHAVIOUR AND CONCEPTUAL UNDERSTANDING OF GRADE 7 MATHEMATICS LEARNERS.

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This study reflects on the online use of concept cartoons to facilitate exploratory talk and its effects on the problem-solving behaviour and conceptual understanding of grade 7 mathematics learners. The research questions focused on adapting concept cartoons for teaching mathematics online, as well as its effects on the problem-solving behaviour and conceptual understanding of grade 7 learners. Data was collected using a mixed method approach. The results show that as learners become used to working online, and with some adaptations to the concept cartoons, similar effects as during face-to-face engagements are possible. Although the results are promising, it cannot be generalised due to the small scale of this study. There is clearly a need for further research and development in the online use of concept cartoons.

Keywords: concept cartoons; exploratory talk; misconceptions; conceptual understanding; problem-solving

Background

In line with the conference theme of *reimagining mathematics in a digital space*, this paper reflects on one case of using concept cartoons online with a group of grade 7 mathematics learners. The data was collected during the COVID-19 pandemic, when strict lockdown restrictions necessitated a shift from the face-to-face small-group engagements with learners originally planned, to online small-group engagements via Zoom meeting software. Prior to the pandemic, much had been written about how concept cartoons can be used to stimulate participation and argumentation in face-to-face teaching and learning. At the time of doing this research, however, we could not find any examples in the literature of a similar use of concept cartoons in online teaching and learning; this presented an opportunity to contribute towards narrowing this identified gap.

Concept cartoons were created by Keogh and Naylor in 1991 to elicit learners' ideas, challenge their thinking, and support them in developing understanding (Keogh & Naylor, 1993). A concept cartoon is a visual representation of a problem, usually taken from the learners' everyday lives, and includes cartoon characters with various viewpoints on the solution of this problem (Naylor & Keogh, 2013). The teacher typically encourages the learners to discuss the various viewpoints presented by the cartoon characters, because this has the potential to stimulate argumentation and facilitate co-construction of arguments (Naylor & Keogh, 2013). In this socio-cultural approach (Mercer & Littleton, 2007; Mercer, 2008) to teaching and learning, what is sought is a type of argumentation known as 'exploratory talk', meaning critical and constructive discussion featuring varying thoughts and opinions about the given problem that usually culminates in a well-reasoned solution (Mercer, Wegerif, & Dawes,

1999). Exploratory talk differs from ‘unproductive dispositional talk’ (Mercer, Wegerif, & Dawes, 1999; Symons & Pierce, 2015), which is all about individual competitiveness and disagreement; and also from ‘cumulative talk’ (Mercer, Wegerif, & Dawes, 1999; Edwards & Jones, 2001; Patterson, 2009; Symons & Pierce, 2015), which is about stitching together suggestions provided by peers, without careful evaluation, into a mutually agreed-on argument, in a non-critical and non-judgemental way. Exploratory talk does not necessarily happen spontaneously and must often be taught and sustained (Wegerif, Littleton, Dawes, Mercer, & Rowe, 2004; Mercer, Hennessy, & Warwick, 2019; Dawes, 2007).

This research aimed to explore the potential of concept cartoons to stimulate exploratory talk in small groups of grade 7 learners meeting face-to-face, and the resulting effects of exploratory talk on their problem-solving behaviour and conceptual understanding. When the COVID-19 pandemic and subsequent lockdowns put a temporary stop to all face-to-face classroom teaching and learning activities and almost all activities shifted online, it gave rise to the following research questions:

- (1) What are the challenges of adapting concept cartoons for use in an online mathematics classroom?
- (2) What are the effects of using concept cartoons in an online classroom on the problem-solving behaviour and conceptual understanding of grade 7 mathematics learners?

Conceptual Framework

The research paradigm is situated in the Participatory Inquiry Paradigm. In this paradigm the focus is on the process through which a participatory reality is shaped, and on the value of participation on individuals. The Participatory Inquiry Paradigm does not perceive people as separate entities, but as part of the living world as a whole (Heron & Reason, 1997). Humans are connected through collaborative inquiry.

The conceptual framework in this study is situated in Vygotsky’s Zone of Proximal Development (ZPD). Social interaction with peers begins at a very young age in children’s lives. Vygotsky believed that a child’s low mental functions develop first, and their higher mental function develops later in life through social interaction with peers (Doolittle, 1995). ‘Low mental function’ consists of children’s perceptions and associative learning, while ‘higher mental functions’ involve their problem-solving skills. Vygotsky named this process ‘internalisation’. Learners’ attitudes and perceptions are first experienced in their social environment, and form part of their mental operations (Doolittle, 1995). This process is important for children’s social development, as their skills are developed at a social level (Kurt, 2020).

The ZPD is central to the cognitive development theory. It is divided into a lower part and an upper part; the lower part is based on the actions children are able to accomplish individually. The upper part is based on things children can accomplish when helped by competent peers, including their teachers. The ZPD focuses on what children can do on their own, rather than what they can do with peer or teacher assistance (Cherry, 2021). The ZPD also joins cognitive development and learning together. Cognitive development in children occurs when they must perform tasks linked to the upper level of their ZPD. Learners require assistance in completing

these tasks; cognitive skills are only developed when the child gradually moves from being assisted less with tasks to a point at which no assistant is required (Doolittle, 1995).

The ZPD also focuses on interdependence. A child's ZPD exists in various social contexts and requires tasks that stimulate the ZPD to influence the child's development positively. Teachers must provide learners with activities to stimulate their ZPDs. They must also provide learners with resources that will ensure successful development. Vygotsky regarded social interaction as a very important contributing factor to children's learning and development (Doolittle, 1995). According to Doolittle's study (1995), Vygotsky stated further that in a child's developmental phase of social interaction, there is an external stage through which the higher mental function must proceed first, as it is seen as a social function. Classroom teaching should be focused on developing higher levels of ZPD, to keep learners interested and engaged in their classroom activities (Zhou & Brown, 2015).

Social interaction itself should be focused; learners must engage in active learning through interaction with peers, and a collaborative partnership must exist between teacher and learner in the interaction process (Zhou & Brown, 2015). In a cooperative learning and teaching environment, interactive and small group social skills form the basis for acquiring knowledge in cooperative learning groups. Cooperative learning groups require learners to learn the academic tasks, and to develop interactive and teamwork skills (Science Education Resource Center, 2021). The Science Education Resource Center (2021) further states that "given the complexity of these skills, teachers can encourage much higher performance by teaching cooperative skill components within cooperative lessons". Through further development of these skills, learners become able to perform these tasks more effectively than their previous tasks.

Vygotsky believed that everyone in a group is responsible for their own ZPD development (Doolittle, 1995). Instruction should therefore be meaningful and focused. Learners must be provided with resources that allow them to go beyond the understanding of the task at hand. Vygotsky also considered the development of individual growth in collaborative groups to be of great importance (Doolittle, 1995). Individual accountability focuses on developing learners in cooperative groups by enabling them to transfer their ZPD past given tasks.

Research Methodology

A mixed methods approach was used to collect quantitative data about learners' conceptual understanding, using multiple-choice-based pre- and post-tests administered before and after using the concept cartoons online; and, qualitative data about the learners' participation, argumentation, and problem-solving behaviour by recording and transcribing all verbal interactions that took place during the online sessions. The population, consisting of all grade 7 learners (n=67) in a primary school in Qqeberha the Eastern Cape Province of South Africa, was divided into candidates and participants. Both groups wrote the same pre- and post-tests, but only the participants took part in the intervention sessions, during which designated groups discussed concept cartoons: one Afrikaans group (n=5) and two English groups (n=10).

The tests provided insight into the candidates' and participants' misconceptions, and their existing mathematical knowledge. The misconceptions that became evident from the pre-test

were used to create mathematics activities in the form of concept cartoons that were used in the intervention sessions. Qualitative data was collected about the participants' participation, argumentation and problem-solving behaviour, by recording and transcribing all the interactions that took place in the online group intervention sessions. All transcripts were analysed through coding in Atlas.ti, a computer software program. The group intervention sessions focused on identifying Mercer's three types of talk, and what problem-solving behaviours arose when participants engaged in the group activities. Another focus was on observing and testing the participants' ability to sustain the three types of talk once the concept cartoons were removed and they were reintroduced to worksheet-based activities, and whether conceptual understanding arose through this problem-solving process.

The quantitative data findings were used to determine if the purpose of the qualitative data findings had been met. Candidates and participants wrote the post-test simultaneously, and their scores were recorded and compared to their pre-tests on the Percentage Frequency Table. The purpose of the comparison was to determine the successfulness of the intervention session, and whether conceptual understanding had taken place.

Ethical issues concerned the fact that the study involved underage learners, all between 12 and 13 years old. Confidentiality and privacy can easily become a problem if the researcher does not have the necessary safety measures in place to protect the identities of those involved in a study. Due to the sensitive nature of the anonymity, confidentiality and privacy issues relating to the learners involved in the study, permission had to be obtained from the school principal and the Department of Education. Parents and learners were given consent forms explaining the study, what data collection methods and tools were to be used and the safety measures in place to protect their identities, and asking for their consent to being included in the study. To protect the learners' identities, names and any other indicators that would enable their identification were not to be written on any physical media (eg the tests). These instructions were repeated when they received their test papers; the instructions on both tests stated this clearly. Both candidates and participants were given pseudonyms by the researcher, in the form of alphanumeric codes. As the learners involved were very young and might change their minds about participating, they were given permission to withdraw from the study at any time. The researcher made it clear that final consent to take part must be sought from the learners themselves, as they were the ones participating in the study. The research study was based on face-to-face group sessions.

Then the COVID-19 pandemic occurred, and the risk of infection was very high. It was compulsory for all school learners and educators to wear masks to reduce this risk. Social distancing was enforced. The pandemic-related rules and regulations made it impossible to obtain clear video and audio recordings, due to social distancing and the wearing of masks. It limited camera angles and clear audio recording during discussion sessions. The Zoom online social platform was the best alternative, but had its own limitations and risks. Video recording, especially online, can make learners uncomfortable, which may affect their decision to participate in discussions. However, the learners involved in the study were used to being recorded, as there were cameras in classrooms pre-pandemic, and recordings were made at sports events. They were happy to have their participation video recorded. The researcher used

ongoing consent to ensure participants were comfortable during the Zoom sessions. Participants could easily identify each other by name; since transcripts were to be made of the video and audio recordings, the participants were instructed to use pseudonyms for confidentiality and privacy.

There are also social risks in studies involving learners, as they may feel they lack knowledge, work pace and reasoning skills compared to the rest of the group. They may also experience negative feelings when they get answers wrong. However, the researcher stepped in when necessary to guide the discussions, and participants were assured that mistakes are normal and that there would be no limit to the number of mistakes that could be made. Group support was encouraged, and the groups managed themselves very well. There are always safety risks when working with data, especially electronic data. Thus, electronic data was password protected and stored safely. All data collected was stored in a safe.

Data analysis

A Percentage Frequency Table was used to present the data collected from the results of the pre- and post-tests of both the candidates and the participants. Concept cartoons were used to initiate talk in each group. Data was extracted from transcripts made from the Zoom video and audio recordings. The data collected from the Percentage Frequency Table supported the data acquired from the video and audio recordings and transcripts. This provided sufficient data to enable the researcher to do a thorough and detailed data analysis.

Transcripts were analysed through coding and peer coding. The transcripts were uploaded to Atlas.ti, which performed the coding process and was used for qualitative research analysis. The researcher allocated specific codes to applicable phrases or sentences through highlighting them and assigning specific codes. Phrases and sentences can be coded using more than one code, which made the allocation of the different types of codes much easier, especially when Mercer's types of talk and Muir's problem-solving behaviours had to be coded. The researcher was able to draw conclusions regarding how exploratory talk affects learners' problem-solving behaviours, which was assisted by the coding process. The codes allocated in Atlas.ti were based on the themes of the research study: the characteristics of Mercer's three types of talk (disputational, cumulative and exploratory), Muir's problem-solving behaviours, and the indicators of conceptual talk.

Mercer's three types of talk can be described as follows. Disputational talk occurs when individuals take an individual approach to solving problems (Webb & Treagust, 2006; Wegerif, 2018). Individuals who engage in this type of talk are either oblivious to the ideas and suggestions made by others or choose to disagree with others rather than engage with their ideas. Cumulative talk is characterised by individuals being allowed to make suggestions in order to construct shared ideas and conclude a coherent and joint agreement, without judgement or critique from peers (Edwards & Jones, 2001; Mercer, Wegerif, & Dawes, 1999; Patterson, 2009; Symons & Pierce, 2015). This type of talk is characterised by repetition, confirmation of statements, sometimes without thorough explanations (Patterson, 2009; Symons & Pierce, 2015). Exploratory talk, on the other hand, is characterised by individuals engaging with their peers' ideas and thinking in a constructive and critical manner (Mercer & Littleton, 2007). It shifts the focus away from obtaining answers, relying solely on communication and the

knowledge shared in the group, and the evidence of the reasoning and problem-solving that arise because of their discussion. Teachers are encouraged to ask their learners for the reasons behind the answers given in class by using ‘why’ questions, as this develops exploratory talk during the discussion and problem-solving process (Mercer & Littleton, 2007).

‘Problem-solving behaviours’ is the process of solving problems by connecting problems to previous knowledge (Muir, Beswick, & Williamson, 2008). People’s ability to make these connections determines whether they are successful in the problem-solving process. Learners must be able to recognise and understand the mathematical structure that exists within the problem to successfully understand and resolve the problem. The authors identified three categories of problem-solving behaviours in learners: naïve, routine and sophisticated. In the current study, qualitative data was not collected based on these categories of behaviours but on various problem-solving behaviours, including metacognition, implementation of various strategies, confidence, use of alternative methods, identification of similar problems, verbal communication, and evidence of Polya’s problem-solving plan. The study used this type of analysis because it was focused on determining the effect that exploratory talk has on learners’ problem-solving behaviours, not on the categories of problem-solving behaviours it affects.

‘Conceptual understanding’ means the concept of ideas in mathematics being understood at a deeper cognitive level, in which people’s perceptions move beyond their awareness of basic mathematical methods (National Research Council, 2001). Individuals who can apply mathematical concepts across various contexts have conceptual understanding. They are able to structure their mathematical knowledge, and to connect new mathematical knowledge to their previous mathematical knowledge. To build effective conceptual understanding, mathematical concepts and methods must be practised, with a good mathematical understanding which will make the process of remembering and implementing methods and ideas easier (National Research Council, 2001). Conceptual understanding is most evident when mathematical concepts can be portrayed in various ways, and their usefulness can be shown across varying purposes. The qualitative data indicators for conceptual understanding are:

- Restating a mathematical concept, reviewing the necessary or sufficient conditions from the mathematical concept in question;
- Learners knowing the most suitable representation for specific situations;
- Choosing and using certain procedures or operations applicable to the given task, applying concepts or algorithms to problem solving;
- Learners' ability to represent mathematical situations in a variety of ways;
- Giving examples and non-examples of concepts;
- Classifying objects according to certain characteristics or properties; and
- Developing the necessary or enough requirements from a concept, and being able to connect one mathematical concept to another.

(Mahendra & Purwati, 2019; Susilaningsih, Fatimah, & Nuswowati, 2018, p. 89).

Results, discussion and findings

(1) What are the challenges of adapting concept cartoons for use in an online mathematics classroom?

It is well known that concept cartoons can address learners' misconceptions, if these are included in the 'speech bubbles' of the cartoon characters (Naylor & Keogh, 2012), and if the learners are encouraged to discuss them. Misconceptions occur when learners' concepts – i.e. their beliefs, theories, explanations, and meanings – conflict with accepted mathematical theories, beliefs and explanations (Egodawatte, 2011). Therefore, it is important to know what misconceptions learners are likely to have, so that these can be included in speech bubbles during the design of the concept cartoons. One of the researchers in the current study is an experienced teacher with good understanding of the misconceptions that grade 7 learners are likely to experience, which was helpful in designing a multiple-choice pre-test that was administered to the entire population (n=67).

The pre-test was used to collect data about the misconceptions held by the learners prior to the concept cartoon interventions and to identify which were most prevalent, as these would then be used in the speech bubbles of the cartoon characters. The test items consisted of three sets of five questions each: the first set focused on percentage calculations, the second set on laws of exponents, and the third set on the concept of time. This paper will focus only on the results of the first set, regarding percentage calculations. The pre-test questions asked in this set are shown in Figure 1.

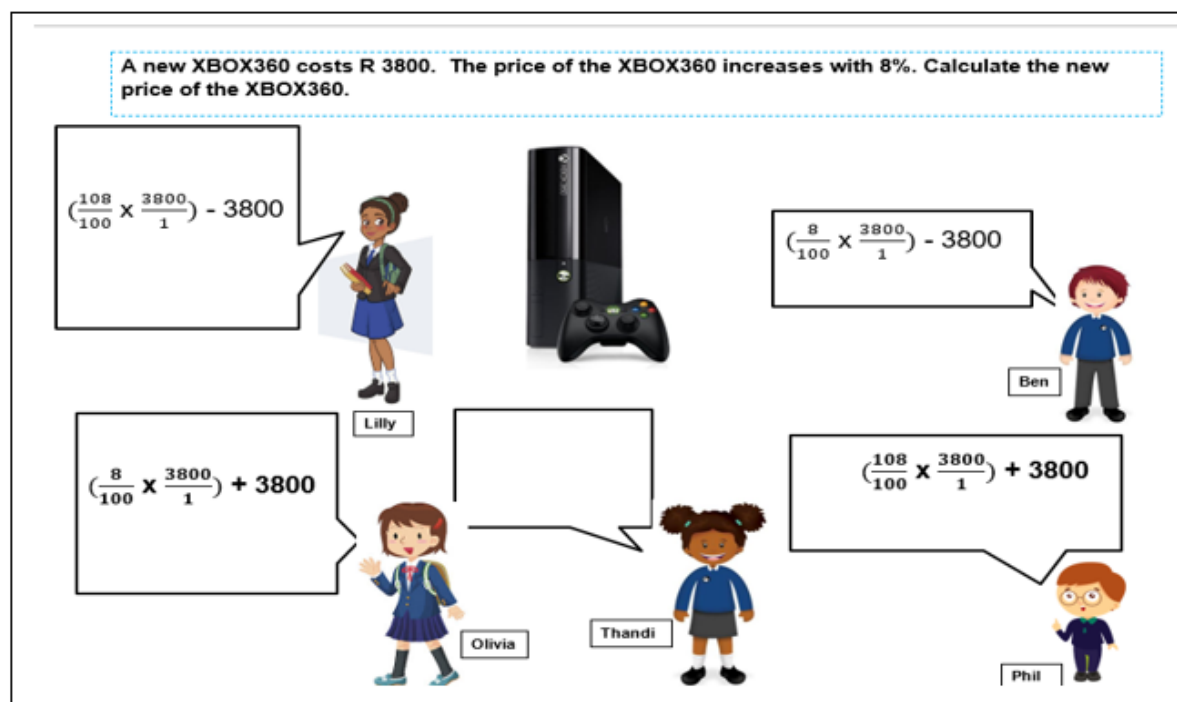
Figure 1: Set 1 of the pre-test questions dealing with financial mathematics focussed on percentage calculations.

1. A bicycle is bought for R 3600. The bicycle is sold for R 2880. What is the percentage loss that occurred?
A. $\frac{2880}{3600} \times 100$ B. $\frac{3600-2880}{3600} \times 100$ C. $\frac{3600-2880}{2880} \times 100$ D. $\frac{3600}{2880} \times 100$ E. None of the above.
2. You purchase a cell phone for R 4400. After a year, you sell the phone for R 3300. What percentage loss occurred?
A. 38% B. 75% C. 133% D. 25% E. None of the above.
3. A shirt is on sale with a 20% discount. This means:
A. Find 20% of the original price and subtract it from the original price.
B. Find 120% of the original price and add it to the original price.
C. Find 20% of the original price and add it to the original price.
D. Find 120% of the original price and subtract it from the original price.
E. None of the above.
4. 10% increase to 250 is calculated as follows:
A. $(\frac{10}{100} \times \frac{250}{1}) + 250$ B. $(\frac{10}{100} \times \frac{250}{1}) - 250$ C. $(\frac{110}{100} \times \frac{250}{1}) + 250$
D. $(\frac{110}{100} \times \frac{250}{1}) - 250$ E. None of the above.
5. Interest is added to the amount of money you borrow from the bank. A percentage amount is added to the amount of money borrowed. Altus borrows R680 from the bank at 15% interest for the year. The total amount owed by Altus to the bank:
A. Find 15% of R680 and add it to R680. B. Find 15% of R680 and subtract it from R680
C. Find 115% of R680 and add it to R680 D. Find 115% of R680 and subtract it from R680
E. None of the above.

Fifteen learners, hereafter referred to as participants, were purposively selected from among all the grade 7 learners who wrote the pre-test (n=67) to participate in this study, based on their SASAMS (South African School Administration and Management System) marks. This was to ensure variance in mathematical competence across all participants. Note that their pre-test scores were not considered in their selection. The selected participants were then divided into three groups of five learners per group. The participants in each group varied in their mathematical competence, but were reasonably fluent in the same language, with either Afrikaans (one group) or English (two groups) as their first language. The data for this study was collected during the strict lockdowns of the COVID-19 pandemic, when face-to-face group sessions were not allowed; thus the online Zoom platform was used to facilitate the group sessions. All participants had access to a computer connected to the internet and able to run Zoom. Participants were invited to the group sessions through links sent via email. Active participation in the synchronous online teaching and learning environment is different from passively receiving tasks and worksheets via email. This was new to the session participants, who all needed step-by-step guidance on how to use Zoom to connect to the group.

To minimise any problems relating to the use of the new technology, a trial run was conducted prior to the actual research group sessions. During this trial run a simple worksheet was employed with problems unrelated to anything in the concept cartoons, so that the real data would not be contaminated. The purpose of the trial run was to familiarise the participants with the use of Zoom before the actual research sessions – for example, how to mute and unmute their microphones, share their screens, etc.

Figure 2: The concept cartoon dealing with misconceptions highlighted by question 4 in the pre-test.



The results of the pre-test were useful in determining which questions and related misconceptions to target in the concept cartoons. The concept cartoon in Figure 2 above is

based on question 4 of the first set of questions on percentage calculations in the pre-test. About 53% of all the candidates ($n = 52$) and 40% of the participants ($n = 15$) answered this question correctly. The problem in the concept cartoon shown in Figure 2 is relevant to the everyday lives of the participants, and different possible solutions are presented in the speech bubbles of the cartoon characters. Among the options is the correct answer, $\left(\frac{8}{100} \times \frac{3\,800}{1}\right) + 3\,800$, as suggested by cartoon character Phil. Apart from the single correct answer and the empty speech bubble (see cartoon character Thandi), the other options are all typical misconceptions related to percentage increase problems. The empty speech bubble signals to participants that there may be other potential solutions that have not been included in the speech bubbles, and is an open invitation to explore such alternative solutions (Naylor & Keogh, 2013). The cartoon characters were given names for ease of reference during the online group discussions because unlike in face-to-face interactions where participants could point at a particular character, in the online environment it is easier to use the character's name as a reference. Moreover, it removes the need to refer to cartoon characters in terms of physical traits, which might be offensive. To comply with ethics protocols the identities of all participants had to be protected, so they were named PA1 (Participant 1, Afrikaans-speaking), PA2 (Participant 2, Afrikaans-speaking), PE1 (Participant 1, English-speaking), etc. During the Zoom sessions, participants were encouraged to keep their cameras off to further protect their identities.

(2) *What are the effects of using of concept cartoons in an online classroom on the problem-solving behaviour and conceptual understanding of grade 7 mathematics learners?*

The following two transcripts are exemplars of the types of talk that transpired during the online engagements with the concept cartoons, and how this affected participants' problem-solving behaviour and conceptual understanding. The first transcript here (which also preceded the second one in real time) shows how one of the Afrikaans groups approached the question posed in the concept cartoon in Figure 2 dealing with the misconceptions highlighted in question 4 of the pre-test. English translations are offered in italics. Initially there appeared to be agreement on selecting the option presented by the cartoon character named Ben, which is not the correct answer.

PA1: Mmm, hy verhoog met 8%, hmm... (*Mmm, it increases by 8%, hmm...*)

PA3: Is dit dan... ek sal sê dit is miskien 'n 8 op 'n 100 ... Maal 3 800 op 1, en dan sal ek sê... ja, iets soos dit. (*Is it then... I would say it's perhaps 8 over a 100... times 3 800 over 1, and then I'd say... yes, something like that.*)

PA1: Ja. (*Yes.*)

PA4: Ja. Ja, dit is definitief daai (*Yes. Yes, that's definitely it.*)

PA1: As ek daai... ek weet nie hoe julle dit geleer het nie. Ek weet nie hoe presies julle dit geleer het by juffrou nie, maar ek onthou dit altyd dis 3 800 op 1, en dan is dit... en dan is dit 8 op 'n 100, so... as jy 1 maal 'n 100, is dit 100. (*If I... I don't know how you have learnt it. I don't know exactly how you have learnt it from the teacher, but I remember it is always 3 800 over 1, and then it's... and then it's 8 over 100, so... if you times 1 by 100, then it's 100.*)

PA2: Dan moet ons net uitwerk... (*Then we must just work out...*)

PA1: Ja, persentasie op 'n 100 maal prys oor 1. (*Yes, percentage over 100 times price over 1.*)

PA3: Ja. (*Yes.*)

PA2: Ja, dis hoe ek ook geleer het. En dan, die ander een is anderste om. Verskil op oorspronklik maal 100 oor 1. (*Yes, that's also how I learnt it. And then, the other one is the other way round. Difference on original times 100 over 1.*)

PA4: Ja. (*Yes*)

PA2: Ja, ek dink ek stem saam met julle. (*Yes, I think I agree with you.*)

PA3: Ja, ek stem saam met Ben, want dis jou... ja, okay. Ja. Ek verstaan. Ek verstaan. (*Yes, I agree with Ben, because that's your... yes, okay. Yes. I understand. I understand.*)

PA4 & PA5: Ja. (*Yes.*)

PA3: Wat sê julle? (*What do all of you say?*)

PA1: Ja. (*Yes.*)

PA4: Ek stem saam met jou. (*I agree with you.*)

PA3: Okay, so ek sê Ben. Wie sê nog Ben? (*Okay, so I say Ben. Who else also says Ben?*)

PA5: Ben.

PA1: Ben.

PA2: Olivia, Ben dan...? (*Olivia, Ben then...?*)

PA3: Ben.

PA4: Ben.

In the transcript above, the discussion is dominated by cumulative talk (Mercer, Wegerif, & Dawes, 1999; Edwards & Jones, 2001; Patterson, 2009; Symons & Pierce, 2015; Mercer, 1996), which is characterised by mutual agreements and acceptances of other participants' views, sharing knowledge through talk in an uncritical manner, and elaborating on ideas without careful evaluation. The group was seemingly more focused on remembering the formula for calculating percentages that they were taught, ignoring the context and selecting to focus their attention on the first option in a speech bubble that caught their attention as the closest resemblance to what they could recall from memory, and neglecting to examine all the options carefully and critically. Moreover, it seems that that they were reading speech bubbles across the screen from left to right, as one would read regular text; because the correct option was completely ignored at first. It is in the next speech bubble when reading from left to right (Olivia's), and is very similar to the previous one (Ben's), with the first part of the expression in brackets showing how to calculate 8% of R3 800. However, both PA1 and PA2 seem to realise that there is a little more to this problem than simply calculating a percentage. PA1

mentioned that “*it increases by 8%*” and that “*I don’t know how you have learnt it. I don’t know exactly how you have learnt it from the teacher*”, but is overruled by the group consensus, which seems to be strongly influenced by PA3’s view that “*I would say it’s perhaps 8 over a 100... times 3 800 over 1, and then I’d say... yes, something like that*”, followed by “*Yes, I agree with Ben, because that’s your... yes, okay. Yes. I understand. I understand*”. PA3 then gives the opportunity for P1 and P2 to agree: “*What do you say?*”; and finally, “*Okay, so I say Ben. Who else also says Ben?*” Even though P2 initially agrees (“*Yes, I think I agree with you*”), he or she must have noticed some similarity between the options offered by Ben and Olivia, which they tried to raise with the group: “*Olivia, Ben then...?*”

The second transcript, below, shows how by re-focusing the group’s attention on another plausible solution (Olivia’s) – in which the first part of the expression in brackets also shows how to calculate 8% of R3 800, like their initial preferred solution (Ben’s) – P1 and P2 shifted the group talk from cumulative talk to exploratory talk. The latter is characterised by critical and constructive discussion about thoughts and opinions, and the generation of a well-reasoned conclusion to the problem at hand (Mercer, Wegerif, & Dawes, 1999). In this instance, the group not only found the speech bubble containing the correct answer to the question (Olivia’s), but even took it further by calculating the numerical value of the answer.

PA5: Okay.

PA1: Of dis Olivia? (*Or it’s Olivia?*)

PA5: Hmm... Olivia?

PA2: Ek dink dis meer Olivia. (*I think it’s more Olivia.*)

PA5: Maar die enigste verskil is, hier plus sy. (*But the only difference is, she plusses here.*)

PA4: Ja. (*Yes.*)

PA2: Ja, so... hierdie is ‘n positiewe getal en die ander is ‘n negatiewe getal of iets. (*Yes, so... this is a positive number and the other is a negative number or something.*)

PA5: Ja. (*Yes.*)

PA2: Ek dink dit sal eerder die positiewe getal wees, so dit moet Olivia wees. (*I think it’s the positive number rather, so it must be Olivia.*)

PA3: Okay, ons werk hom dan uit. (*Okay, let’s work it out then.*)

PA5: Ja, kom ons werk maar Olivia dan uit. (*Yes, let’s work Olivia out then.*)

PA2: Okay... Dit is 8 oor ‘n 100 maal 3 800 oor 1; dan kry jy ‘n 100... Die 8 maal 3 800 is dan 30 400. Dis wat ek kry. (*Okay... It’s 8 over 100 times 3 800 over 1; then you get 100... The 8 times 3 800 is then 30 400. That’s what I get.*)

PA3: Maar moes ons nie, soos... moes ons nie, soos, die nulle uitgehaal het nie? Dan sou dit 8 maal 38 gewees het. (*But shouldn't we have, like... shouldn't we have, like, taken out the zeros? Then it would have been 8 times 38.*)

PA2: O, ja (*Oh, yes.*)

PA3: En dan is dit 304, en moet ons nog net die 3 800 saam met dit plus. (*And then it's 304, and we must still plus the 3 800.*)

PA1: Ja. (*Yes.*)

PA4: Ek kry 4 110. (*I get 4 110.*)

PA5: Mens kan ook... (*One could also...*)

PA3: So is die prys dan verhoog. Onthou, dit is mos met 8%, so is dit dan 4 104? (*So the price is now increased. Remember, it's by 8% percent, so then it's 4 104?*)

PA4: Ja, ek het dit ook gekry. (*Yes, I also got that.*)

PA1: Ek ook. (*Me too.*)

PA4: Ja. (*Yes.*)

PA5: Ja, ja. (*Yes, yes.*)

PA2: PA3...

RESEARCHER: Okay, wat is julle antwoord? (*Okay, what's your answer?*)

PA2: PA3?

PA1: 4 104.

RESEARCHER: 4 104?

VARIOUS: Ja. (*Yes.*)

In this second transcript above, PA1 suggested an alternative option to Ben (who the group had previously agreed on): “*Or it's Olivia?*” This was supported by PA2: “*I think it's more Olivia.*” This in turn prompted PA5 to examine the differences between these two options: “*But the only difference is, she [Olivia] plusses here*” and “*Yes, let's work Olivia out then*”. The use of exploratory talk enabled the group to change their superficial method of interaction to a more engaged and deeper discussion. The participants encouraged each other, engaged everyone in the group, and worked together on solving the problem. Ideas were challenged with constructive criticism, as participants felt comfortable pointing out the calculation mistakes of other participants; for instance, PA2's suggestion (“*It's 8 over 100 times 3 800 over 1; then you get 100... The 8 times 3 800 is then 30 400.*”) was challenged by PA3: “*But shouldn't we have, like... shouldn't we have, like, taken out the zeros? Then it would have been 8 times 38.*” Note the use of the pronoun ‘we’, indicating a joint rather than an individual problem-solving effort. Constructive criticism like this,

Figure 3: Set 1 of the post-test questions dealing with financial mathematics focussed on percentage calculations.

Choose the correct answer by selecting the appropriate letter.

1. A fitness watch is bought for R 2200. The watch is sold again for R 1600. What is the percentage loss that occurred?
 - a. Divide the selling price by the purchasing price.
 - b. The rand difference between the purchase price and the selling price must be calculated. The rand difference must be written on top of the purchase price in fraction form and the fraction must be divided by 100%.
 - c. Calculate the percentages of each price and subtract the percentages from each other.
 - d. The rand difference between the purchase price and the selling price must be calculated. The rand difference must be written on top of the purchase price in fraction form and the fraction must be multiplied by 100%.
 - e. None of the above.
2. You purchase an electric scooter for R 2600. After a year, you sell the electric scooter for R 1200. What is the percentage loss that occurred?
 - a. 53,8%
 - b. 14%
 - c. 0,01%
 - d. 2,2%
 - e. None of the above
3. Earphones are on sale with a 30% discount. This means:
 - a. Find 130% of the original price and subtract it from the original price.
 - b. Find 30% of the original price and subtract it from the original price.
 - c. Find 30% of the original price and add it to the original price.
 - d. Find 130% of the original price and add it to the original price.
 - e. None of the above.
4. A 5% increase to 400 is calculated as follows:
 - a. Find 105% of 400 and add it to 400
 - b. Find 5% of 400 and subtract it from 400
 - c. Find 105% of 400 and subtract it from 400
 - d. Find 5% of 400 and add it to 400.
 - e. None of the above
5. The original price of a product before adding VAT (15%) is calculated as follows:
 - a. Find 15% of the original price and add it to the original price.
 - b. The original price divided by 100/115. The answer obtained is deducted from the original price to determine the price before adding VAT.
 - c. The original price is multiplied by 100/115 to determine the price before adding VAT.
 - d. Find 15% of the original price and subtract it from the original price.
 - e. The original price multiplied by 100/115. The answer obtained and the original price are added to determine the price before VAT is added.

and the way in which it is done, helps participants restructure their understanding of the problem-solving process, and establishes trust. By using ‘we’ rather than ‘you’, PA3 is aiming the question not only at PA2 but at the whole group. This not only alleviates the pressure on

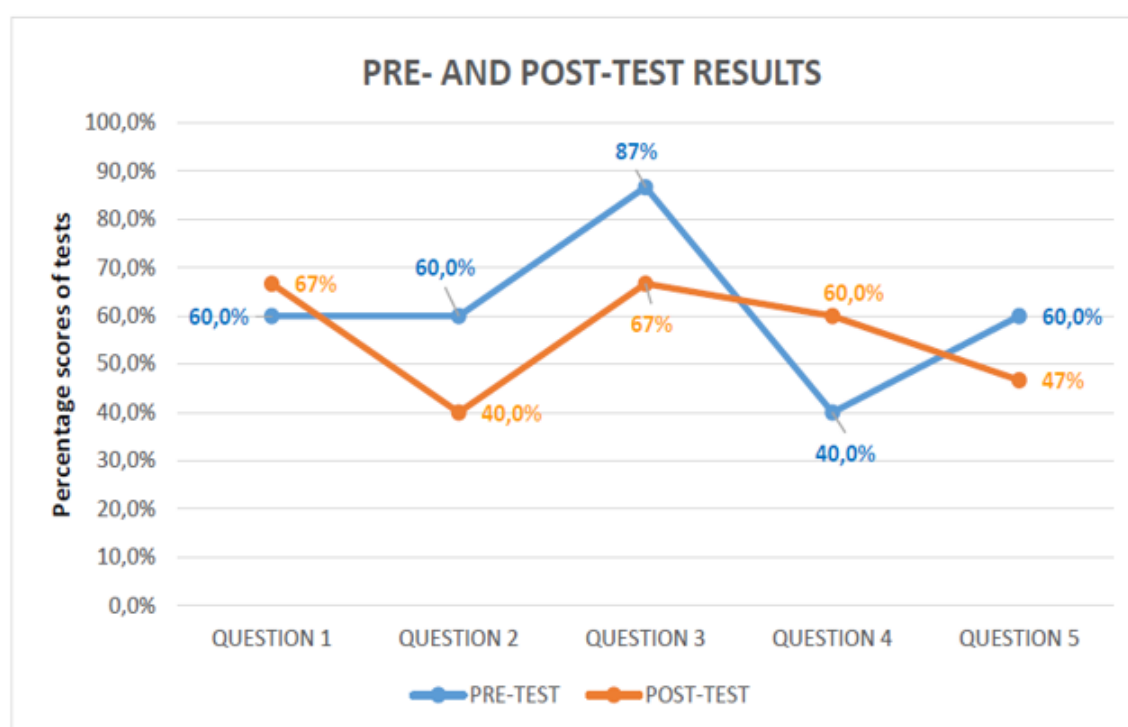
PA2, but is also an indicator that group input is valued. PA3's question "So the price is now increased. Remember it's by 8%, so then it's 4 104?" is a request for validation of PA3's answer. PA3's question indicates that group agreement was sought before a final decision was made on the correct answer. An important goal of exploratory talk is to reach group agreement within the discussion process, after carefully and critically evaluating all contributions.

The exploratory talk used here in the discussion of Olivia's solution engaged more participants than the cumulative talk used in the discussion of Ben's solution. This shift from cumulative talk to exploratory talk occurred when it was brought to the group's attention that there were alternative and equally viable options to consider, and that a more critical approach was necessary; this helped the participants to find the appropriate solution to the problem.

To assess the efficacy of having used concept cartoons in the online group sessions to address the misconceptions identified in the pre-test, it was necessary to follow up with a post-test, using different questions but testing for understanding of the same concepts shown in Figure 3.

When the results of the pre- and post-test are compared, as they are in Figure 4, the effects of using concept cartoons can be seen.

Figure 4: Comparison of correct answers selected by participants (n=15) in the pre- and post-test of set 1 on Financial Mathematics.



According to Figure 4, for the participants the percentage of correct post-test scores for questions 2, 3 and 5 are lower than the percentage of correct pre-test scores. Questions 2 and 3 stand out with a difference of 20 percentage points each. In question 4 the difference between the percentage of correct pre- and post-test scores is also 20 percentage points, but here the post-test score is higher (60%) than the pre-test score (40%). Likewise in question 1, although the difference between the percentage of correct pre- and post-test scores is only 7 percentage

points, the post-test score (67%) is higher than the pre-test score (60%), which is comparable with the post-test score of question 4 (60%). It is worth noting that questions 1 and 4 were both converted into concept cartoons, while questions 2, 3 and 5 were not.

The results show that for at least 20% of the participants, conceptual understanding has taken place in question 4, and for 7% of them in question 1. In question 4 this may be because of the shift from cumulative talk to exploratory talk, shown in the excerpts above, which was facilitated by using the concept cartoon. Due to space limitations, and because the change was

Table 1: Results of the pre- and post-tests for the participants (n = 15) showing which answers they have selected for each question as an absolute value and as a percentage*

		Question 1					Question 2					Question 3					Question 4					Question 5				
Multiple choice options		A	B	C	D	E	A	B	C	D	E	A	B	C	D	E	A	B	C	D	E	A	B	C	D	E
Pre-test results	Responses	1	9	2	2	1	1	2	2	9	1	13	0	1	0	1	6	3	2	1	3	9	5	0	0	1
	Percentage	7	60	13	13	7	7	13	13	60	7	87	0	7	0	7	40	20	13	7	20	60	33	0	0	7
Post-test results	Responses	1	3	0	10	1	6	1	0	6	2	1	10	2	0	2	2	3	0	9	1	7	1	6	1	0
	Percentage	7	20	0	67	7	40	7	0	40	13	7	67	13	0	13	13	20	0	60	7	47	7	40	7	0

* The correct answers are printed in bold face.

less dramatic, the effects of using a concept cartoon in question 1 are not discussed in detail, but they were similar. Based on the positive results with questions 1 and 4, it could be surmised that concept cartoons would have similar effects on questions 2, 3 and 5, if the design principles as discussed in the method section were followed. For instance, in question 2 of the post-test, participants may have experienced a misconception relating to the incorrect use of the formula for calculating percentage change. Unlike in the options for questions 1 and 4, where the calculations are shown, in the options for question 2 the calculations are hidden behind pre-calculated answers. As shown in Table 1, for question 2, 40% of the participants selected options A and D. Option A (40%) is the correct answer, calculated as $(\text{selling price} - \text{original price}) \div \text{original price} \times 100$, while the answer in option D (40%) was incorrectly calculated as $\text{original price} \div \text{selling price}$. To address this misconception, a concept cartoon must be designed that will generate exploratory talk around why option A is correct while option B is not. In question 3 there is also a 20% difference between the pre-test score (87%) and the post-test score (67%); however, based on the relatively high pre-test score, this question was not deemed suitable for conversion into a concept cartoon. In retrospect, it seems that attention must be given to questions in which methods or solutions in the options given are expressed in words. Likewise for question 5, where there is a 13-percentage point difference between the pre-test score (60%) and the post-test score (47%). Both question 4 and question 5 were based on a percentage price increase; but in question 4 the options consisted of formulas, while in question 5 the options consisted of verbal expressions. The difference in the scores seems to indicate that most participants understand concepts presented as formulas or numbers more readily than concepts presented as verbal expressions.

In question 5 of the pre-test, it is stated that “Interest is added to the amount of money you borrow from a bank. A percentage amount is added to the amount of money borrowed”. It is likely that for the five participants who selected option B (33%) and the one who selected option E (7%), the notion of *interest* is a new and poorly understood concept. Option B seems

to suggest that if interest is ‘*added*’ to money borrowed from a bank, then the outstanding amount should be less, and therefore the amount of interest should be calculated as a percentage of the original loan amount, then subtracted from it. This is a misconception held by 33% of the participants, and shows the importance of ensuring conceptual understanding when new terminology or concepts are introduced in mathematics. This is where well-designed concept cartoons can be a useful tool for facilitating conceptual understanding, by engaging learners in exploratory talk as they make sense of new knowledge.

The researchers concluded that exploratory talk was facilitated by using concept cartoons. Researchers concluded from the results that its use in group intervention sessions helped the participants to engage constructively and co-operatively in their respective groups while engaging in problem solving and in the reasoning process. Jan T’Sas (2018) conducted a pilot study during his doctoral studies, and observed that after he conducted an intervention session with the pupils he studied, there was an increase in their problem-solving skills. An increase was also observed in the usage of keywords related to exploratory talk; pupils started taking turns, and were engaged in longer discussions; and the quality of their arguments improved (T’Sas, 2018). T’Sas (2018) concluded that the type of talk that pupils used affected their problem-solving skills. Vygotsky’s theory is reflected in these results. Learning first occurs in the social context and is taken up by the individual afterwards, which indicates that the use of exploratory talk within a collaborative setting develops the group’s reasoning strategies (T’Sas, 2018). T’Sas (2018) connected these findings with Vygotsky’s theory, stating that new knowledge and skills had been integrated at individual level. Concept cartoons create opportunities for positive reinforcement when learners communicate with one another to accomplish a mutual goal (Yin Yin, Wai Bing, Yusof, & Zakariya, 2016). In this study participants formed a better understanding of the various mathematical topics presented to them. Participants’ conceptual understanding improved, according to the results obtained after their intervention sessions in which concept cartoons and exploratory talk were used confirming Bhowon et al. (2014) who claimed that concept cartoons assist performance and positively influence conceptual understanding.

Concept cartoons were useful tools for promoting exploratory talk and addressing misconceptions; they enabled study participants to identify their own misconceptions, and assisted them to construct new knowledge and understanding and to look at problems from different perspectives, engaging them in active learning (Yin Yin & Fitzgerald, 2017; Birisci, Metin, & Karakas, 2010). Different problem-solving behaviours were displayed through using concept cartoons and exploratory talk. The format in which the different concept cartoons were presented to the participants encouraged them to use exploratory talk, as they had to engage constructively with the content, especially the misconceptions that arose during these discussions. Misconceptions relating to the problems were addressed and overcome, and group members were confident in asking questions relating to the problem; their ideas regarding the problem were addressed, and alternative suggestions and support were provided by the group.

Conclusion

The process of problem-solving facilitated by exploratory talk and concept cartoons assisted participants to construct knowledge and adapt their understanding of the problem presented,

and they began to evolve into independent thinkers. After the study, positive responses were received from participants regarding the use of concept cartoons in conjunction with exploratory talk. The important conclusion to be drawn from these responses was that they learnt more and their understanding advanced through the process, and their communication in groups improved, which helped them to ascertain the correct answers; and they had fun while learning. According to Yin Yin et al. (2016) who conducted an exploratory study using concept cartoons for peer learning, concept cartoons impacted learners' communication skills, enhanced their interest in learning, promoted their creativity and assisted them to engage with their learning material when in a group setting.

Recommendations for further studies: involve teachers from grades 4 to 6 and introduce their learners to concept cartoons. The types of talk used by the teacher should be evaluated while the learners are busy with the concept cartoons and fully engaged in the problem-solving process. Teachers should be introduced to and assisted in using exploratory talk and implementing it while their learners use concept cartoons. Liang and Fung (2020) stated that teachers should teach their learners to use exploratory talk when engaging in group discussions. Concept cartoons are an effective teaching method that advances learners' social skills, which is a necessity in the reasoning process, creating opportunity for collaboration to take place and encouraging learners to engage in joint discussion and problem solving (Yin Yin, Wai Bing, Yusof, & Zakariya, 2016). Teachers should be able to create their own concept cartoons based on the misconceptions their learners experience in certain mathematical topics. Learners should then be divided into groups, and the teacher should encourage them to use exploratory talk when working with the concept cartoons. Teachers should be allowed to repeat these sessions and then be interviewed on their experiences of using concept cartoons and exploratory talk in the classroom, and how their learners' reasoning, understanding of mathematics, problem-solving behaviours and knowledge have changed after the learners used concept cartoons and received guidelines on using exploratory talk in their groups. Learners could also be interviewed on their experiences of the new method of doing group work and of using concept cartoons and exploratory talk when discussing mathematical topics. Concept cartoons and exploratory talk can also be used in whole-class discussions of certain mathematical topics. Concept cartoons can be used to address misconceptions about the teacher's choice of topic, while the teacher guides learners to start using exploratory talk. Concept cartoons enable classroom teachers to monitor their learners' understanding and ideas to avoid the same misconceptions recurring (Yin Yin, Wai Bing, Yusof, & Zakariya, 2016).

The teacher should take on a facilitating role, rather than just teaching the topic. Teachers can be interviewed on how their teaching was influenced by this process, and what they have noticed regarding how their learners conducted themselves and engaged with their groups. Tests can be written to determine the learners' results.

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IMPLEMENTING AN ADAPTED LESSON STUDY AS AN APPROACH TO IMPROVE MATHEMATICS TEACHERS' CONTENT KNOWLEDGE IN GEOMETRY OF THE CIRCLE

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This study explores teachers' experiences regarding the implementation of an adapted Lesson Study as an approach to improve teacher content knowledge in circle geometry. It discusses significant findings derived from a bigger study on teachers' understanding of lesson study after implementing the first two stages of lesson study, diagnostic analysis and lesson planning process through qualitative data. The researcher collected data through semi-structured interviews, document analysis, field notes and lesson observation. Analysis of the data generated suggests that the use of instructional materials during lesson preparation has a huge impact on the content knowledge of teachers. However, participants identified time as one aspect that may impede the implementation of lesson study effectively. The main findings of this study suggest that contrary to the authority initiated in teacher development programmes, lesson study appears is an ideal professional development programme that enhances the content knowledge of mathematics teachers in circle geometry in the North West province of South Africa.

Key words: Mathematics, Euclidian geometry, circle geometry, professional development, lesson study

Introduction

Teacher competence in Euclidian geometry has been a severe challenge supporting the call for the continuous professional development of South African mathematics teachers. Lesson study is a platform where teachers work with others to evaluate and criticise one another's teaching approaches as part of a process in gradually improving their teaching methods (Misnasanti, Dien, and Azizah, 2018). It is a popular professional development strategy among Japanese teachers, in which they perform a systematic critique of their educational practices by examining their lessons in detail (Fernandez, 2002; Saito & Atencio, 2013). The remarkable academic achievement of Japanese learners in TIMSS has been attributed to the practice of lesson study (Pang & Ling, 2012; Stigler & Hiebert, 2009; Ono & Stols, 2016). In this article we therefore argue that, based on its effectiveness in Japan, lesson study could assist South African mathematics teachers if implemented as a professional development strategy. We further argue that the use of instructional materials during lesson planning generates benefits on the content knowledge of teachers in circle geometry.

This article begins with a discussion of curriculum modification in the South African education system that came with the re-introduction of Euclidian geometry and the implementation of an adapted lesson study approach to improve the content knowledge of mathematics teachers in circle geometry. The study draws on the results of a research project conducted with four

mathematics teachers from four secondary schools in the North West Province of South Africa. The article argues that based on its effectiveness in Japan, lesson study could be a viable option for ongoing training for teachers where lesson planning is done from an informed position and the use of instructional materials appears to have a huge impact on the content knowledge of teachers.

The current article seeks to explore the teachers' experiences of the implementation of the first two stage of lesson study process, diagnostic analysis and the lesson preparation. The article addresses the following question:

- How do teachers give meaning to the concept of lesson study during the lesson planning processed?

Curriculum and Assessment Policy Statement (2011) and the re-introduction of Euclidian Geometry

The advent of South African democracy in 1994 resulted in the South African education system undergoing a number of curriculum modifications commencing from the 2002 National Curriculum Statement (NCS), followed by the 2009 Revised National Curriculum (RNCS), and finally Curriculum and Assessment Policy Statement (CAPS) that was inaugurated in 2012. All of these have brought many instructional challenges to teachers, in particular secondary school mathematics Euclidian geometry. Euclidian geometry is just but one of the assessment standards which was optional from 2006 to 2011 but was brought back in the CAPS in 2012. Riizo (2016) argues that Euclidian geometry is one of the hardest areas of mathematics for teachers as much as it is also the most difficult to learn for the students. Chimuka (2017) asserts that circle geometry, which is part of Euclidian geometry, is a topic that was previously optional. This was made optional based on the realisation that it is highly problematic in secondary school mathematics. Dhlamini (2012) equally underscores the point that even though teachers had studied Euclidian geometry in high school and at a university level, they found this section challenging.

Taylor (2013) indicates that in the case of CAPS, there is a risk that in the absence of solid subject knowledge on the part of the teachers, the strategies outlined in the policy document could be misinterpreted as learning outcomes, obscuring the knowledge capacities they strive to develop. The success of teaching geometry is contingent on the teacher's content knowledge of this construct and how to teach it properly (Jones, 2000). However, Siyepu and Mtonjeni (2014) indicate that in South Africa, central to the learning of Euclidian geometry is memorisation, which according to Masilo (2018: 19), leads to passive rote learning where students' problem-solving abilities are not honed for success in the classroom or in real life. Kanyongo and Brown (2013: 107) add that subject knowledge is an adequate proxy for teacher quality, which can influence student achievements in significant ways.

The inclusion of Euclidian geometry in the CAPS (2011) constituted a significant shift for both teachers and learners, which warrants that teachers receive correspondingly high-quality continuing professional development in this specific topic. According to Bridge News (2017), CPTD ensures that teachers maintain a strong knowledge of the discipline by keeping up to speed on latest research on how children learn, developing technologies in teaching, and new

instructional materials. According to Darling- Hammond, Hyler, Gardener, and Espinoza (2017), a comprehensive and efficiently executed PD scaffolds an integrated teaching and learning process that helps students gain the insight, techniques, and capacities they need to succeed in this dynamic world.

To ensure a coherent system that supports teachers across the entire professional continuum, PD should cultivate teachers into reflexive practitioners, affording them time to discuss with colleagues the successes, challenges and strategies for classroom practice. Scholars further indicate the standard of education improves when teachers engage cooperatively to review their classroom instruction (Achinstein, 2002; Grossman, Wineburg, & Woolworth, 2001; Gutierrez, 1996; Louis & Kruse, 1995; McLaughlin & Talbert, 2001; Westheimer, 1998; Witziers, Sleegers, & Imants, 1999). However, Lam & Lam 2002 indicate that CPTD programmes that were meant to bring teachers out of isolation did not yield any meaningful results. Similarly, Burney (2004) asserts that while teachers were given time to interact in groups through workshops and conferences to analyse their instructional practices in order to improve, putting what they learnt into practice has always been a struggle.

Joyce & Showers, (1982) add that basic practices such as peer dialogue, observation, and rigorous assessment should all be included in such a joint effort for it to be beneficial. Ajani (2018) suggests teachers ought to be refreshed and expanded on a periodic basis in order to strengthen their professional abilities of teaching methods in their content knowledge.

Researchers also emphasise lack of grasp of mathematics transference and adoption (Reddy, 2004, Ross & Bruce, 2005, Angelo & Cross, 1993). In line with the aforementioned, a survey of South African literature indicates that there is still a gap between substantial content and pedagogical competence (Reddy, 2004). Other international research highlights comparable concerns, implying that the issue is global sore (Gess-Newsome et al., 2019; Erwin & Rustaman, 2017; Graven, 2004).

In response to the concerns raised by researchers in mathematics education, the Japanese devised a corrective approach to effectively grasp mathematical concept transfer and adoption, consequently enhancing mathematics classroom instruction. Lesson study in the Japanese context has been successful in ongoing teacher professional development (Ermeling, & Graff-Ermeling 2016; Nami et al., 2016; & Verhoef et al., 2015). However, Ono & Stols (2016) indicate that lesson study is not an overnight solution for educational problems – educational challenges call for thoughtful engagement.

What is Lesson Study?

Lesson study is a platform where teachers work with others to evaluate and criticise one another's teaching approaches as part of a process to gradually improve their teaching methods (Misnasanti, Dien & Azizah, 2018). It is a professional development method wherein teachers prepare, teach, watch, analyse, and adjust live class lessons together (Maeda & Ono, 2019). Along similar lines, Ono & Stols (2016) define lesson study as a professional development model that is informed by what happens in the classroom in which involved educators collaborate to prepare and present the lesson, thereafter evaluate and modify it.

A team of three to five teachers, generally from the same grade level, gathers on a constant basis to explore a research lesson intended to improve student outcomes (Fernandez, Cannon & Chokshi, 2003; Puchner & Taylor, 2006; Cheung & Wong, 2014). The work group collaborates to establish a curriculum objective within a content domain and establish improvement goals for their learners (Saito & Atencio, 2013; Puchner & Taylor, 2006). Lesson study, according to the National Science Education Standards (NSES), emphasises the notion of teachers improving their topic or content knowledge, which must be at the heart of many of these professional development initiatives (NRC, 1996). Scholars emphasise that a well-organised development programme could assist new teachers integrate more quickly into the institutional cultural setting (Helms-Lorenz, Van de Grift & Maulana (2016). Coenders & Verhoef (2018) assert that lesson study can indeed be part of one such professional development programme. Da Ponte et al, 2017; & Ni Shuilleabhain, (2017) are also convinced that the lesson study approach is *the* programme for inexperienced teachers.

Lesson study process

The stages of cycle lesson study process differs from one context to the other. Based on the context implemented, one cycle comprises at least four and at most eight stages (Fernandez & Chokshi 2002, Lewis 2002, Stepanek, Appel, Leong, Mangan, & Mitchel, 2007; Stigler & Hiebert 1999, Weeks & Stepanek 2001, Yoshida 1999, Fujii, 2014).

Within a South African context, the University of Pretoria has formally established a Departmental Focus Area in the Department of Science, Mathematics and Technology Education (SMTE) dedicated to the Lesson Study. One cycle at UP involves a five step process starting with diagnostic analysis and ending with lesson improvement (www.up.ac.za/lesson-study/article/2725780/lesson-study).

The diagnostic analysis is a significant characteristic of the South African Lesson study, which was inspired by a loophole in the newest edition where there was no indication of how the challenging topic/concept was recognised. Lim, Teh & Chiew (2017) argue that a comprehensive analysis of the content where teachers are supposed to have a much firmer understanding of the content related to the topic involved even before conducting the scheduled lesson is nearly like conquering half the problem. Within the backdrop of South Africa, the diagnostic analysis involves review of instructional material to identify the inconsistencies and misconceptions made by students on a particular topic.

Theoretical Framework

Vygotsky's social constructivist theory, supported by Wenger's social learning theory and Lin Lo's variation theory, provided a framework for the implementation of lesson study process as a means of improving teacher professional knowledge and growth in four schools in the North West province. Vygotsky and Wengers' theories were employed as they both acknowledge the importance of teacher collaboration and these provided the researcher with insights from slightly different perspectives. Variation theory was also used as it acknowledges the centrality of content and provides teachers with advice on what content to choose and how to handle it in order to help pupils learn.

Walqui (2006: 160) lists five premises as the key principles of Vygotsky's SCT, however only two of them (social interaction and the Zone of Proximal Development (ZPD)) are explored in this study. The two concepts were chosen because they complement each other in terms of the importance of human engagement in growth and learning, and because optimal learning takes place in the Zone of Proximal Development (Vygotsky, 1896).

Vygotsky (1978) argues that effective learning occurs when two or more individuals with varying degrees of competence and knowledge interact socially. This implies that teachers acquire knowledge over time while collaborating with more experienced others. Verenikina (2010) concludes that social constructivism emphasises the importance of social engagement in development and learning, with effective learning prevailing within the Zone of Proximal Development (Vygotsky, 1986). Kadri, Roth, Gil and Mateus (2017) define ZPD as a one-way motion in which an inexperienced individual is supported in learning a culturally relevant expertise. The learning individual is said to have developed when the individual can demonstrate this knowledge without support.

According to Perry, Lewis and Akiba (2002), one of the key pathways to improvement is strong collegial networks. Teachers can build communities of practice centred on planning and teaching using the Lesson Study technique. In such communities, teachers develop, integrate, share, and improve their understanding of the lesson (Leskie; 2009). Campbell, Osmond-Johnson, Faubert, Zeichner, Hobbs-Johnson (2017), found lesson study to be a collaborative learning experience valued by educators. Teachers' increased eagerness to engage in targeted discussion about particular areas of teaching and learning promotes such engagement (Rock and Wilson 2005), which has a significant impact on participants' expertise, abilities, and practice.

Content as well as how to handle it should not be chosen at random, but rather thoughtfully planned to accomplish desirable learning goals. Marton & Tsui (2004) outline the three different objects of learning as follows:

- Intended
- Enacted and
- Lived object of learning

The focus of this investigation was solely on the intended learning object. A precise teaching process and the desired object of learning (Lo; 2012: 129) should drive the lesson. The desired object of learning is defined by Marton and Tsui (2004) as whatever the teacher wants the students to learn, and this intent is reflected in the choice, arrangement, and preparation of teaching materials.

The scope of the teacher's expertise and experience defines the desired object of learning. This, according to Kullberg, Kempe & Marton (2017), is the teacher's specific aim and intent for the students' learning. Consistent with lesson study, before the teachers can collaboratively engage in lesson planning, they carefully study instructional materials and curriculum, which in Japan is called *Kyouzai kenkyuu* (Fuji, 2016). Scholars agree that one critical feature of lesson study

is that *kyouzai kenkyuu* or the study of materials for teaching requires a substantial amount of time (Fernandez, Cannon, and Chokshi (2003); Takahashi and McDougal 2016; Watanabe, Takahashi and Yoshida (2008). Through *kyouzai kenkyuu*, teachers acquire a better understanding of mathematics and student reasoning (Lewis, Perry, & Friedkin, 2011; Takahashi, Watanabe, Yoshida, & Wand-Iverson, 2005).

While preparing the lesson, teachers anticipate how students are likely to respond to different activities and this necessitates reflection and re-evaluation of their teaching techniques (Coenders & Verhoef, 2018). Watanabe, Takahashi, & Yoshida (2008: 133) contend that through *kyouzai kenkyuu* lesson study leads to the strengthening of daily teaching.

Research Methodology

Qualitative research approach was used in order to understand and interpret the perspectives of participants concerning the implementation of the adapted lesson study as a strategy designed to enhance teacher content knowledge in circle geometry (Brikci & Green, 2007; Denzin, & Lincoln, 2000; Fouche, & Delport, 2002). This study involved a case study methodology, which included a full description of the place and individuals, followed by data analysis for emerging themes (Creswell, 2014). The choice was driven by the assumption that reality is constructed socially rather than objectively. Gall, Gall & Borg (2005) argue that through case studies, the real-life experience of participants in the research project is unveiled.

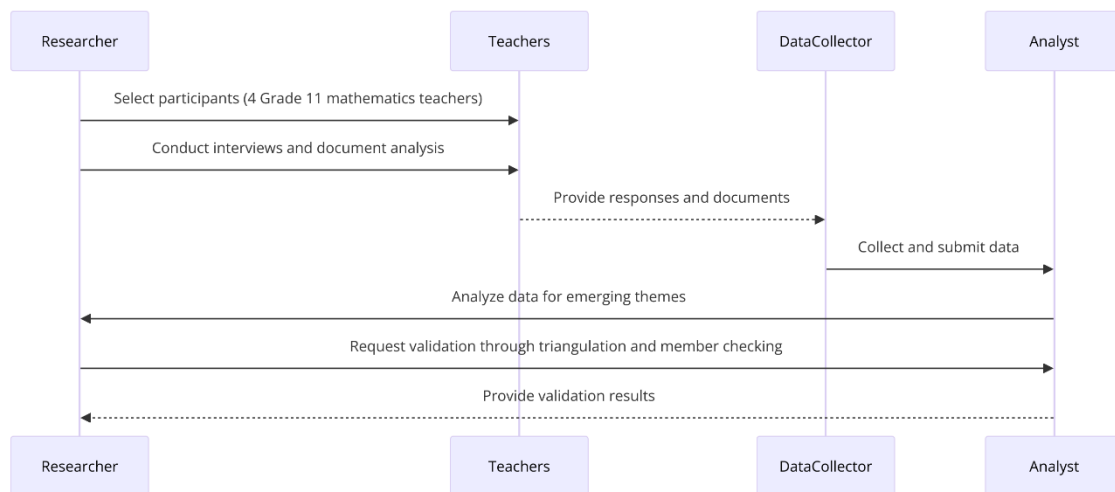
The case study's main strength is that it enables the researcher to investigate the phenomena comprehensively in its naturalistic environment, employing a variety of sources and approaches in generating data, such as interviews, document analysis, observation, preserved documents, and journals (Springer, 2010: 406). The main data collection methods used in this study were document analysis and interviews.

Research Design

A qualitative research design is used in this study. According to McMillan and Schumacher (2006; 315), when the investigation is on social issues, a qualitative research design is particularly useful. This research was conducted with four Grade 11 mathematics teachers in the Further Education and Training (FET) phase from the selected four schools in Ngaka Modiri Molema district, North West province. The sample size was too small but the number is supported by Fernandez et al., (2003); Puchner & Taylor, (2006); Cheung & Wong, (2014) who argue that a team of three to five teachers, generally from the same grade, participates in lesson studies. These teachers get together on a regular basis to examine a research lesson that enables students obtain improved results.

The design of the study reported in the diagram below, Figure 1, a sequence used by the researchers to achieve the research goals:

Figure 1. Research Design sequence diagram



The sequence diagram above illustrates the qualitative research methodology used to study the implementation of the adapted lesson study aimed at enhancing teacher content knowledge in circle geometry. The researcher begins by selecting participants, specifically four Grade 11 mathematics teachers. Following this, the researcher conducts interviews and document analysis to gather data. The teachers provide their responses and documents, which are then collected and submitted by a data collector. The analyst receives this data and analyses it for emerging themes. To ensure the credibility of the findings, the researcher requests validation through triangulation and member checking. Finally, the analyst provides the validation results back to the researcher. The profile of participants is summarised in the table below:

Table 1. Profile of participants.

Teacher	Gender	Rank	Experience	Qualification
WH	Female	Departmental Head	32	Bachelor's' Degree
NT	Female	Departmental Head	25	Master's Degree
MO	Male	Post Level 1	6	B. Ed
MM	Male	Post Level 1	3	BSc (Computer Science) plus PGCE

Implementation of lesson study

Underperformance in Euclidian geometry, notably circle geometry, which is offered in Grade 11, prompted this research. Grade 11 geometry forms the foundation for Grade 12 Euclidian geometry and the goal of doing a customised lesson study was to establish a teacher development platform that would improve teachers' subject knowledge of circle geometry.

The findings of the diagnostic reports helped in selecting which theorem to prepare for the lesson study implementation. Mathematics CAPS document specifies the topics that must be addressed in each level during the academic year, in this instance Grades 10 to 12. The four participants, together with the researcher, all had both soft and hard copies of the 2016 to 2018 mathematics diagnostic reports and the mathematics CAPS document.

The diagnostic reports helped to identify the misconceptions and errors made by learners on Euclidian geometry (circle geometry and geometry of the tangent) and formed the first stage in the of implementation process which is identifying the problem (Ono & Stols, 2016; 14). The findings of the diagnostic on circle geometry the same common errors and misconceptions throughout 2016 – 2018 are presented in Table 2 below.

Table 2. Findings of the Diagnostic Reports (2016 – 2018)

Year	2016	2017	2018
Common Errors and Misconceptions	• Generally, candidates either lost marks for incorrect or incomplete reasons or for naming angles incorrectly • In Q 8.1.1, candidates failed to state which lines were parallel when providing a reason. A number of candidates assumed that SR was a tangent and gave the tan-chord theorem as a reason	• Generally, candidates either lost marks for incorrect or incomplete reasons or for naming angles incorrectly • In Q 8.1, some candidates still omit to state the parallel lines when dealing with corresponding angles, alternate angles or co-interior angles. Candidates use the whole $\angle C$ instead of $\angle C_2$. Some candidates assumed that F is the centre of the circle.	• In Q 8.1.1 – 8.1.4, candidates either lost marks for incorrect or incomplete reasons or for naming angles incorrectly. • In Q 8.1.2, some candidates failed to identify the angles in the semi-circle. Candidates failed to mention which lines are parallel when corresponding angles are equal

These findings of the diagnostic reports helped in selecting which theorem to prepare for the lesson study implementation. There are seven theorems in Grade 11, five of them are geometry of the circle, and two are tangents (DBE Mathematics CAPS 2011: 39). The topic chosen for the lesson preparation was on Theorem 1 (*the line drawn from the centre of a circle perpendicular to the chord, bisects the chord*) from Grade 11 circle geometry. Theorem 1 was chosen, as it was the first to be taught in Grade 11 according to the Annual Teaching Plan. The Annual Teaching Plan is informed by CAPS document and it outlines all the topics that must be covered from Grade 10 to 12.

Collaborative lesson planning

After the study of the instructional materials, the researcher developed the lesson study-planning document together with the lesson study observation sheet guided by lesson study experts (Coe, 2015, Ono & Stols, 2016). These lesson study-planning documents were introduced to the participants on 23 January 2020.

Paulsen (2009: 254) indicates that proper preparation is a cornerstone of lesson study process. Fuji (2016) complements this by stressing that lesson planning is the bedrock of lesson study. The analysis of students' background knowledge and challenges, particularly in ideas relevant to the topic, constitute an essential part of this planning (Da Ponte et al, 2017). The two documents, diagnostic reports and Mathematics CAPS documents helped in identifying the content for teaching. This is in line with Lin Lo (2012: 19) who argues that we cannot talk about learning without first talking of what to learn. The “what” of learning is referred to as

the object of learning or content to be dealt with (Lin Lo, 2012: 19). Runesson (2005) argues that content, as well as how to deal with it, should be carefully planned with the goal of accomplishing specific instructional objectives.

The meeting of the lesson planning was held on the 7th February 2020 at 14:00 at one of the participants' school. The intention was that the implementation of this stage should align with the time when Euclidian geometry is taught in Grade 11 as per Annual Teaching Plan (ATP), commonly known as syllabus. Each participant was given data generating documents to direct him or her throughout the planning process. The lesson plan was done on Theorem 1, which forms part of those that are examinable in Grade 12. The lesson planning session ended at 17:00. After the completion of the lesson plan that took almost 3 hours, a semi-structured one-on-one interview was held with all the four participants. The process of the lesson planning was video recorded.

Data Generation and Analysis

The main methods of data generation for this study was document analysis, interviews, observations, research lesson documents (descriptions of study lesson, observation focus forms, post-lesson discussion forms). If the researcher uses multiple data collection instrument as indicated by Esterberg (2002: 176), their analysis and conclusions are more likely to be stronger than when they rely on only one source of evidence. However, the main data generation tools for this article were document analysis, field notes and semi-structured interviews. Following the recommendations by Creswell (2014: 248), the participant's responses were coded combining both predetermined and emerging codes.

Because the qualitative researcher is frequently regarded as a research instrument, they must guarantee that the data they report is accurate, not oversimplified or misconstrued (Lewis, 2009: 7). Guba (1981) suggests that for ensuring credibility, raw data items should be collected. This was addressed through recording video for all discussion and interviews. The recordings were uploaded on the DVDs and thereafter converted word by word into the written form and saved electronically. All the documents used in this study were compiled into a manuscript in a bid to ensure the trustworthiness of this study.

Ethical considerations

The North West Department of Education, Tshwane University of Technology and the principals of the participants gave prior permission to conduct this research. In addition, all the participants were informed of the research and its purpose and their participation in the interviews was voluntary. The participants further gave permission to video-record the interviews. Confidentiality and anonymity were maintained at all times and pseudonyms were used to protect the interviewees.

Findings and Discussion

The findings are presented based on the themes derived from the research questions. The themes are as follows:

- Implementation of the instructional materials (in this case diagnostic reports and mathematics CAPS documents)
- Relevance for lesson planning

Implementation of the instructional materials

The first stage of lesson study, diagnostic analysis, is an important one in the lesson study but as to the implementation, most teachers do not use the diagnostic reports and other instructional materials to inform their lesson plans. MO and MN indicated that at the beginning of the year, they get printed reports at their schools, keep them in file for use when external monitors visit the school. MN and MO further indicated that their lesson planning is informed by textbooks, ATPs and sometimes CAPS document.

Although WH uses the diagnostic report during lesson planning and assessment, she indicated:

I find it strange that the analysts' findings are the same every year. All of the three years' reports indicate, "Generally, candidates either lost marks for incorrect or incomplete reasons or for naming angles incorrectly." One is starting to doubt if indeed schools are making use of the reports to make sure that the same errors or misconceptions are not committed again.

The vignette above suggests all participants have copies of the instructional materials and despite the rich information contained in the diagnostic reports and other materials, these are rarely used to inform lesson planning and assessment hence the same errors and misconceptions committed by learners on circle geometry each year. These support the findings by Fung (2000), Fuji (2015) and Quaresma et al, (2018). For example, Fuji, (2015) contends that whenever teachers had to verify the purpose of a lesson within the larger curriculum, they rarely turned to the National Course of Study (NCS).

MN and MO further indicated that the textbook often informs their lesson planning, supporting the finding by Looney (2004) & Nojiyeza (2019) that textbooks are frequently used to bridge the gap between instructional aim and implementation, a matter that Quaresma et al, (2018) strongly criticise, citing that the use of textbooks to explicate how to teach is actually a taboo.

Relevance of instructional materials during lesson planning

Participants in this study agree that the study of diagnostic reports and other instructional materials such as CAPS document and examination guidelines inform the lesson planning. NT submitted that it was only through the planned session of diagnostic analysis that he realised the critical value of examiner's report to the crafting of each lesson. NT further adds:

Although I must indicate that going through the diagnostic analysis indeed took a lot of time but yes, it was worth it. The way we interacted with the reports and the CAPS document was really a revelation especially to us who should monitor curriculum implementation of mathematics.

Participants further agreed that the diagnostic reports also provide suggestions for improvement on each error or misconception identified and all indicate those aspects that could have been addressed by learners.

The above indicates the usefulness of studying the instructional materials first before planning any lesson, endorsing previous findings that the key component of lesson study is lesson preparation (Fuji, 2016). This equally demonstrates the efficacy of the thorough examination of educational content and teaching resources, which in Japan is called *Kyouzai kenkyuu*. Scholars agree that *Kyouzai kenkyuu* assists teachers in acquiring knowledge and understanding of mathematics and appreciating the reasoning that learners engage (Lewis, Perry, & Friedkin, 2011; Takahashi, Watanabe, Yoshida, & Wand-Iverson, 2005).

NT indicated that that going through the diagnostic analysis indeed took a lot of time but it was worth the effort. This endorses findings by various scholars that one critical feature of lesson study is the substantial quantity of time used on *kyouzai kenkyuu* or “study of materials for teaching” (Fernandez et al. 2003; Lewis, Perry and Hurd 2009; Takahashi and McDougal 2016; Watanabe et al., 2008). Watanabe et al. (2008: 133) confirm that lesson study adds to the enhancement of daily teaching specifically through study of instructional materials. It is therefore logical to recommend that the study of instructional material should form the basis of every lesson preparation.

Collaborative lesson planning of Lesson Study

The researcher and the participants held lesson-planning meeting on the 7th February 2020 from 14:00 to 17:00 hours. Participants could not believe that just one lesson plan for a single session could take such a time and realised the importance of content when planning the lesson and not just content but the importance of planning with the aim of addressing learner errors and misconceptions. NT submitted the following:

The process of this lesson study planning sessions was informed by the mathematics diagnostic reports and we were addressing key aspects of the content as identified in the reports... I have realised that if we can get the content right from the planning stage, the rest will be much easier to manage.

Participants realised that when preparing the lesson, the focus should be on the learners’ errors and misconceptions. Learners’ prior knowledge and challenges, particularly in concepts linked to the topic, constitute a crucial part of this planning (Quaresma et al., 2018)

All participants agreed that lesson study is an effective teacher professional development platform. For example, MO said:

Lesson study lesson is the best teacher professional development programme that I have ever come across. All of us contribute to the lesson planning.

The above meaning suggest that lesson study is a collaborative teacher professional development session where all participants are involved in the study of materials that should inform the lesson planning which supported previous studies (Dudley, 2013; Ono & Stols, 2016; Takahashi & McDougal, 2016: 513). As advocated by Lewis (2002), the processes of

lesson study allow teachers to create clear, well-directed lesson plans that have been shown to promote student learning. Misnasanti et al. (2018) indicate that the purpose of lesson study is to increase capacity, competence, and understanding to improve teaching and learning across a wide range of subjects and fields, not just to generate well-designed lessons.

Furthermore, findings show that the availability of experienced mathematics teachers or departmental heads who are mathematics specialists is seminal in the implementation of lesson study in order to mentor and coach inexperienced teachers. Van Driel et al (1998) indicate that the contribution of years of practice made by competent teacher yields strengthened PCK knowledge, beliefs and abilities in terms of teaching and learning certain topics. The impact of the experienced teacher helping the inexperienced one points to the manifestation of Vygotsky's Zone of Proximal Development (ZPD).

Conclusion

Various scholars have recommended that improvement of learner performance in mathematics demand that South Africa examine different angles in the way mathematics or different concepts related to mathematics are taught. Kösa (2016) argues that the effective teaching of mathematics, especially Euclidian geometry demands that mathematics teachers explore different strategies to generate deep learner understanding. Masilo (2018) concludes that the implementation of inquiry-based learning culminates in thorough, realistic, analytical, self-driven learning that is likely to yield competence in Euclidean geometry.

This study sought to implement an adapted lesson study, which provides teachers with a platform to explore their instructional practices for the identification of what features are key for students' learning. Participants deemed lesson study an effective instructional tool that strengthens their content knowledge not just in circle geometry but also in all other difficult areas in the mathematics classroom. Diagnostic analysis utilising diagnostic reports is a beneficial process, which allows teachers to plan lessons from a position of knowledge. Diagnostic analysis is useful not only for developing learner-centred approaches, but also for assisting teachers in identifying the key elements of each programme of study.

Collaboration within all the stages of lesson study cycle proved a valuable attribute of lesson study. South African mathematics teachers are under so much pressure to turn the situation around in terms of learner performance in mathematics and that can only be realised provided practitioners are well developed in curriculum and instructional expertise. In addition, teachers particularly those just entering the teaching system, need to be exposed to on-job training programmes that accentuate content, teaching methodology and ways of strengthening student achievement, and this should be part of the school culture.

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INTEGRATING LITERATURE IN TEACHING FRACTIONS: A QUALITATIVE STUDY OF GRADE 7 MATHEMATICS TEACHERS IN SOUTH AFRICA

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Introduction

The implementation of the National Curriculum Statement in elementary schools has been a transformative step in shaping the educational landscape over the last three decades. This change, as highlighted by Denzin and Lincoln (2011), has brought an increased emphasis on the teaching of common fractions. This shift is significant, given the complex nature of instructing learners in fractions, a topic that remains a challenging area in mathematics education.

Common fractions, tracing their origins to ancient Egyptian mathematics around 1000 BCE, are more than just mathematical entities; they represent essential divisions of wholes, a concept critical to various mathematical practices. As students transition to the senior phase, notably by grade 7, a solid grasp of these principles becomes crucial. The importance of this understanding, as suggested by Sandelowski (2000), lies in its application in real-life scenarios and its role in reinforcing fundamental mathematical ideas.

Recent research, including the work of points to a trend where the instruction of fractions, particularly in grade 7, is sometimes downplayed (Sandelowski, 2000; Creswell & Miller, 2000). This neglect can significantly impact students' future mathematical proficiency and confidence. The comparative analysis by Addanki, Sabitha, and Venkataraman (2023) further confirms that fraction instruction often receives less attention compared to other mathematical topics, despite its significance in the curriculum. This situation is not necessarily due to an inherent difficulty in understanding mathematics but is often a reflection of inadequate instructional methods or common misconceptions, a notion supported by Orb, Eisenhauer, and Wynaden (2001).

The challenges in teaching fractions are also linked to the pedagogical approaches employed by teachers. Evans, Johnson and Whitfield (2017) argue that insufficient subject knowledge or pedagogical content expertise among teachers can hinder effective fraction instruction. This underscores the need for further research into teachers' perspectives and instructional approaches regarding fraction teaching.

In South Africa, the task of teaching fractions presents notable complexities. The 2019 Annual National Assessments report revealed that only a small percentage of grade 7 learners achieved satisfactory scores in mathematics, with fractions being a particularly challenging concept. This aligns with Nkosi and Mji, (2018)'s findings, which indicate a correlation between difficulties in understanding fractions and challenges in grasping more complex mathematical concepts. The poor performance of South African learners in international assessments, such as the TIMSS, further highlights the urgent need for improvement in this area.

The primary aim of this study is to investigate the strategies and motivations behind Grade 7 mathematics teachers' integration of literature into their fraction lessons. This involves exploring their pedagogical methods, understanding their motivations, and identifying the benefits and challenges of merging literature with fraction teaching. This study seeks to contribute to the field of mathematics education by highlighting the relationship between content knowledge and pedagogical approach, particularly in the context of Pedagogical Content Knowledge (PCK).

The study is guided by two main research questions: how teachers integrate literature in teaching fractions and what drives their objectives, and what benefits and challenges they identify in this approach. The objectives include exploring methods and techniques for incorporating literature, understanding teachers' motivations, and identifying perceived benefits and challenges in the current curriculum framework.

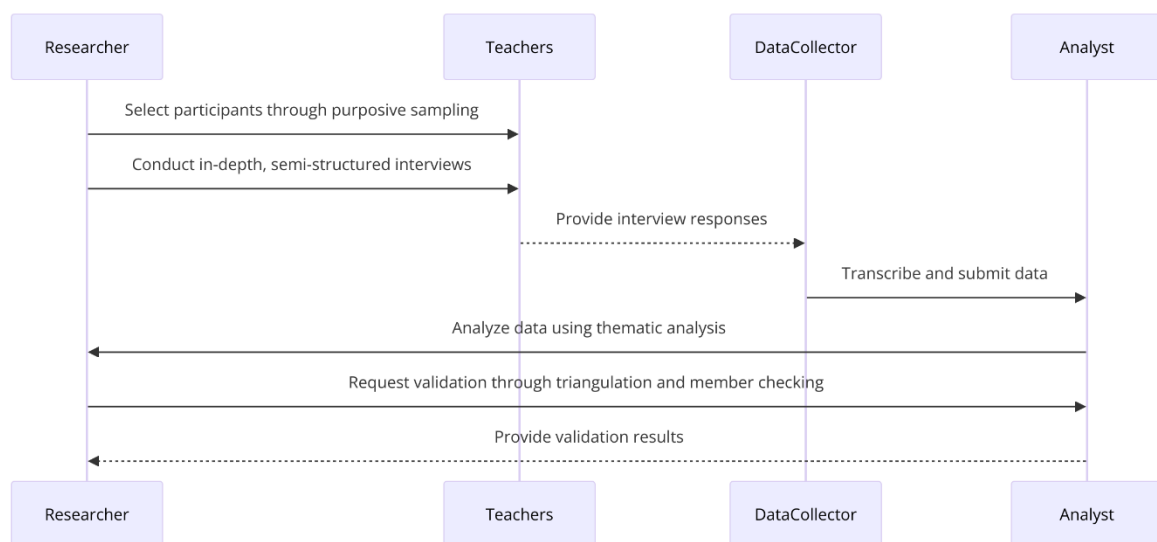
Method

The study exclusively employed a qualitative research method. This approach is particularly effective in gaining deep insights and understanding the complexities of human experiences and perceptions (Denzin & Lincoln, 2011). In this case, it allowed for an in-depth exploration of teachers' perspectives and strategies in integrating literature into fraction lessons (Addanki, Sabitha & Venkataraman, 2023).

Research Design

The research was designed as a qualitative descriptive study. This design is suitable for obtaining a detailed understanding of phenomena within their natural settings (Sandelowski, 2000). In this context, it enabled a thorough exploration of the pedagogical approaches of Grade 7 mathematics teachers. The figure below, Figure 1, provide a sequence diagram of the research plan:

Figure 1. Sequence diagram illustrating the study's research design (Source: Sepeng, 2024).



The sequence diagram above illustrates the qualitative research methodology employed to explore teachers' perspectives on integrating literature into Grade 7 fractions lessons. The researcher begins by selecting participants through purposive sampling, targeting teachers with relevant experience. In-depth, semi-structured interviews are conducted to gather detailed insights. The responses from the teachers are collected by a data collector and then transcribed and submitted to an analyst. The analyst performs thematic analysis on the transcriptions and provides the analysis results to the researcher. The researcher then requests validation through triangulation and member checking, ensuring the credibility and trustworthiness of the findings. Finally, the analyst provides the validation results back to the researcher.

Research Procedure

Participant Selection: Identify and select Grade 7 mathematics teachers through purposive sampling. The selection aimed at teachers who have experience in integrating literature into their mathematics instruction. **Data Collection:** Conduct in-depth, semi-structured interviews with the selected teachers. The interviews focused on their methods, motivations, and experiences in blending literature with fraction teaching. **Data Analysis:** Transcribe the interviews verbatim. Analyse the transcriptions using thematic analysis to identify key themes and patterns related to the study objectives.

Testing and Validation

Given the qualitative nature of the study, validation was ensured through methods like triangulation, member checking, and maintaining a comprehensive audit trail (Creswell & Miller, 2000). These techniques help in enhancing the credibility and trustworthiness of the qualitative findings.

Data Acquisition

Data was acquired through semi-structured interviews. Semi-structured interviews and focus groups were chosen for their ability to elicit detailed, context-specific information from the participants. This method allows for open-ended discussions that encourage participants to express their thoughts, experiences, and rationales related to integrating literature into Grade 7 Fractions instruction. These interviews were audio-recorded and then transcribed meticulously. The transcriptions were subsequently coded and analysed for themes and patterns relevant to the research questions.

Sampling Strategy

The population for this study comprises all Grade 7 mathematics teachers in the Greater Taung District. Given the specific focus on teachers' experiences with integrating literature into Fractions instruction, a purposive and stratified random sampling strategy is employed. The sample size consisted of 30 Grade 7 teachers, ensuring data richness without compromising the depth and diversity of experiences. Purposive sampling is utilised to deliberately select teachers who have actively integrated literature into their Fractions lessons, providing nuanced and contextually significant insights. On the other hand, stratified random sampling is employed to

ensure the inclusion of diverse perspectives by categorising the population based on factors such as years of teaching experience and school profiles.

Ethical Considerations

The preservation of ethical integrity in research is a vital aspect that underpins the legitimacy and acceptance of such research within the academic community. Consistent with this premise, the present investigation rigorously conformed to recognised ethical norms. Before collecting data, thorough clearance was rigorously sought and granted by the Postgraduate Committee's ethical committee at Sol Plaatjie University. Including this step served not just as a procedural need but also as a declaration of the study's dedication to carrying out research responsibly and ethically (Addanki, Sabitha, & Venkataraman, 2023).

The notion of informed consent played a significant role in the ethical issues. Each mathematics teacher participating in the study received a comprehensive informed consent form that clearly outlined the research's aims, methodology, and possible consequences. This procedural measure ascertained the participants' comprehensive comprehension, enabling them to make a well-informed decision about their engagement. Addanki, Sabitha, and Venkataraman (2023) highlight the significant importance of informed consent in research, emphasising the participant's entitlement to voluntary and well-informed involvement.

Furthermore, the identity of the participants was kept confidential during the duration of the research. To guarantee confidentiality, all identifying information was encoded or pseudonymised. According to Orb, Eisenhauer and Wynaden (2001), this approach protected participants' privacy and facilitated the sincere and transparent exchange of experiences, free from concerns of retaliation or prejudice.

Results and discussions

In this section, it is explained the results of research and at the same time is given the comprehensive discussion. Results can be presented in figures, graphs, tables and others that make the reader understand easily. The discussion can be made in several sub-sections.

Table 1. Teacher A responses to the interview questions

Line	Interviewer's Question	Teacher A's Verbatim Response
L12	The first question is on the background and teaching experience. Can you briefly describe your teaching experience and background in the teaching of mathematics as a teacher?	"I have 16 years of experience teaching mathematics. But did you study in school? Different grades. Started teaching mathematics, mathematical literacy, the grade 11s I think for three years. Then I went on to teach at middle school. We.

		Yes, great night and great eight I think also for three years then I thought great. 746. This is 16 years of experience. So, I can say. I taught mathematics for 16 years of my life."
L13	Well, you are one of those we refer to as veterans.	"Yeah, I."
L14	OK. Then the second question, thank you very much. The second question, how do you approach the teaching of fractions like the techniques or methods maybe? That you apply in your teaching.	"I usually use things that are relevant to the learners that I am teaching. Things like dividing Pizza or sharing candles, even sharing sweets amongst them, or even fruits or whatever. Sometimes I even. Like using measuring ingredients in a recipe, the simple recipe may be like making cookies at home, or something like that. That is what I normally do. I normally use methods that are relevant to the Greek that I am teaching. But using the examples that are relevant to the individuals that are."
L15	Thank you very much for your insightful information. I like this.	(No verbal response)
L16	Then the third one, like their experience with literature integration like I can see that you have previously integrated literature in your math. Lesson if so. Like, why did you decide to do it? What are? The benefits?	"I think for cultural relevance. Even for learners to understand or apply mathematics in their real-life situations, if I have to say that, or if I get your question right."
L17	Like the next question, can you describe a memorable lesson where you integrated literature? Maybe? How did you weave it into your lesson? Can you take me through that?	"I think yes. I have for instance. I said the literature concept into the introduction where I used story involving something that my learners like, something like pizza. You know, they like food. Like dividing a pizza clean, introduce the concept. Of fractions. And it was like. Relevant for them since they like pizza since they lack

		everything, they were all concentrating the visual and narrative elements of the story. Help the students grasp their idea. Of dividing a hole into equal parts." "They responded positively. To the use of literature. It was something that they liked, so they responded positively because the story was relevant to them. And that did challenge the literature provided the context. And it gave the real-world experience for mathematics. Concept. It also helped my students understand. The relevance and practicality of what they are learning. I think I am." "It provides a real-world context. For fractions. Making abstract math. Concepts more tangible and relatable. It also helps them to analyze mathematical problems within the context of a story. Which encourages critical. I am OK. It is fractions. It just helps liners or literature. It helped them to relate to their real-life situations."
L18	Thank you very much. Like I can see that you have like. That was question number four, OK? Like how the student have responded to that lesson of yours when you were integrating literature in your math lesson?	
L19	That is remarkably interesting. As you go deeper like now, I want to check the benefits and the challenges of you. So, I will be taking you through this lesson. So, it will be just a link, one question to the other. In your opinion, what are the potential benefits of using the tutor to treat fractions?	
L20	Yes, like storytelling, at least they can now remember when you talk about the fractions, especially the circular infractions, they now can relate to.	"Yes, please."
L21	So OK, this is. What we did yesterday, we were baking. We were measuring. Those are the fractions. OK, the 7th question.	"Yes, yes."
L22	Are there any challenges or drawbacks you encountered while using this literature?	"Yes, it was. Since I realized that it required an additional time. And it also. We also face constraints within the curricula. Striking balance covering. Essential mathematical content and

		incorporating literature can be challenging to both teachers and learners. Traditional methods of assessment may need to be adapted to evaluate students' understanding when literature is integrated into math lessons."
L23	Yes, thank you very much. Like this integration of the literature may take time like you have mentioned that the time constraint and it is not we are crashing. We are facing with the 8 P there is no time for this.	"Yes, it is time-consuming."
L24	Years and the language of one teacher talking about it. Our learners are a barrier to them. OK, like now have you attended any workshop like regarding the integrate they have?	"Yes, it is a barrier in. No, no. I have never been to any training."
L25	Have been size. That is the gap that I am trying to close because I am also a mathematics teacher. There has never been such so like when you come into the resources, they take a book. Are you using something for teaching literature?	(No verbal response)
L26	Yes, you may proceed. Sorry about that. Like the resources maybe when you are integrating this literature, are there any sources that you use maybe that you can recommend?	"Ohm, I think book stories. I I I. I am using books and stories to integrate literature into my math lessons. OK, but now there are no prescribed books, so I normally use Google for stories and math's manipulatives physical tools like. Action bars and circles can complement literature. By providing. By providing experience to reinforce the concepts introduced in stories."
L27	OK.	(No verbal response)

L28	Yes, like this one. I will repeat it for them because it appears here like you are saying: you have never undergone any training.	"I have never undergone any training or any professional development."
L29	OK, bye.	(No verbal response)

Table 2. Teacher B responses to the interview questions

Line	Interviewer's Question	Teacher B's Verbatim Response
L36	So if you are ready, I'm ready.	"OK so. I'm ready, man."
L37	OK. Thank you very much. I'll start with the first question. Can you briefly describe your teaching experience and background in the teaching of mathematics?	"Yeah."
L38	Generally not specifically on the fractions. By teaching mathematical mathematics in general.	"I'm thank you for that question, ma'am. I'm apparently I am. I have 12 years teaching experience name in primary education, but teaching mathematics? I only have something like one year, six months. Did you mathematics in general? Because of, you know, in the primary school it's a primary and shortage of teachers. So now we have to juggle around like you teach everything in primary school. So in mathematics is only one year, six months, but unfortunately I have done mathematics in my high school and in my speech level, so that's why I'm so competent with mathematics, especially the one of grade 7. Thank you so much."
L39	How do you approach the teaching of fractions? What techniques or methods do you often employ?	"Thank you, ma'am. E generally, when teaching functions, they are quite tricky. They're not difficult for kids. They are quite tricky, so I teach mathematics in grade 507. So one thing that they understand mostly is when you make everything practical. When she called by so meaning I'm whenever you are giving your problem on the chalkboard. If you don't understand it or they have a problem with it, try to make it practical. Like if it's a one fifth times $\frac{1}{5}$ or any other fortune, just try to make that on the box. Draw and share the pieces so that they can see. This is how this thing it it looks like when you see the one fifth because sometimes these kids, they do know things, but then in terms of expressing their own ideas on

		how to solve that, it's quite a problem. So normally I always. Do practicals. Whenever I teach fractions, or if ever there's something that you don't understand, maybe I can go and collect objects so that I can show them exactly what this is what I mean."
L40	Yeah, he thank you very much.	"Yeah. Thank you, ma'am."
L41	So in short, you need to everyday life.	"Yeah. Don't."
L42	OK.	"Just you hand. Yeah, don't just depend on the textbook or whatever material that you are using. Sometimes just try to and explore that. Teach them with everyday life things that they know by then it's quite difficult for them to write them because you know sometimes you know something. But then in terms of writing it, the problem now, so show them that even if in a real-life situation this is how these things have been done."
L43	OK, say food very much please.	"Yeah. They do that, he."
L44	Like now your experience with literature integration and moving to the next question like you have already covered this, but I want you to tell me like I can see that it previously you have integrated literature in your math lesson like you are relating with every day's life.	"OK."
L45	So how normally do you do it? Why did you decide maybe to relate to this topic with the literature?	"I'm. OK. Thank you so much for that question, ma'am. Unfortunately, in teaching there is what we call a local parent is the name Umm, it's sometimes. Sometimes you just have to forget who you are, but then just try to assist this little ones. Think of their future on their behalf, so that is why always when I go to a classroom. I always integrate my lessons with the everyday life, like the way they are living, the way they are doing things you cause

apparently. I'm. I was sampled for teams international side. The trend in mathematics and science study, meaning he didn't eat a national study whereby all the learners are going to write the same paper and on our web shops they have likes the. Perhaps like this like I've of how are we doing this in class? Sometimes we fail our learners in a way that. We only teach them for the sake of teaching them. So we have those ones who are much quiet, uh. Clever than other. So I'm there was this question whereby they need to estimate. So on the estimation some, they didn't get it and then someday just say normal, this is what we do on the everyday life when we are at the field shepherding the sheep or cows, we know that a cow is this much. And then the ship is this much so when you bring that with classroom, they don't know. You must do this this thing, so it's quite simple for them. So that is why I always I always integrate my lessons if they don't understand, I go to grade three grade four and go there apparatuses so that I can use them in my classroom. So now in short, I can just say I'm. Let's not let's just not follow the Lesson plan. Everything rather photo elephant playing, but if it doesn't work, think of something else that will add the bills that lesson plans so that the kid can understand what is, what is what is important here is that they keep need to understand what you are talking about and after teaching them they need to express their own views. Thank you, ma'am."

Contextual Learning

Implication: Teacher A utilizes real-world examples such as pizza, candles, and sweets in teaching fractions [L14], effectively bridging the gap between abstract mathematical concepts and students' everyday experiences. This approach enhances understanding and retention [L19]. Teacher B emphasizes practical learning in mathematics by integrating real-life situations into the teaching of fractions [L14]. This approach makes abstract concepts more tangible, aiding comprehension and retention.

Broader Impact: Emphasizes the importance of contextual learning in education, encouraging a shift towards teaching methodologies that are more relatable and engaging for students. Both teachers demonstrate how experiential learning can deepen understanding and kindle student interest in mathematics. Highlights the effectiveness of contextual learning in mathematics, fostering a deeper understanding by relating theoretical concepts to everyday experiences.

Both teachers emphasized the use of practical illustrations and narrative techniques in the instruction of fractions. Teacher A's utilization of common objects such as pizza or recipe ingredients is consistent with the research conducted by Smith and Smith (2021), which underscores the significance of contextual learning in the field of mathematics. Implementing this method helps in concretizing abstract ideas, a tactic endorsed by Jones, Smith, Lee and Johnson (2022), who discovered that pupils exhibit enhanced comprehension and memory when mathematics is taught using practical, real-world illustrations.

Cultural Relevance in Education

Implication: Teacher A highlights the significance of cultural relevance in education [L16], showing the importance of making learning materials relatable and meaningful to students' lives and experiences, thereby enhancing their engagement and motivation. Teacher B's methods, though not explicitly stated as culturally relevant, implicitly acknowledge the importance of making learning relatable by tying mathematical concepts to students' daily lives [L14].

Broader Impact: Indicates a growing recognition of the need for culturally responsive teaching methods (Lee, 2023), particularly in diverse educational environments. Emphasizes the need for education to be adaptable and reflective of the diverse backgrounds of students, enhancing engagement and comprehension.

Teacher A's focus on cultural relevance and Teacher B's incorporation of narratives align with Lee (2023)'s research, indicating that culturally responsive teaching approaches greatly improve student engagement and motivation. Teacher B's method of connecting mathematical ideas to students' everyday experiences is a tangible demonstration of cultural relevance in education. This aligns with the assertions made by Jones, Smith, Lee and Johnson (2022) and Garcia and Ortiz (2023) on the beneficial influence of relevant material on students' academic achievements.

Student Engagement and Response

Implication: The positive student response to literature-integrated methods used by Teacher A [L18] suggests that engaging students through storytelling and relatable contexts makes learning more enjoyable and meaningful. The integration of mathematics with practical, real-life examples as practiced by Teacher B likely results in increased student engagement and better understanding [L16].

Broader Impact: Highlights the need for educational research and policies that prioritise student engagement and consider diverse learning styles. Underscores the importance of adopting

engaging teaching methods that resonate with students' experiences, making education more relatable and impactful.

Challenges in Implementation [of Innovative Teaching Methods]

Implication: Teacher A identifies challenges such as time constraints and curriculum balance [L18], reflecting systemic issues where rigid structures and traditional assessment methods can hinder innovative teaching approaches. Teacher B's approach, focusing on practical applications and everyday life examples [L17] (Jones, Smith, Lee & Johnson, 2022), might face challenges in traditional educational settings that emphasize textbook-based learning.

Broader Impact: Suggests the need for systemic reforms in education that allow for more flexibility and creativity in teaching methods. Suggests a need for educational systems to evolve and support innovative teaching approaches that go beyond conventional methods.

The issues emphasised by the teachers, namely about limited time and the need for a balanced curriculum, demonstrate a prevalent worry in educational advancement. Hamilton and Richardson (2024) argue that inflexible educational frameworks and conventional methods of assessment might hinder the implementation of new pedagogical approaches. Therefore, it is essential to implement a comprehensive transformation, as proposed by Patel and Kumar (2023), in order to grant teachers more adaptability in their instructional methodologies.

Professional Development and Resource Gaps

Implication: Teacher A's absence of specific training in integrating literature into math teaching [L4], O'Connor and Fernandez (2022) point to a gap in professional development and available resources, potentially limiting the ability to implement innovative teaching strategies. Teacher B's improvisation and adaptation in teaching indicate a potential gap in professional development and resource availability [L19]. This gap may limit some teachers' ability to implement similar innovative strategies effectively.

Broader Impact: Indicates a broader need for targeted professional development programs and resources that support teachers in adopting new pedagogical approaches. Points to the necessity for ongoing professional development programs that provide teachers with the tools and skills to innovate in their teaching methods.

The absence of specialised training in incorporating literature into the teaching of mathematics for Teacher A underscores a deficiency in professional development. This conclusion aligns with the study conducted by O'Connor and Fernandez (2022), which advocates for the implementation of comprehensive professional development programmes for teachers. These programmes aim to educate teachers with the essential skills required to effectively use new teaching practices. If this gap is resolved, it might result in a more extensive use of efficient teaching techniques, such as those shown by Teachers A and B.

Assessment and Evaluation Challenges

Implication: Adapting traditional assessment methods to evaluate students' understanding effectively when integrating literature into math lessons is a significant challenge highlighted by Teacher A [L20]. It requires a rethinking of how student learning and comprehension are measured. Teacher B's adoption of practical and experiential teaching methods suggests a need for alternative assessment strategies that align with these teaching approaches (Thompson & White 2023).

Broader Impact: Calls for re-evaluating assessment strategies in education to accommodate diverse teaching methods and learning experiences. Calls for a re-evaluation of assessment techniques in education to accommodate diverse and interactive teaching methods, ensuring a comprehensive understanding of students' learning experiences.

Both teachers' experiences indicate the need of reassessing conventional evaluation techniques to accommodate varied and interactive ways of teaching. Thompson and White (2023) share the same views, supporting the use of different evaluation methodologies that are more compatible with creative teaching methods. These evaluations have the potential to provide an improved assessment of students' comprehension and involvement with the subject matter.

Answering the research questions

Research Question One: Integration of Literature in Teaching Fractions

Research Objective 1.1: Explore methods and techniques for incorporating literature into fraction lessons.

Findings: Teacher A uses real-world examples, such as dividing pizza or measuring recipe ingredients, to teach fractions. These examples are tailored to be relevant to the learners, suggesting a method of integrating literature that makes abstract concepts [L19] more tangible and relatable (Sandelowski, 2000). However, according to Teacher B, he weaves mathematical principles seamlessly into narratives, creating a unique and imaginative approach to fraction instruction (Creswell & Miller, 2000).

Research Objective 1.2: Understand the motivations behind integrating literature with fraction teaching.

Findings: Teacher A's approach is driven by the desire for cultural relevance [L16] and to make mathematics applicable to real-life situations, aligning with Smith and Smith (2021)'s findings on the importance of context in mathematics education. This indicates a pedagogical choice to enhance student engagement and understanding by linking mathematical concepts to familiar experiences. Teacher B's motivation for integrating literature with fraction teaching stems from a nuanced understanding of the profound impact that stories and narratives can have on the learning experience. Teacher B's use of stories and narratives reflects Evans, Johnson and Whitfield (2017)'s emphasis on the emotional and cognitive impact of literature in learning. Teacher B transforms fraction lessons into captivating narratives, fostering a connection

between the abstract nature of fractions and the emotional engagement derived from literary content.

Research Question Two: Benefits and Challenges of Merging Literature with Fraction Lessons

Research Objective 2.1: Identify the benefits of incorporating literature into fraction lessons.

Findings: The benefits identified by Teacher A include providing a real-world context for abstract mathematical concepts [L19], making these concepts more tangible, and helping students analyse mathematical problems within a narrative context. Teacher A's use of literature to contextualize mathematical concepts aligns with Denzin and Lincoln (2011)'s findings on enhancing student engagement and critical thinking. The findings suggest that incorporating literature into fraction lessons, as demonstrated by Teacher B, can enhance student engagement by tapping into their creativity and imagination. This approach deepens conceptual understanding and cultivates a love for learning that extends beyond the confines of traditional mathematical instruction. Teacher B's narrative approach is consistent with Orb, Eisenhauer, and Wynaden (2001)'s observations on the power of storytelling in deepening conceptual understanding.

Research Objective 2.2: Uncover challenges in merging literature with fraction teaching within the existing curriculum framework.

Findings: Teacher A points out several challenges, such as the additional time required for integrating literature into lessons and the constraints of the existing curriculum. Balancing essential mathematical content with integrating literature and adapting traditional assessment methods are also significant challenges, echoing Addanki, Sabitha, and Venkataraman (2023)'s concerns about balancing curriculum demands with innovative teaching methods. Teacher B grapples with the constraints of time. The traditional educational setup often demands adherence to fixed schedules, leaving limited room for incorporating innovative approaches. Teacher B finds it challenging to balance covering the required curriculum and dedicating time to more dynamic and engaging teaching methods. The necessity of adapting traditional assessment methods to accommodate these new approaches aligns with Nkosi and Mji (2018)'s research on the challenges of educational innovation.

Integrating literature into mathematics instruction, as shown by Teachers A and B, is a viable method for improving student involvement and comprehension of fractions. The problems revealed, however, indicate the existence of more extensive structural flaws in education that need attention. Subsequent studies might prioritise the creation of tactics to surmount these obstacles, thereby enabling the widespread use of cutting-edge pedagogical approaches.

Conclusion

This research aimed to investigate the strategies and motivations of Grade 7 maths teachers in South African schools when it pertains to incorporating literature into their fraction classes. The qualitative study technique yielded comprehensive insights into the educational strategies

used by the teachers, their underlying motives, and the perceived advantages and difficulties associated with this novel teaching strategy. The results indicate that instructors use diverse and innovative approaches to include literature into fraction instruction. The strategies used by the interviewed instructors include using real-life illustrations and contextual situations that strongly connect with students' own experiences. This method not only helps to make abstract ideas such as fractions more concrete and relevant, but also increases student involvement and enthusiasm for mathematics. The research emphasises the crucial significance of cultural relevance in education (Garcia & Ortiz, 2023), demonstrating that learning materials have a greater effect when they are relevant and meaningful to students' lives and experiences.

A crucial result of this study is the focus on the significance of Pedagogical Content Knowledge (PCK) in the successful teaching of fractions. The integration of literature into mathematics training exemplifies a pioneering method that merges subject expertise with teaching abilities, in accordance with the principles of Pedagogical Content Knowledge (PCK). This technique is crucial in cultivating a deeper comprehension of mathematical topics among learners. In addition to the advantages, the research also reveals difficulties such as limited time and the need to harmonise the requirements of the curriculum with inventive teaching methods. These problems emphasise the need of implementing systemic educational changes that facilitate adaptable and innovative teaching methodologies. The research suggests implementing focused professional development programmes and providing tools to assist instructors in embracing novel educational approaches. Moreover, it proposes a reassessment of evaluation methodologies to accommodate a wide range of different and dynamic instructional approaches.

This study makes a significant contribution to the area of mathematics education by emphasising the use of literature in teaching fractions. It emphasises the need of educational research and policies that give priority to student participation and consider various learning styles. The results indicate that using captivating teaching strategies that align with students' own encounters might render education more relevant and influential, thereby augmenting learning achievements.

This study provides opportunities for future research, namely in investigating the enduring effects of incorporating literature into mathematics teaching and analysing comparable approaches in other educational settings and age ranges. Additional research on the structural modifications required to facilitate these new instructional approaches would also be beneficial. In conclusion, this study highlights the importance of innovative and contextually relevant teaching methods in mathematics education. Through integration of literature into fraction lessons, teachers can significantly enhance students' understanding and appreciation of mathematical concepts, paving the way for more engaging and effective mathematics education.

Recommendations

The study reported in this article proposes multiple recommendations that may improve the teaching methods, develop policies, and guide future research in this field. The purpose of the

following recommendations is to tackle the issues that have been identified and take use of the advantages of creative teaching approaches:

- (i) Enact specific professional development initiatives that prioritise the use of cutting-edge teaching techniques, such as incorporating literature into mathematics lessons. It is advisable to arrange workshops and training sessions to provide teachers with practical skills and ways to integrate real-world examples and storytelling into their classes.
- (ii) Modify the mathematics curriculum to enhance adaptability, enabling teachers to integrate literature and real-life situations while maintaining the comprehensive covering of fundamental mathematical concepts.
- (iii) Promote a curriculum design that fosters a harmonious blend of conventional mathematics education and inventive, contextually-driven learning approaches.
- (iv) Create and disseminate educational resources that integrate mathematical principles with literary elements and cultural allusions, enhancing their relevance and appeal to pupils.
- (v) Allocating resources towards research to develop and assess these products, guaranteeing their efficacy in improving student comprehension and involvement.
- (vi) Policymakers in the field of education ought to weigh the advantages of incorporating literature into the teaching of mathematics and provide financial assistance and regulatory modifications to promote such projects.
- (vii) Policies should promote schools to engage in the exploration of alternative pedagogical approaches and ensure the provision of necessary resources and assistance.
- (viii) Create and execute assessment processes that are in line with cutting-edge teaching methods. These evaluations should test not just the comprehension of mathematical concepts but also the ability to apply them and think critically.
- (ix) Providing teachers with training on alternate assessment methodologies that align with literature-based and practical teaching practices.
- (x) Promote the engagement of parents and the community in educational endeavours, particularly those that establish connections between mathematical principles and practical situations and cultural backgrounds.
- (xi) Conduct community workshops and parent-teacher meetings to elucidate the advantages of literature-integrated teaching techniques and their potential to enhance children's learning at home.

If these recommendations were to be implemented, the teaching and learning of fractions is most likely to become more engaging, relevant, and effective, potentially leading to improved outcomes in student understanding and interest in the subject.

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IN-SERVICE HIGH SCHOOL MATHEMATICS TEACHERS' EXPERIENCES WITH THE BLENDED TEACHING APPROACH

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Teaching mathematics in the Fourth Industrial Revolution era allows in-service mathematics teachers to apply integrated learning strategies. This study investigates in-service high school mathematics teachers' experiences when using a blended teaching (BT) strategy to teach mathematics. Eight in-service teachers were purposively selected from four schools in the Gauteng Province's Sedibeng West District to participate in this qualitative case study. The community of inquiry model served as the study's analytical framework for examining the data. Semi-structured and focus group interviews were used to gather data on the in-service teachers' experiences of the use of BT to teach mathematics in high schools. The findings of the study were that the in-service mathematics teachers used a variety of technological tools such as GeoGebra (free downloadable software), YouTube videos, projectors and computers to supplement their everyday teaching strategies, and each of these technological tools brought the teachers new experiences. These findings contribute to the body of knowledge on the use of the BT approach for teaching mathematics in high schools. This study could be repeated using a bigger sample and teachers from different quintile levels, to achieve a broader understanding of in-service mathematics teachers' experiences of the use of BT. Future studies could focus on investigating the impact of the use of BT as a teaching strategy, and of its associated technological tools, in the teaching and learning of mathematics in schools.

Keywords: Blended teaching, teachers, in-service, community of inquiry

Introduction

Blended teaching (BT) is the blend or amalgamation of teaching modes that include the experience of online technology, including (but not limited to) text, cooperative learning, streaming video, audio, self-paced education and the online school room, to accomplish an educational objective (Cronje, 2020; Pule & Raxangana, 2024). In the context of this study, BT is a teaching strategy that includes a combination of face-to-face teaching approaches and the use of Information and Communications Technology (ICT) tools to support the teaching and learning of mathematics in high schools (Fitri & Zahari, 2019; McKenzie et al., 2022). Moreover, Fitri and Zahari (2019) asserted that 'blended' refers to a mixture of equipment and traditional teaching methods. 'Traditional teaching methods' refers to face-to-face instructional teaching methods that are mostly preferred and utilised currently, due to the direct interaction of in-service teachers with learners (Bates, 2018; Pule & Raxangana, 2024). In this study, the BT approach utilises different forms of e-learning, such as GeoGebra (free downloadable software) and YouTube videos, as well as the traditional approach to teaching. For example, Pule and Raxangana (2024) averred that to achieve an educational objective, BT combines various web-based technologies (eg audio, live online classroom, streaming video, interactive

board, etc). With this background in mind, this study investigates the in-service teacher's experiences with the BT approach to mathematics.

BT embodies the transformation of teaching and learning from the traditional approach to the integration of technology to teach mathematics (Mal & Adhya, 2020). An international study conducted in New York by Culbertson (2018) indicated that as schools moved to the BT approach to meet the learning needs of learners, a lot would still need to be studied on how to provide the best BT format. A 2018 study commissioned by the US Department of Education concurs with these conclusions, stating that BT reduces in-class seat time for learners (Becker et al., 2019). That study also states that schools must be well-resourced with technology and have qualified staff for BT to succeed.

However, Bates (2018) pointed out that in the global South, high schools in countries such as South Africa face resource challenges, especially in public high schools. This study therefore differs from other studies as it focuses not only on online teaching and technology, but on various devices that in-service teachers could use to promote BT in teaching mathematics, such as GeoGebra, projectors, computers, YouTube videos, etc. Thus, this study aims to respond to the research question and corresponding objective of exploring in-service high school mathematics teachers' experiences with the BT approach.

Horn and Staker (2017) asserted that a complete online learning experience for learners also comprises non-teaching factors, such as convenience, learner characteristics, and readiness. Convenience is an enormous non-quality factor for learners, as it allows flexibility and comfort in studying at a convenient time for the learner (Artino, 2010). Moreover, the BT approach necessitates embracing online learning, and this was especially noticeable during the COVID-19 pandemic; whereas the teacher-centred approach attempts to transfer information to learners through one-sided, downward communication, and mostly passive listeners must work individually without questioning the educator's knowledge (Pule & Raxangana, 2024). Gwaltney (2012) claimed that the South African Department of Basic Education (DBE) has been forced to modify inflexible teaching methods, due to the changing demands and opportunities that face 'millennial' instructors. Similar opinions were voiced by Bates (2018), who stated that traditional face-to-face education methods have fast become obsolete in the 21st century.

Nonetheless, BT continues to be the most popular and commonly used teaching strategy in Africa. It enables teachers to instruct and communicate with learners face-to-face (Anderson, 2016). However, Hlathi (2021) asserted that in-person instruction has reached its peak; and that though it has had some success, the COVID-19 epidemic and technical advancements may ultimately lead to its destruction. Notwithstanding Hlathi's remarks, Bates (2018) emphasised the drawbacks of online learning for education. Moreover, Bates (2018) thought that mixed non-quality elements in online learning would make it challenging to determine the contextual roles of the instructors in quality when it comes to online learning with mixed non-quality characteristics. Considering this background, BT seeks to be more learner-centred; and learners should learn from each other, so that they do not easily forget what they have learned

(Bouilheres et al., 2020). The current study thus responded to the following research question: What are in-service mathematics teachers' experiences on the use of BT to teach high school mathematics?

Theoretical framework

Community of Inquiry Model Construct

This section discusses the theory that underpins the study. The study uses the Community of Inquiry (CoI) as its theoretical framework. A critical thinking and practical inquiry paradigm is the foundation of the CoI framework (Garrison, 2022). The dynamic model of a BT method provided by the CoI framework encompasses both in-person and online learning experiences, altered by the subsets in Figure 1 below, and can be used to enable in-service teachers to incorporate technology into the curriculum to support student learning. Additionally, the CoI proposes a theoretical framework that adds to the complexity that virtual learning is able to achieve (Padilla & Kreider, 2018). The CoI model recognises that learning occurs within the 'community' through the cooperation of three main elements: teaching presence, social presence, and cognitive presence. These main elements of CoI are described in the following section. Within the CoI framework, as it relates to BT, cognitive presence is closely associated with the development of critical thinking and reflective practice, and promotes higher-order thinking processes in both teachers and learners (Garrison et al., 2001). The most fundamental component of success in high school mathematics learning is cognitive presence, as its components (supporting discourse, educational experience and selecting content) are the core business of teaching mathematics (Garrison, Anderson & Archer, 2010). The CoI model was first developed in a course in which an instructor was setting out to teach in-service high school mathematics teachers online. Figure 1 below shows the CoI model.

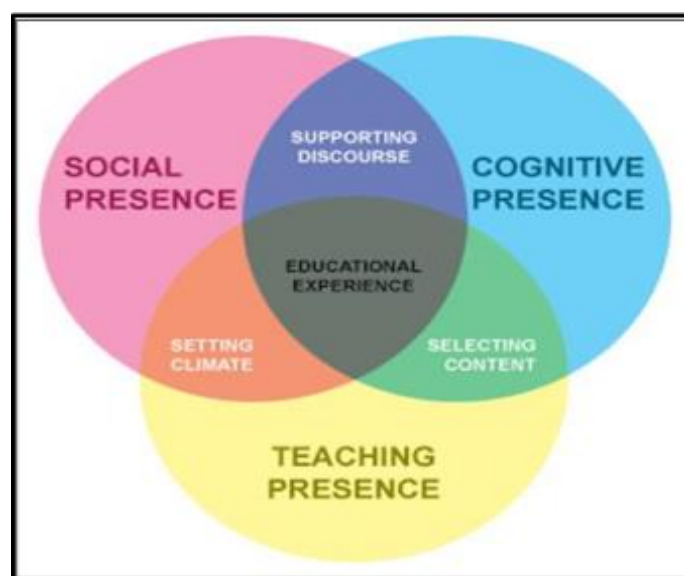


Figure 1: Community of Inquiry (CoI) framework

Source: Adopted from Garrison, Anderson and Archer (2010, p. 6)

In their analysis, Garrison et al. (2010) determine certain indicators, and show the correlation of those indicators with the three core elements.

Community of Inquiry Model on BT Approach

According to Garrison et al. (2010), the CoI framework has gained popularity as a paradigm for studying BT, and it can help to explain some of the challenges, perspectives, and experiences of in-service teachers in teaching high school mathematics. According to Mutambara and Tsakeni (2022), a blended (online learning) environment is enabled by the interaction of three factors: social presence (a feeling of belonging to a group that promotes learning), cognitive presence (the capacity to create meaning for learning), and teaching presence (what the instructor does to support learning). Several other acts – such as choosing the material, establishing the tone, encouraging discussion, and developing a communication channel – also assist these elements (Mutambara & Tsakeni, 2022). The current study examined in-service high school mathematics teachers' experiences of the BT approach using the CoI lens.

Implementation of the Community of Inquiry Model

According to Garrison et al. (2010), the CoI model in Figure 1 divides teaching into:

1. *Teaching presence*: everything the in-service teacher does – instructing the subject, providing lectures and creating rehearsal opportunities (Garrison, 2022). In-service teachers first teach their learners how to use a BT approach; and in that process, mathematics is learned in the classroom. In addition, during BT, the teaching presence in the CoI framework (the dimensions of which are design, facilitation and direction is seen “as a significant determinant of student satisfaction, perceived learning, and sense of community” (Garrison & Arbaugh, 2007, p.163).
2. *Cognitive presence*: In-service teachers provide opportunities for learner interaction with the material in robust, thought-provoking ways (Garrison, 2022). Moreover, this opens a way for learners to navigate the different BT tools, such as GeoGebra (free software), and construct new knowledge that could make the concept taught not easily forgotten, as the learners have visualised the content.
3. *Social presence*: Building a learning community that incorporates learner-to-learner and learner-to-teacher interactions (Garrison, 2022). Social presence allows collaboration between in-service teachers, so that they can learn from each other without necessarily being guided by knowledgeable others (e.g. subject advisors or facilitators) who are subject specialists in the field. This collaboration would mean teachers interacting with each other and sharing good practices through peer learning circles.

This CoI model uses the three components of Garrison's 2010 model (cognitive presence, social presence, and teaching presence) interacting dynamically to produce a meaningful and productive online learning environment (Garrison et al., 2010). Moreover, according to Garrison et al. (2010) the CoI paradigm in online learning environments supports deep learning and knowledge production by encouraging active involvement, learner engagement, collaborative inquiry and reflective inquiry. Meaningful online learning occurs at the CoI

model's junction of the three presences (Mutambara & Tsakeni, 2022). A dynamic and captivating learning environment results from the successful intersection of cognitive presence, social presence, and teaching presence (Garrison et al., 2010). This is what transpires at this overlap of presences, according to Garrison et al. (2010):

1. **Comprehensive Education:** Deep learning is supported when the three presences come together. Learners use reflective inquiry, critical thinking and group discussion to create new knowledge and meaning. As they engage with peers and teachers, they draw from their experiences and viewpoints, which helps them comprehend the material more deeply.
2. **Engagement and Motivation:** Learner engagement and motivation are improved when the presences intersect effectively. Social contacts foster community and belonging, and motivate people to participate actively in conversations and events. In addition to encouraging learners to remain involved and interested in the learning process, teaching presence guarantees that learning experiences are well-designed, pertinent and stimulating.
3. **Construction of Knowledge:** Knowledge is constructed at the intersection of the presences through cooperative inquiry and discourse. Learners offer a variety of viewpoints, question one another's theories, and collaborate to co-create knowledge. While social presence fosters the interaction and discourse necessary for knowledge production and sharing, cognitive presence powers the process of meaning-making and sense-making of inquiry.
4. **Effective Communication:** Social presence combined with instructional and cognitive presence makes for effective communication. Learners are at ease sharing their ideas and opinions, which results in insightful conversations and knowledge sharing. The teaching presence promotes a favourable learning environment by ensuring that all lines of communication are open, that feedback is given promptly by in-service teachers, and that misconceptions are cleared up.
5. **Community Building:** Lastly, a strong learning community is developed due to the intersection of the presences. Learners build relationships with the teacher and one another, fostering a cooperative and encouraging environment where they are respected and appreciated. This feeling of community enhances the learning process by encouraging collaboration, peer learning, and group problem-solving. Garrison (2022) argued that the fundamental dynamic of a CoI is critical thinking, which is focused on creating personal meaning and shared understanding. However, Garrison (2018) confirms that separating the responsibilities of teachers and learners violates the framework's integrity.

In summary, deep learning, learner engagement, knowledge construction, successful communication and community building are all characteristics of effective BT that occur at the intersection of the three presences (cognitive, social and teaching) in the CoI model. At the intersection of the social presence and the teaching presence, in-service teachers must set up a climate conducive to delivering educational experiences. In contrast, at the intersection of the cognitive and teaching presences, there has been a selection of content that will support the discourse of social and cognitive presence.

Research methodology

This case study uses a qualitative approach. The sample for this study included eight purposely selected in-service high school mathematics teachers from four selected schools in the Sedibeng West district of Gauteng Province. The data collection was through semi-structured interviews and focus group discussions. The purpose of this study was to gain an in-depth understanding of the in-service teachers' experiences of the use of BT in the teaching of high school mathematics. Data was collected through six focus group interviews; these allowed the researchers to gain deep insight into the experiences of the teachers selected. The recorded interviews and focus group discussions were later transcribed verbatim, and common themes were identified and categorised from the transcriptions. Due to limited resources, this research was limited to a small sample of eight in-service mathematics teachers from Gauteng's Sedibeng West district. The researcher sought permission and ethical clearance from Gauteng's Sedibeng West Department of Education, whose employees were part of this research; since all participants were adults, only the Education Department was consulted as part of the ethical clearance process.

Findings

Data was collected through semi-structured and focus group interviews with the participating in-service high school teachers. This study explored the experiences of in-service teachers using BT to teach mathematics. It sought to answer the following research question: What are in-service mathematics teachers' experiences of the use of BT to teach high school mathematics? The views of in-service teachers regarding their experiences when implementing BT are discussed in the following section. In the context of this study, 'moderators' refers to the researchers.

Experiences of in-service teachers when using BT

The excerpts below attest that there is an increased engagement between learners and in-service teachers when BT is used, as asserted by Deschacht and Goeman (2015). This increased engagement is beneficial in the sense that the teachers could steer learners towards being critical independent thinkers. As Teacher 4 stated, "Learners develop a different lens on the concept; they discover a way to understand a particular image, and most things, by themselves." Teacher 4 thus concurs with CoI model application perspective (Mal & Adhya, 2020). In addition, these excerpts demonstrate that the learners were able to express their experiences as they became fascinated by their teachers' BT approach. In agreement, Teacher 2 said:

"Today's learners are technologically advanced and technology-inclined. When using/blending traditional teaching with technology, I contribute to that [*the BT approach*], and BT makes teaching enjoyable; they [*the learners*] just get excited. Also, it removes boredom in the classroom. Learners can be motivated by the use of technology such as GeoGebra and YouTube videos."

The findings of this study agree with Teacher 2's statement that "learners can be motivated by the use of technology", and also that today's learners are easily distracted, for example when there is loadshedding (power outages). Distraction could also be due to a lack of visual elements (BT tools) in the classrooms. BT tools keep learners' attention until the end of a lesson, and even motivate them to practise at home (as Teacher 2 commented) without being supervised by their instructor/teacher.

Moderator: What were your teaching and learning experiences when BT was used to recall prior knowledge in your lesson?

Teacher 2: Learners just get excited. Also, it removes boredom in the classroom. Learners can be motivated by the use of technology such as Youtube videos and smartboards.

Teacher 4: Learners develop a different eye on the concept; they discover a way to understand a particular image, and most things, by themselves.

Moderator: Is BT assisting you, as a teacher, in teaching mathematics?

Teacher 4: If I show them a visual using GeoGebra, it will help them for exam purposes; and [they] discover on their own that they want to relate to real-life situations such as 2 Dimension diagrams of Grade 9 shapes.

Teacher 2: *Ja*, mathematics, by nature... they say it is challenging. The learner has to use many senses to understand and use [the] visual aspects; I cannot over-emphasise concepts. Moreover, through BT, there is a visual aspect, even the use of colour, such as in GeoGebra when plotting graphs using different colours. In that way, it assists learners – even those with short listening spans. That visual aspect, with the motions and audio, plays a role.

Learner discipline during BT approach

The following excerpt summarises the in-service teachers' experiences of how BT promotes discipline in the teaching of high school mathematics.

The excerpt below indicates that there are BT challenges that cause learners' attention spans to shorten, leading to disciplinary concerns. This is attested by Teacher 1 who says, "Discipline, to learners, is for a limited period." Their inability to answer questions on the smartboard during the process makes them more likely to ask their friends, which disrupts the class and makes them lose discipline during that time. Mathematical learners who lack discipline run the risk of failing the learning area and failing the subject altogether. This goes against Kilis and Yildirim's (2018) support for the Teaching Presence aspect in the CoI model. Since learners take advantage of not suffering any consequences for late submissions, timely submission is a critical concern. Learners require communication with their instructor or the person giving the course online. Teacher 4 agrees, saying that "the traditional way of teaching makes learners make noise". Additionally, it may imply that the integrated learning strategy draws the learners' attention and so curbs classroom insubordination. Niekerk and Webb (2016) further asserted that managing learner behaviour should take precedence. It is also possible to address

behavioural concerns quickly in this context – another benefit of blended teaching. The benefit of discipline is that learners may more readily absorb knowledge from in-service teachers without wasting class time. Learners who lack self-control during a BT session risk upsetting more self-aware learners and hindering their ability to study mathematics. For in-service educators, maintaining discipline is difficult when using BT to teach mathematics. Some problems lead to learners misbehaving. This is how the teachers in this study experienced this topic:

Moderator: Are learners disciplined during teaching with BT?

Teacher 3: *Ja*, they are.

Moderator: Explain.

Teacher 3: I think it even makes them more focused, unless disrupted by something. It entices them – unlike teaching, using the traditional way of teaching.

Teacher 6: Teaching becomes exciting, and they are enthusiastic when I use blended teaching, like videos.

Moderator: How did BT equip you, as a teacher, so that you can teach mathematics even in the absence of using BT tools? Please elaborate and give examples.

Teacher 6: Some learners become active in class and more curious, unlike when I use the chalkboard and textbook.

Moderator: What strategies do you employ to execute BT? List them.

Teacher 1: I will watch the video first, then take it to learners, and afterwards include learners and add what the online tutors teach them. I always watch relevant videos for my learners.

Teacher 2: I try to use videos online, videos which triggers the visual aspect of learners. I want to say something other than ‘a learner-centred approach’, where I engage learners in my teaching.

Teacher 3: Teacher-learner interaction... what is it called? *Ja*, ‘learner-centred approach’. I try first to find their level of understanding, and build on that.

In-service teachers point out that when learners are motivated to learn mathematics, boredom in the classroom is eliminated by their enthusiasm for these tools. Moreover, BT stimulates learners' visual elements, enabling them to develop conceptual comprehension of real-world scenarios independently. In-service teachers stated that software such as GeoGebra (which is inadequately addressed in the literature) is helpful, in that it allows learners to plot functions (graphs) using multiple colours (Pule & Raxangana, 2024). Moreover, learners are disciplined in BT implementation of BT, according to the experiences of in-service teachers; visual features stimulate their efficiency, as they develop creative methods of teaching mathematics.

This study's findings align with assertions by Persky and McLaughlin (2017) that learners' attitudes change in the classroom as they are motivated by the use of technology such as GeoGebra and smartboards, and are thus in agreement with Teacher 6, who said, "Teaching becomes more interesting, and learners are enthusiastic when I use blended teaching, like videos." However, the effectiveness of a learner-centred approach advocates that learners should be in charge of their knowledge construction or learning. As Teacher 3 puts it, "I will watch the video first, then take it to learners and then include learners. It makes learning easier if in-service teachers prepare thoroughly and watch the video before the learners can view it. In addition, the learner-centred approach entices the learners and puts them in charge of their own learning. This could help them remember the concepts, as they have developed them from their prior knowledge.

Tools that teachers use for BT

In the section of the findings, the study summarises the technological tools that in-service mathematics teachers used to carry out BT when teaching mathematics. The excerpt below summarises the discussion between in-service teachers and the moderator about the types of technological tools used to facilitate teaching and learning during mathematics lessons.

Moderator: What do you understand about BT regarding the learning of mathematics? What technological tools are used to facilitate BT in lessons?

Teacher 1: BT combines online and in-person teaching, using technology and manual material.

Teacher 5: BT uses smartboards and devices such as (thinking) projectors to teach mathematics.

Teacher 6: BT uses technology in mathematics, and not just technology like cell phones. It is also about using different materials in education, like a projector – it can be a digital or overhead projector. If I use something other than a textbook, it can be a smart board or whiteboard, so I use different materials, not just a book.

Teacher 8: Understands that BT teaching involves the use of smartboards, in most cases also using other technological material.

These respondents agree that BT incorporates technology with traditional ways of teaching. Teacher 1 avers that "BT combines online and in-person teaching, using technology and manual things". BT is understood by Teacher 8 "Understands that BT teaching involves the use of smartboards, in most cases also using other technological material." Similarly, Teacher 5 says that "BT uses smartboards and gadgets such as projectors to teach mathematics". And Teacher 6 says that BT uses various technology-related materials, and understands that "BT uses technology in mathematics, and not just technology like cell phones ... It is also about using different educational materials, like a projector".

Benefits of BT

In this section, the study summarises the experiences of in-service teachers regarding the benefits of the use of BT to teach high school mathematics. This is how the in-service teachers responded to the following moderator question.

Moderator: How does the use of BT tools benefit the teaching experience of in-service high school mathematics teachers?

Teacher 5: It reminds me of the concepts I have forgotten, and helps my visual learners.

Teacher 8: And yes, it did; it feels... like I said, it did add value, it should. If I do not understand specific topics, I check them, and then get them to start doing different examples. It does help. There is even GeoGebra, which shows how to get an angle quickly. They [BT approaches] do enhance one's (learners) knowledge.

The main objective of BT is to incorporate virtual teaching with traditional ways of teaching. There might be hindrances to implementing BT; however, if in-service teachers are innovative enough, by using BT in learning mathematics could benefit high school mathematics learners. The excerpts above show that BT is beneficial in learning mathematics, as learners gain insight from this approach. In addition, with BT there is increased engagement between learners and in-service teachers, as studies have asserted (Deschacht & Goeman, 2015; Graham, 2019). This increased engagement benefits the teacher, because it leads to learners becoming independent and critical thinkers. As Teacher 4 said, "Learners develop a different visualisation prospect on the concept, as they discover a way to understand a particular image – and most things, by solely themselves." This excerpt also alluded to the fact that learners have experience with technology, which then makes them fascinated by the execution of BT approaches. As Teacher 2 stated: "Today's learners are technologically advanced and technology-inclined. If they blend traditional teaching with technology, the teacher contributes to that and makes teaching enjoyable; they just get excited. Also, it removes boredom in the classroom. Learners can be motivated using technology." Regarding Teacher 2's comment that "Today's learners get distracted easily", distraction in classrooms might indicate a need for more visuals. BT tools keep learners' attention to the end of the lesson, and even motivate them to practise at home without being supervised by their instructor, the in-service teacher).

Usefulness of BT in teaching mathematics

The in-service teachers responded to questions about the usefulness of BT when teaching mathematics.

Moderator: Do you find BT useful for your learners to understand mathematics? If yes, how?

Teacher 2: A lot; it does help me, especially with real-life problems.

Teacher 2: It does help me; I do not want to lie.

Teacher 3: BT has contributed to conceptual understanding instead of an algorithm when I teach platonic solids in Grade 9; in the past, a learner used to see a 3D shape in a 2D plane.

Teacher 3: I think it expanded my knowledge, and even if there are no BT tools, I can teach without them.

Teacher 4: BT is time-consuming because sometimes you must move from one app to another, from the smartboard to the laptop; it takes time. But prompt and thorough preparation helps learners and is beneficial, because limited time is provided.

Teacher 5: As an in-service teacher, it reminds me of the concepts I have forgotten, and helps my learners to visualise concepts using tools such as GeoGebra.

Teacher 2 states that BT tools help represent real-life problems. Visualising mathematics through BT approaches benefits mathematics instruction, as learners can then visualise mathematical concepts, even in the absence of BT tools.

This study discovered that BT advantages the learners of this generation, as they can see things that some could not see, the in-service teachers said. BT execution by in-service teachers could stimulate learners' prior knowledge. In support of this (not shown in the excerpt above), Teacher 5 says: “Well, as I said, it reminds of the forgotten concepts and aids learners to visualise concepts using GeoGebra.”

These narratives from in-service teachers allude to the fact that learners show their experience with BT tools as Teacher 3 attests that it expanded their knowledge with technology, which then stimulates the learners' fascination with the use of BT approaches. If I blend traditional teaching with technology, I contribute to that and make teaching enjoyable; they just get excited. Also, it removes boredom in the classroom. Learners can be motivated using technology.”

Solutions to the challenges on the use of BT

The following excerpt summarises the solutions teachers gave to the challenges of using BT in the teaching and learning of mathematics in high schools.

Moderators: Which strategies do you use to counter the impact of loadshedding when implementing BT in the teaching of mathematics?

Teacher 2: Unless I am innovative enough, I have to be prepared by using an alternative – which is the traditional method of teaching – when the power goes off. When I am most innovative, I can have a plan B.

Teacher 3: Eh, usually when they announce loadshedding, we make other plans; we improvise, so that the lesson does not get disturbed.

According to the teachers quoted, in-service teachers should be innovative in their BT approach, so they can continue teaching even without electricity. This implies that teachers

could even prepare for alternatives such as whiteboard and chalkboard components of BT as they prepare their lessons.

The literature reveals that senior South African government officials declared COVID-19 a national disaster (Pule & Raxangana, 2024), encouraging in-service high school mathematics teachers to use alternatives such as the BT approach to save the academic year. The findings from this current research suggest that when in-service high school mathematics teachers experience challenges, they should be equipped to teach – even in the absence of electricity (i.e. when there is loadshedding). Electricity is considered the issue that has contributed most to encouraging the implementation of BT to the maximum possible. Prior studies by Pule and Raxangana (2024) describe how in-service high school mathematics teachers experienced the ‘new normal’ (COVID-19) for the first time, with insufficient resources, knowledge and support to remedy the situation. This unprecedented situation resulted in in-service teachers adapting their teaching approaches, changing their teaching methods to implement the BT approach. This study also found that learner discipline is one of the challenges experienced by in-service high school mathematics teachers when performing BT. Surprisingly, reduced learner discipline seems to be triggered by internet connectivity, as attested in this paper; some learners are not used to the internet, which could cause them to misbehave by searching for content that is not educational when connected to the internet. Moreover, pupils in disadvantaged areas such as townships could not access digital resources such as BT approach tools that are required for online learning at home. The literature agrees that in-service high school mathematics teachers and school leaders should be trained to leverage technology effectively to help learners learn. This could help to ensure in-service high school teachers get a thorough orientation on using the BT tools when teaching high school mathematics.

Conclusion

This study aimed to investigate in-service high school mathematics teachers’ experiences when using a BT method to teach mathematics. Eight in-service teachers from four schools in Sedibeng West, Gauteng province were purposively selected to participate in the study. The main findings of the study were that the experiences of in-service mathematics teachers show that the use of various technological tools supplements their instructional strategies. According to the participants' general opinions about GeoGebra, in-service teachers felt the free downloadable software (GeoGebra) was worth it since it produced an engaging learning environment in line with the South African mathematics curriculum. In-service teachers also thought GeoGebra was a valuable tool since it might aid learners in understanding and visualising mathematical concepts. Each of these tools brought new experiences – and at times, challenges – to the teaching of mathematics in high school. The findings of the study contribute to the body of knowledge on the use of the BT approach to teaching mathematics in high schools. This study used a small sample; it could be repeated using a bigger sample as well as teachers from different quintile levels, to achieve a broader understanding of in-service mathematics teachers’ experiences with the use of BT to teach mathematics in high schools. This study suggests that future studies could focus on investigating the impact of the use of BT

as a teaching strategy and its associated technological tools in the teaching and learning of mathematics in schools.

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UNPACKING NC(V) LEVEL 2 TVET COLLEGE STUDENTS' PERCEPTIONS ON THE SOURCES OF THEIR MATHEMATICS DIFFICULTIES

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This study examined NC(V) Level 2 TVET college students' perceptions of mathematics difficulties in South Africa, using Sociocultural Constructivist Theory to analyse social, cultural, and institutional influences on learning. Through semi-structured interviews with students, key themes emerged, including self-blame, lack of practice, teaching methodologies, absenteeism, and negative attitudes towards mathematics. The findings suggest that internalised self-blame, inadequate practice, and negative cultural attitudes significantly impact student performance. Recommendations include reviewing and reinforcing of attendance policy, revising teaching methods, enhancing tutoring quality, and fostering supportive peer networks.

Keywords: self-blame, sociocultural constructivism, absenteeism, mathematics difficulties

Introduction and Background

This study aimed to examine and understand the perceptions of National Certificate (Vocational) [NC(V)] Level 2 students in Technical and Vocational Education and Training (TVET) colleges in South Africa regarding the challenges they face in learning mathematics. TVET colleges in South Africa serve as pivotal institutions for equipping students with essential workforce skills, particularly within the NC(V) Level 2 framework. Despite the critical role these colleges play, mathematics education has emerged as a significant challenge, with student performance frequently failing to meet expectations (Sehole, Sekao, & Mokotji, 2023; Ngoveni, 2018). This underperformance, compounded by a notable lack of active student participation (Buthelezi, 2018; Vimbelo & Bayaga, 2023), emphasises the urgency of investigating the root causes of mathematics difficulties from the student's perspective.

Central to this inquiry is the question: "What are the perceived roots of mathematics difficulties among NC(V) Level 2 TVET college students?"

The NC(V) Level 2 program is strategically designed to align educational offerings with the workforce's evolving needs, aiming to foster student motivation and address the skills gap crucial for economic progress (Remington, 2018). However, challenges such as limited industry support, practical application difficulties (Sephokgole & Makgato, 2019), and alarmingly low completion rates (Zulu & Mutereko, 2020, Ngoveni & Mofolo-Mbokane, 2019) highlight the need for targeted interventions.

The research seeks to explore the complex issues students encounter by gathering their perspectives through interviews. While the primary study involved conversations with both students and educators to understand the underlying reasons for the high failure rates, this investigation concentrates on the viewpoints of students. Nonetheless, the opinions of lecturers are employed to corroborate the themes that arose from the student interviews.

Emphasising the need for targeted interventions, such as review and adherence to attendance policy and academic advising (Papier, 2009), this research seeks to enhance student engagement and achievement within the TVET sector. By integrating Sociocultural Constructivist Theory into our understanding of NC(V) Level 2 students' mathematical difficulties, the study contributes to improving the effectiveness of the NC(V) program and advancing vocational education's broader objectives in South Africa.

Theoretical Framework

Utilising Sociocultural Constructivist Theory (SCT) as its foundation, this study delves into the challenges faced by NC(V) Level 2 TVET college students in mathematics, anchoring its analysis in the seminal contributions of Lev Vygotsky (1978). SCT advocates that learning is intrinsically a social endeavour, deeply rooted in interactions that occur within specific cultural and institutional frameworks (Vygotsky, 1978; John-Steiner & Mahn, 1996). These interactions, facilitated by more knowledgeable others (MKOs), including educators and peers, play a pivotal role in shaping students' mathematical learning experiences and outcomes.

Central to SCT, and directly relevant to this study, are the Zone of Proximal Development (ZPD) and the concept of scaffolding. ZPD illuminates the realm where learners can advance with assistance beyond their independent capabilities (Vygotsky, 1978), whereas scaffolding represents the structured support provided by MKOs to navigate this space (Wood, Bruner, & Ross, 1976). This scaffolded learning, emerging from dynamic social interactions, is instrumental in constructing knowledge and overcoming mathematical challenges.

The emphasis on social interaction extends beyond the classroom to encompass the broader cultural and institutional contexts of TVET colleges. Through collaborative learning activities, peer discussions, and teacher-student interactions, mathematics education becomes a shared, community-driven process. This approach aligns with SCT's assertion that cognitive development is significantly enhanced through social engagement and the use of cultural tools, such as language and digital resources (Dong, Lee, Wang & Shannon, 2023; Kaplan-Berkley, 2022).

By foregrounding social interaction in the theoretical framework, this research underscores the critical need for educational strategies that leverage the social fabric of the learning environment. It advocates for practices that not only engage students in meaningful mathematical discourse but also foster a supportive, inclusive community, thereby enhancing students' learning experiences and academic success in mathematics.

Literature Review

In South Africa, TVET colleges play a pivotal role in readying students for the job market, notably through initiatives like the National Certificate (Vocational) Level 2. A significant hurdle within this curriculum is mathematics, a subject that many students find daunting. This investigation seeks to understand the perceptions of NC(V) Level 2 TVET college students regarding the challenges they face in mathematics, focusing on issues such as self-blame, previous educational experiences, and attitudes toward the subject. Matiti (2024) emphasises the importance of a solid foundation in mathematics for both academic success and future

opportunities. The difficulties in transitioning from mathematical literacy to pure mathematics are due to cognitive, personal, organisational, and social factors, which necessitate supportive measures for students and lecturers (Barton & Frank, 2001; Tall, 2008).

Vinerean et al. (2023) highlight the critical need for a comprehensive understanding of basic mathematics to successfully manage this transition, drawing attention to the obstacles encountered by both students and educators. The lack of elementary mathematical skills in secondary education significantly hampers performance in higher education settings, as demonstrated by Tshabalala and Ncube (2016). Mathematics anxiety, stemming from fear and negative attitudes, poses a substantial barrier to learning. Furner and Duffy (2022) suggest that psychological and pedagogical strategies are essential in addressing mathematics anxiety, with emotional hurdles such as lack of confidence and adverse prior experiences influencing students' attitudes toward STEM fields. The Mathematical Resilience Toolkit, as introduced by Hernández de la Hera (2023), proposes methods to build psychological safety and confidence, offering a solution to the pervasive fear and negativity surrounding mathematics.

Research indicates that self-blame affects students' academic and psychological well-being, with Thalib and Aditya (2022) noting its impact on self-criticism and academic performance. Self-esteem plays a decisive role in academic achievement, as it is associated with motivation and academic engagement. High self-esteem correlates positively with better academic performance (Acosta-Gonzaga, 2023; Britwum et al., 2022). On the other hand, poor self-image and anxiety can negatively affect academic performance. Students with low self-esteem often experience higher levels of academic stress and lower academic achievement (Yu et al., 2022; Masadeh et al., 2021). High self-esteem is correlated with better academic performance, and interventions that enhance self-esteem can potentially improve educational outcomes. Ali and Jameel (2016) stress the necessity of regular mathematics practice for proficiency, pointing out that inadequate practice can prevent students from fully grasping mathematical concepts. Integrating practice activities that reflect students' perspectives, as Pease (2016) suggests, can enhance educational strategies and learning outcomes. Leone and Richards (1989) highlight the importance of regularly completing homework and classwork, advocating for parental involvement to maximise educational benefits.

Teacher training and the adoption of diverse pedagogical techniques play a critical role in enhancing the learning experience in TVET colleges, especially in the context of mathematics education. Traditional teaching methods lacking in interactive engagement may limit students' learning potential, a concern raised by Vimbelo and Bayaga (2023). Meanwhile, Saira et al. (2023) discuss how tutor training and personalised educational strategies can significantly influence the quality of tutoring and student outcomes.

The influence of peer dynamics on academic achievements cannot be understated, with Júnior et al. (2023) noting that positive interactions among students can boost motivation and development. Wu and Cai (2023) add that collaboration and teamwork in mathematics education can lead to improved understanding and performance, emphasizing the importance of peer influence. Lastly, the challenges posed by the diverse student population in TVET colleges, especially regarding classroom management and curriculum delivery, are

compounded by socio-economic conditions. Muchineripi, Akwasi, and Kofi (2022) call for comprehensive support systems within TVET institutions to effectively tackle these barriers.

Addressing the mathematics achievement gap in TVET colleges requires targeted intervention programs designed to improve educational outcomes for NC(V) students, as argued by Vimbelo and Bayaga (2023) and Papier (2021). This comprehensive approach underscores the multifaceted challenges faced by students in mastering mathematics, highlighting the need for systemic changes to support their educational journey. The existing literature does not adequately explore NC(V) Level 2 TVET college students' perceptions regarding the root causes of their mathematics difficulties, nor does it examine the depth of students' understanding of how these perceptions directly impact their engagement and success in mathematics within the TVET educational framework.

Methodology

This study adopts a qualitative research methodology to explore the perceptions of NC(V) Level 2 students at TVET colleges in South Africa regarding their difficulties in mathematics. This approach is chosen for its effectiveness in obtaining deep insights into participants' experiences, beliefs, and attitudes (Creswell, 2013).

Seventeen NC(V) Level 2 students from a TVET college campus in Tshwane, South Africa participated in this study. Participants were selected using purposive sampling to ensure a diverse representation of the programmes and student abilities (Patton, 2015). This method allowed me to collect data that was rich and provided insights into the issues under investigation (Merriam, 2009).

Data were collected through semi-structured interviews, which allow for flexibility in probing deeper into participants' responses and facilitating a more comprehensive understanding of their perspectives (Kvale, 2007). The interviews explored students' personal experiences with mathematics, including challenges faced, perceptions of teaching methods, and the impact of their socio-economic and cultural backgrounds on their learning.

The data analysis followed Braun and Clarke's (2006) thematic analysis approach, which involves the identification, analysis, and reporting of themes within the data. This method was chosen for its flexibility and suitability for examining the complexities and nuances of qualitative data (Braun & Clarke, 2006).

Throughout the analysis, the principles of Sociocultural Constructivist Theory informed the interpretation of the data, focusing on how students' perceptions of their mathematics difficulties are shaped by their interactions within the social, cultural, and institutional contexts of TVET colleges.

The themes identified from these discussions, which aligned with those from the student interviews, were emphasized to corroborate the perspectives of the students.

This research is a segment of a broader study that received ethical approval from a higher education institution and was carried out in accordance with the institution's specified regulations.

Integrated Findings and Discussion

In this integrated findings and discussion section, I delve into the perceptions of NC(V) Level 2 students from a TVET college, identified as S1 to S17, about their struggles with mathematics. Retention rates at Level 2, the entry-level, which averaged 11.2% between 2019 and 2021, offer valuable insights into an institution's annual performance by indicating the percentage of students who continue to enrol in consecutive years (Khuluvhe, 2023, p. 4). This highlights the importance of focusing on this level, as it appears to be where the majority of challenges are encountered. Additionally, insights from five lecturers marked as L1 to L5, aimed to triangulate perspectives on the themes emerging from student responses. It is important to clarify that this study primarily concentrates on the students, not the lecturers. While the lecturers' viewpoints have been addressed in a separate article, it was essential to validate the student findings by briefly referencing the lecturers' insights, without going into a detailed analysis. Students were prompted with the question, "What would you regard as the reasons for your poor performance in mathematics?" while lecturers addressed the question, "What can you attribute poor performance for your students to?" These questions served as the initial point of inquiry and were open-ended in nature. This allowed for the asking of follow-up questions to probe deeper into the students' perspectives. This section aims to bridge students' perceptions with lecturers' observations, weaving in references from existing literature to frame these experiences within the broader educational landscape of South Africa. It is important to mention that not all the themes that were brought up by students were reflected in the comments made by the lecturers.

The analysis not only uncovers the multi-layered challenges faced by students but also aligns these findings with theoretical insights, offering a comprehensive understanding of the factors influencing mathematics performance in the TVET setting. In this discussion, the selected comments from students and lecturers (where applicable) are provided before the discussion takes place. Finally, it is worth mentioning that the themes used in this discussion emerged from the data.

Self-Blame and Educational Background

Students generally do not attribute their difficulties in mathematics to the lecturers or the institution. While they found it challenging to explicitly blame themselves, it was evident that they felt the responsibility lay with them. Here are some of their statements: S₆: *Lecturers do everything right to help us pass mathematics. They give us homework, extra-work and also volunteer to give us extra classes.*

S₁₁: *I did mathematical literacy in high school. It is difficult for me to cope with pure mathematics. I do regret doing mathematical literacy at high school.*

The phenomenon of self-blame among students, as implied by S₆ and S₁₁, quoted above, resonates with the findings of Thalib and Aditya (2022), who discuss how self-criticism impacts students' academic stability. This self-perception issue, compounded by choices made between mathematical literacy and pure mathematics, aligns with the challenges highlighted by Morska and Rogerson (2023). They emphasize the importance of a strong mathematical foundation, suggesting the need for interventions that address both psychological aspects and

educational gaps. According to Vygotsky (1978), scaffolding these foundational gaps through more knowledgeable others could significantly improve students' self-efficacy and mathematical understanding.

Fear and Negative Attitude Towards Mathematics

S₁₇: I define myself as a person who enjoys using his hands and not a numbers person. I just don't like mathematics.

L₅: These students have convinced themselves that mathematics is difficult. They just have a negative attitude towards mathematics.

The pervasive fear and negative attitudes toward mathematics, highlighted by S₁₇ and L₅, give emphasis to the societal and educational barriers identified by Adams (2014). Sociocultural Constructivist Theory suggests that transforming the educational narrative around mathematics into one of accessibility and relevance could alter students' trajectories in mathematics education. Implementing strategies that foster a positive learning environment, as discussed by Dweck (2012), could mitigate these fears and negative attitudes.

Lack of Practice and Homework Engagement

S₅: I believe that practice is the only key to passing mathematics. You can only excel in mathematics when you give yourself enough time to practice. ... students say that this is college level and homework is not important.

S₁₂: I do not do homework. I am not the only one as the majority of us do not do it.

L₃: A few students do their homework. You will fight a losing battle if you want to push them to do their homework. About 3 out of 40 students do their homework in my class.

The theme of Lack of Practice and Homework Engagement stresses a pivotal challenge within mathematics education, highlighted by Student S₅'s assertion that "Practice is the only key to passing mathematics." This recognition of the importance of consistent study and practice starkly contrasts with the reality of students' engagement, as noted by Lecturer L₃, who observed that "Many students see homework as optional." This observation suggests a systemic undervaluation of homework's role in reinforcing learning, a point also stressed by Ali and Jameel (2016), who highlight the necessity of regular practice for mastering mathematical concepts.

Furthermore, the reluctance of students like S₁₂ to actively engage with homework highlights a broader issue of motivation and relevance in their educational journey. The gap between the recognised importance of practice and students' actual engagement habits calls for a pedagogical reevaluation, as advocated by Sociocultural Constructivist Theory, which emphasises the need for active engagement and culturally relevant pedagogy. By making homework and classwork more engaging and supportive, educators can better motivate students to value and commit to regular mathematical practice, thus bridging the gap between ideal study behaviours and students' real-world practices.

Pedagogical Gaps

S4: There is no clear understanding between me and my lecturer. There is no joy in class as the lecturer does not make learning interesting. The lecturer sometimes just comes to class and says to us, do an exercise on page "x" without showing us how it should be done.

S7: Our lecturer is too fast, does not teach, just looks at the problem and just thinks we all understand. Our lecturer does not explain step by step. If you ask for clarity from our lecturer, the next explanation will be more confusing than the previous one.

Students such as S4 and S7 express dissatisfaction with the current pedagogical approaches, highlighting a gap that is also noted in the literature. For instance, Motsoeneng (2022) delves into the effectiveness of teacher training programs, suggesting that many educators lack proficiency in engaging students. This insight calls for a shift towards more interactive, student-centred learning environments that foster social interaction and collaboration, fundamental principles of Sociocultural Constructivist Theory.

Absenteeism

S5: Attendance at our college is very poor. When you abscond, the lecturer may continue with other students and complete a chapter or start another one in your absence. There is nothing that you will understand on your return.

L2: Absenteeism is a big issue at our campus. Even the implementation of the policy which says a student must attend 80% of the classes to qualify to write the exams did not help much to address the issue. Our students just do not care.

The challenge of absenteeism, as discussed by S5 and lecturers like L2, reflects broader institutional and socio-economic challenges. Garcia and Weiss (2018) discuss the impact of absenteeism on academic achievement, underscoring the need for comprehensive strategies that not only make mathematics lessons more engaging but also address external factors contributing to absenteeism. Sociocultural Constructivist Theory supports the notion that regular participation in social contexts of learning is crucial for cognitive development.

Peer Influence, Collaborative Learning, and Tutoring

S15: Mathematics is not a subject in which you can just work alone. You have to involve yourself in study groups to become successful.

S4: Extra classes should not be offered by tutors who are just previous students. We require experienced lecturers to offer these additional classes.

S12: They should organise competent people who are sure about their story to offer these extra lessons. They might have completed Level 4, but they are not in a good position to help us pass. That is the reason why we do not attend those classes.

Incorporating student perspectives alongside academic literature enriches the discussion on Peer Influence, Collaborative Learning, and Tutoring. Student S15 highlighted the vital role of study groups, saying, "Success in mathematics often comes from working together," emphasising the significant impact of peer support. This insight aligns with Vygotsky's concept

of the Zone of Proximal Development, which posits that learners achieve more through social interactions than they could independently. However, the discussion on tutoring reveals a gap, with students expressing concerns about the quality of tutoring received. For instance, Student S₄ lamented, "Tutors need to be more than just former students; they should bring real understanding," pointing to a need for more experienced or better-trained tutors. This concern was supported by Lecturer S₁₂'s observation, "... organise competent people. that is the reason we do not attend those classes." This feedback suggests a gap in the tutoring program's ability to meet students' learning needs, underscoring the importance of tutor expertise in achieving educational outcomes. This reflects broader findings in the literature, such as those by Buthelezi (2018), who stresses the importance of qualified tutors in facilitating effective learning. The integration of peer learning with quality tutoring, as advocated by Sociocultural Constructivist Theory, suggests a pathway to enhance mathematics education, emphasising the need for tutoring programs that not only support academic growth but also foster a collaborative learning environment.

The positive impact of peer influence and collaborative learning, highlighted by students like S₁₅, aligns with Vygotsky's (1978) concept of the Zone of Proximal Development. Studies by Júnior et al. (2023) reinforce the significance of peer interactions in academic success. This suggests the importance of creating structured opportunities for collaborative learning and tutoring within TVET colleges, leveraging peer influence to support mathematical understanding and achievement.

Summary of Findings

The exploration of NC(V) Level 2 TVET college students' perceptions of the roots of their mathematics difficulties reveals a complex interplay of personal, pedagogical, and socio-cultural factors that contribute to their struggles. Consistent with Vygotsky's Sociocultural Theory (1978), the findings stress the significant role of social interactions and cultural tools in shaping students' learning experiences and outcomes in mathematics. This study confirms the assertions of scholars like Lave and Wenger (1991) on the importance of situated learning and the zone of proximal development in fostering meaningful educational experiences.

Students' struggles with self-blame and educational background, as well as their experiences of fear and negative attitudes towards mathematics, highlight the psychological barriers that need to be addressed alongside the curriculum content. These findings are in line with Boaler's (2016) research on the impact of mindset on mathematics achievement, suggesting that interventions aimed at shifting students' attitudes and perceptions about their capabilities could enhance their engagement and success in mathematics.

Furthermore, the lack of practice and homework engagement identified in this study points to a misalignment between students' learning needs and the current pedagogical approaches. This observation aligns with Ali and Jameel's (2016) emphasis on the necessity of regular, meaningful practice in achieving mathematics proficiency. Pedagogical gaps and absenteeism further exacerbate these challenges, calling for a reevaluation of teaching methods and attendance policies within the TVET colleges.

Peer influence, collaborative learning, and the quality of tutoring emerged as critical factors in students' learning experiences, reinforcing Vygotsky's (1978) emphasis on the social foundation of learning. The positive impact of collaborative learning environments on students' mathematical learning suggests that fostering a supportive peer culture and ensuring the quality of tutors are essential strategies for improving outcomes.

In conclusion, this study answers the research question by identifying key factors that contribute to NC(V) Level 2 TVET college students' mathematics difficulties. Addressing these challenges requires a holistic approach that encompasses both curricular and instructional reforms, targeted psychological interventions, and the cultivation of a supportive educational environment. By integrating the insights from Sociocultural Constructivist Theory and relevant empirical research, this study contributes to the ongoing efforts to enhance mathematics education within the TVET sector.

Recommendations

1. Develop professional development programs emphasizing sociocultural constructivist approaches to teaching, focusing on engaging students through relatable and context-specific mathematics education.

Implement curriculum changes that bridge the gap from mathematical literacy to pure mathematics, incorporating real-world applications to make mathematics more appealing and relevant to students.

Ensure tutors are well-trained and capable of addressing students' specific needs, potentially involving more experienced educators or peer tutors with proven success in mathematics.

Encourage the formation of study groups and peer learning sessions to exploit the benefits of collaborative learning, underpinned by Vygotsky's theory of social development.

Adopt a multifaceted approach to improve attendance, including addressing external barriers to attendance, enhancing classroom engagement, and establishing clear consequences for absenteeism.

Limitations

The study's focus on a single TVET college and a specific student cohort may restrict the generalizability of the findings to other contexts or educational levels.

The qualitative nature limits the ability to quantify the impact of identified factors on mathematical performance, suggesting the need for complementary quantitative research.

Recommendations for Future Research

Conduct broader studies across multiple TVET colleges to assess the universality of the identified challenges and recommendations.

Implement quantitative methods to measure the impact of specific interventions on students' mathematical achievements and attitudes.

Explore the long-term effects of pedagogical changes and interventions on students' engagement, performance, and attitudes toward mathematics.

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CONTRIBUTING FACTORS TOWARDS THE EXCLUSION OF MATHEMATICS PROBLEM-SOLVING TASKS IN GRADE 7 GAUTENG NORTH DISTRICT

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The present South African mathematics curriculum namely, Curriculum and Assessment Policy Statement (CAPS), requires learners to be empowered with skills to be systematic, to generalise, solve problems, and think critically. Hence, it is important to use the problem-solving tasks to develop learners with such abilities. While research acknowledges the role of problem-solving in promoting critical thinking and reasoning skills, research also acknowledges that problem-solving entrenches the complexities that often cause teachers to exclude it from the assessment tasks. Thus, the qualitative case study reported in this article describes the contributing factors towards the exclusion of problem-solving tasks in grade 7. Data were collected through semi-structured interviews and classroom observations. Thematic analysis was used to analyse data. Analysis of data revealed the following as the main contributing factors towards the exclusion of problem-solving in mathematical tasks: learners' inability to make sense of English language, teachers' negative belief towards problem-solving, the unproductive teaching strategies and teachers' inability to engage learners in problem-solving activities. Findings of the study suggest the need for challenges to be understood in context in order for meaningful solutions to be formulated. Thus, the teachers' experiences should be taken into consideration to inform better assessment practices.

Keywords: *assessment strategies, exclusion, mathematics tasks, problem-solving, teaching strategies.*

Introduction and background

Problem-solving is considered as an assessment task that provides an intellectual challenge for enhancing learners' mathematical development and understanding (National Council of Teachers of Mathematics (NCTM), 2014). It relates to higher-order cognitive levels and is considered an integral part of mathematics education (Wakhata, Mutarutinya & Balimuttajjo, 2023). Wakhata et al. (2023) believe that problem-solving is included in mathematics, because of its potential to promote higher-order thinking, which challenges learners' intellectual capacity. On the other hand, problem-solving assessment tasks are regarded as an important component of education, which among others, offer learners the opportunity to participate in tasks, which encourage hard work and require learners to think intensely (Cai, Mok, Reddy & Stacey, 2016). Problem-solving tasks thus play a vital role in promoting deep learning and for this reason, learners need to be assessed well, in order to lay a good foundation. Although problem-solving assessment tasks help learners to develop critical thinking and analytic skills (Henningsen & Stein, 1997), most learners are still deprived of an opportunity to engage in such tasks (Bhekiswayo, 2023). This deprivation happens despite the recommendation and stipulation of the Curriculum Assessment and Policy Statement (CAPS) document for assessment diversification and balancing of cognitive levels' purposes.

A study conducted in Malaysia reveals that teachers lack knowledge and skills of balancing assessment tasks, hence they often exclude problem-solving tasks in their assessments (Zulkifli, Razi & Mohammad, 2021). Zulkifli et al. (2021) further discovered that teachers in Malaysia frequently worry about meeting their own demands, such as a test that learners will be able to pass and obtain good results in. Likewise, in Canada, it was found that the reason problem-solving was excluded in assessment tasks, was due to teachers' inadequate knowledge of problem-solving. When teachers do not have knowledge of what problem-solving is, it becomes difficult for them to assess it (Barham, 2020). Furthermore, teachers who have a negative attitude towards problem-solving often tend to exclude it in their assessment tasks (Barham, 2020).

One of the major challenges identified in the South African context is that teachers use ineffective teaching and assessment strategies in their classroom to assess problem-solving (Mdladla, 2017). For instance, teachers regularly use teacher-centred approaches and lecturing styles in their classrooms, which avert them from engaging their learners in problem-solving tasks (Mdladla, 2017). Additionally, Mahlangu (2021) found the "poor" selection of tasks and sources of assessments to be other major reasons why learners were deprived of the opportunity to participate in more problem-solving tasks.

A body of literature highlights the significance of including problem-solving in the assessment tasks (Ukobizaba, Niseyimana & Mukuka, 2021; Nagle & Styers, 2015). In one accord, Stein and Lane (1996) advocate the inclusion of problem-solving in the assessment tasks, due to its potential to develop learners' conceptual understanding of mathematics. According to Taley (2022), the problem-solving tasks also have the potential to strengthen learners' mathematical calculation skills, which subsequently promote higher-order thinking skills. The present South African mathematics curriculum namely, Curriculum Assessment Policy Statement (CAPS), requires teachers to develop learners' systematic, generalisation, problem-solving and critical thinking skills, which are essential in developing learners' understanding of the mathematical concepts. These skills also enable the learners to articulate the processes used to arrive at solutions, rather than just remembering and applying a set of procedures. Thus, it is important for teachers to continuously strive to include problem-solving tasks in their assessment to promote deep learning and understanding of the mathematical concepts. Despite its significant role, unlike other types of assessments, problem-solving still experiences continuous exclusion from the assessment tasks by the teachers.

Based on this, therefore, this article sought to explore factors that contribute towards the exclusion of problem-solving assessment tasks. Our belief is that the understanding of factors that contribute to the exclusion of problem-solving, will assist in identifying the appropriate/suitable measures to circumvent such exclusion in future. The study will, therefore, help promote the inclusivity of problem-solving tasks in teachers' assessment tasks. It will also help teachers to be cognisant of the factors that contribute to the exclusion of problem-solving tasks. The message that is conveyed in this article is that the exclusion of problem-solving is a serious issue that should not be overlooked, especially where the agenda of deep, meaningful and quality learning is concerned. Thus, an in-depth understanding of an exclusion of such a

crucial aspect and the related factors thereof, warrants attention to elude the undesirable consequences.

The study, thus, highlights the need to engage the four cognitive levels, namely: knowledge (20%), routine procedures (35%), complex procedures (30%), and problem-solving (15%) in the types of mathematical tasks, as outlined in the CAPS document (Department of Basic Education, 2011). The implications of the study are placed in the current debates aimed at promoting the improvement of assessment in mathematics as highlighted by results of some international benchmark assessments, such as Trends in International Mathematics and Science Study (TIMSS). In line with all this, therefore, we ask the question: What are the contributing factors towards the exclusion of problem-solving tasks in Grade 7?

Literature review

Problem-solving tasks are defined differently by various authors. Some authors refer to them as problems that use words or use graphics or are problem-based (Tambychik & Meerah, 2010). Other authors defined them as the illumination of the problem structure – information, objective and action-plan (Zhining, Johnson & Johnson, 1995). According to Ibrahim (1997), there are two main procedural steps in problem-solving: i) transforming the problem into mathematical sentences; and ii) computation of the operational involvement of the mathematical sentences. Chapman (2013) defined problem-solving tasks as tasks that can usually be represented in multiple ways, including by means of manipulative materials, diagrams and symbols, as well as real-life situations. At the core of this, is the demand for learners to analyse task constraints that may limit possible solution strategies and to apply complex, non-algorithmic thinking (Champman, 2013).

DBE (2011) defined problem-solving as non-routine problems, which are not necessarily difficult and higher order understanding and processes are often involved. They are considered Higher-Order Thinking (HOT), as they demand critical thinking and the ability to reason. According to these authors (Zhining, Johnson & Johnson, 1995; Ibrahim, 1997, Tambychik & Meerah, 2010; Champman, 2013) problem-solving tasks are seen as a high-level skill that requires engagement with conceptual ideas and complex thinking. They are an important part of the mathematics curriculum, because of their ability to promote mathematical skills, such as critical-thinking, logical-reasoning and creativity in learners (Mahlangu, 2021). These problems (problem-solving tasks) are an important part of mathematics curricula and should be included in mathematics assessments.

The consistent inclusion of problem-solving in teachers' assessment tasks has been shown to develop learners' conceptual understanding of mathematics (Stein & Lane, 1996). According to Taley (2022), teachers use problem-solving tasks to strengthen learners' mathematical calculation skills, which also promote higher-order thinking-skills. This indicates that the inclusion of problem-solving tasks is critical and teachers should include them in their assessment tasks.

Despite the argument that support the importance of problem-solving tasks in teachers' assessment, as discussed above, there are challenges in the assessing of problem-solving tasks,

since teachers often struggle to include problem-solving in their classroom assessment tasks (Parrish & Bryd, 2021). A study conducted by Mdladla (2017) indicated that teachers exclude problem-solving in their assessment tasks, rather including lower-order tasks and tasks that did not prompt mathematical-thinking. Mdladla's findings are consistent with those of Mahlangu (2021), who discovered that teachers mostly assigned routine procedures and recall-type questions, using pure mathematics, which are characterised as lower-order thinking tasks. This means that when teachers do not include problem-solving questions in their assessment tasks, they do not give learners the opportunity to develop higher-order skills.

Some studies reveal that teachers have a negative view of problem-solving tasks. For instance, Bullock (2015) found that problem-solving tasks are the most feared, disliked and dreaded assessment tasks. A study conducted by Murray (2012), revealed that teachers are concerned about including problem-solving tasks in their assessment, because learners tend to dislike them and become anxious when they have to solve them. Teachers' beliefs and attitudes matter, because they influence teachers' assessment strategies (Martínez-Sierra, García-García, Valle-Zequeda & Dolores-Flores, 2020).

A study conducted in Sweden by Palm, Andersson, Boström and Vingsle (2017), discovered that learners' poor performance in problem-solving was caused by the assessment strategies that teachers used to assess learners. They discovered that the assessment strategies were ineffective and did not help improve the achievement of learners. A study conducted in Singapore by Fan and Zhu (2007), discovered that during their classwork activities, teachers wrote a problem to solve on the board without offering learners an explanation and then asked learners to work out a problem on their own. Additionally, teachers looked at the textbook to find the problems there. Fan and Zhu (2007) argued that teachers need to help learners understand the problem, to assist them in coming to a decision about a possible plan of action that may lead to a solution, to encourage them to 'carry out the plan of action' and to persist with the plan if it is leading to an appropriate solution; but to discard the plan and try a new one if the initial plan does not lead to a solution. Furthermore, a teacher might encourage learners to reflect on the plausibility of their solution and make judgments where necessary.

Adding on challenges of the exclusion of problem-solving tasks, a detailed study conducted by Siswono, Kohar and Hartono (2019), revealed that teachers had inadequate knowledge of problem-solving and what problem-solving entail, and that had a negative impact on their teaching and assessment of problem-solving. They discovered that teachers found it difficult to create problem-solving tasks for learners, which hindered the development learners' problem-solving skill proficiency and critical-thinking skills (Siswono et al., 2019). In the United State of America, Monarrez (2017) conducted a study and discovered that teachers had challenges solving tasks of higher-level cognitive demands, like problem-solving. Stein et al. (2009) established that teachers in their professional development workshop had challenges in categorising tasks at Level 3 (complex procedure) or Level 4 (problem-solving), and that they were only able to categorise Level 1 (knowledge) and Level 2 (routine procedure) tasks. Teachers in the study, conducted by Bardy, Holzäpfel and Leuders, (2021), were found to have

difficulties in terms of categorising their own assessment tasks, as they frequently identified a task as a Level 3 when it was a Level 2.

The consulted literature clearly indicates the need to understand the challenges pertaining to problem-solving tasks. Literature further indicates the need for promoting awareness of the issues involved and extent thereof to enlighten and facilitate understanding of the root causes of the problems. Hence, the purpose of the study was to explore the factors that contribute towards the exclusion of problem-solving assessment tasks.

Conceptual framework

Henningsen and Stein's (1997) model of mathematical task progression was adopted as this study's conceptual framework. This model guides the inclusion of problem-solving tasks in all the mathematics classrooms. The model of tasks progression emphasizes the relationships between the three factors, namely: the learner, the teacher, and the mathematical assessment tasks. Thus, the conceptual framework is built around the following pillars, namely: the mathematics assessment tasks (that is, the cognitive demands and the nature of tasks), and teacher-specific factors (that is, the teaching strategies and approaches), as well as how these two pillars influence classroom engagement and time spent on tasks. Researchers who have used this conceptual framework were able to assess whether the teachers had administered problem-solving tasks in their mathematics assessments or not. The framework is thus adopted for this study to help us understand what problem-solving is and what it entails.

The figure below shows the conceptual framework adapted from Henningsen and Stein (1997).

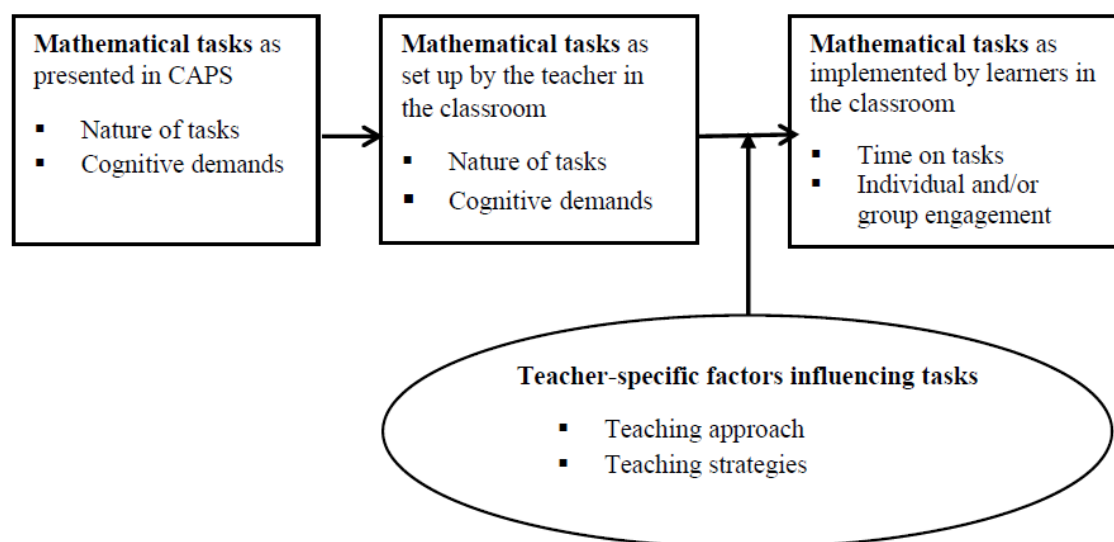


Figure 1.1: Model of task progression

Critical Reflection Theory (CRT) was also adopted as a paradigm to afford the participants opportunities to describe why and how they do “certain things”. It was further adopted, in order to allow the participants an opportunity to narrate their experiences. These narrations were deemed significant to provide learning opportunities to understand the participants’ day-to-day

situations / experiences. CRT will also provide the researcher with an opportunity to ask the participants questions that would then warrant them to explain their actions and why they do certain things in certain ways.

Methodology

This qualitative study was conducted in five schools in the Gauteng North District. A purposive sampling was used, involving five mathematics teachers, who are teaching Grade 7 and had more than five years of experience. Semi-structured interviews and observations were used to collect data. The data collection took place from the month of February to the month of May 2023. The schools where data was collected, use English as a medium of instruction. There are two home languages that were used in the schools: Sepedi and IsiZulu. Five semi-structured interviews were held. Two observations were done for each teacher. The research data was conducted in English.

Ethical clearance was obtained from the University of the Free State. Permission to conduct the study was also obtained from the Department of Education and the school principals where the study was conducted. Participants signed consent forms that further emphasises their confidentiality and rights.

Data collection procedure

The data collection process in this study consisted of two primary stages of data collection, namely: semi-structured interviews and classroom observations. The subsequent sections briefly discuss the instruments used to collect data, based on the specific question(s) aimed at generating appropriate data.

Stage 1: Semi-structured interview

The semi-structured interview asked questions regarding problem-solving (what is it? what is its significance in teaching, learning and assessment?), and examples of problem-solving tasks. Most importantly, questions were phrased in an open manner that allowed for clarification and probing in an attempt to understand and identify factors that contribute towards the exclusion of problem-solving tasks

Stage 2: Classroom observations

Through the use of an observation tool (see Annexure A), it was observed how teachers carry out mathematics lessons and the assessment process. This data collection tool aimed at determining the factors that contribute towards the exclusion of problem-solving tasks.

Data analysis

Data were analysed through a thematic analysis technique. Braun and Clarke's (2006) six steps of thematic analysis were used to analyse data, which include *becoming familiar with the data*, *generating initial codes*, *searching for themes*, *reviewing themes*, *defining themes* and finally,

using the findings for the write-up. Thematic analysis provides accessible and organised procedures for generating codes and themes from qualitative data (Maguire & Delahunt, 2017). The technique made it easy for us to identify the emerging themes and to organise them in line with the objectives of the study. Thus the sub-themes were highlighted.

Ensuring rigour

Measures to ensure credibility were put in place. Member checks, semi-structured interviews and an audio recorder were used. Member checks were used to ensure that data were captured, analysed and most importantly interpreted according to how the participants meant it. For the semi-structured interviews an audio recorder was used and transcribed verbatim, while the observation schedule was completed, with notes being taken as well. This method was judged necessary, in order to adhere to fairness norms and confirm that we did not twist the statements of the participants (Campanella, 2009).

To ensure dependability, triangulation of data was used, whereby the gathering of data was performed from several sources (semi-structure interviews and observations). This strategy was found significant in compensating for the limitations of one data collection method (Rulandari, 2021).

To ensure confirmability, we kept a reflexive journal wherein a record was kept by the researcher to reflect on and organise data collection (FitzPatrick, 2019). We kept a diary in which the personal observations and notes were recorded. Keeping a reflexive diary is seen as vital, since these records aid in cross-checking data and composing the study's final report. The data were safely kept in a safe, which required a code to open and the one on the computer was protected with a password.

Findings and discussion

This section provides discussions on the analysis and interpretation of the data collected during the semi-structured interviews and observations. The subsequent sections thus highlight the five themes that emerged in relation to the factors that contribute towards exclusion of problem-solving tasks in Grade 7 mathematical tasks.

Theme 1: Learners' inability to make sense of English

In the semi-structured interviews, wherein the contributing factors towards the exclusion of problem-solving tasks in Grade 7 were discussed, learners' inability to make sense of English was highlighted as one of the factors that affect the inclusion of problem-solving tasks in teachers' assessments. To that effect, the following was said:

“As we know that problem-solving tasks are normally written in English so we have learners who do not understand English, those learners have a problem of attempting a question written in English. So, when I put the problem-solving, it will be difficult for them to solve as English is not their home language. So that will be a very serious a serious contextual factor.....” (Teacher A)

“Language barrier, you know, when coming to problem-solving many learners are struggling with the language. So, you find that they don't understand it's not like they don't know that maybe the language, they don't understand what is exactly that is required of them.....” (Teacher B)

“Most learners do not attempt to answer such questions, because they cannot read in English they just leave the question just as it is, and as such, mostly they get it wrong.....” (Teacher C)

“.....some of the learners they fail to understand the level of language that is used especially in the word problem-solving. So that makes them to require more time to solve the problem.....” (Teacher D)

From the remarks above, it was apparent that learners lack an ability to make sense of English when solving problem-solving tasks that are word problems. As a result, most learners are unable to translate the word problems into easily solvable equations. Teacher A and Teacher C's statements indicate that a lack of English ability hinders learners from attempting to solve the questions. While Teacher B and Teacher D's statement indicate a lack of English understanding. The statements of all these teachers indicate that learners lack understanding of English, does not only hinder understanding of the problem-solving tasks, it also hinders learners' motivation of working on a problem, as well as procedural fluency. The findings above concur with those of Moleko and Mosemege's (2020) study, where they discovered that learners lack English proficiency, which makes it difficult for learners to comprehend and execute the word problems.

Therefore, teachers who teach word problems to learners should employ a variety of strategy-based approaches in the use of language as a resource and pedagogies that will enable them to teach learners to become efficient problem-solvers in multilingual mathematics classrooms. If teachers do not support learners to overcome language barriers, it will make it difficult for learners to solve word problem-solving tasks. Eventually teachers will exclude these tasks in order to save time and improve learners' achievement by giving them questions that they can solve easily. This act will be against Henningsen & Stein, (1997) model of tasks progression which requires teachers include problem-solving in their daily assessment tasks.

Theme 2: Negative belief towards problem-solving tasks

A second contributing factor towards the exclusion of problem-solving in mathematics tasks was found to be the negative belief that teachers held regarding problem-solving tasks. During the semi-structured interviews, teachers pointed out that they believe that problem-solving is difficult and challenging for the learners. This was evidenced by the comments they made:

“Problem-solving question is much challenging to the learners ...” (Teacher B)

“Because it is difficult for the learner ...” (Teacher C)

“... It provides learners with confidence. It also provides learners with a better view of thinking and applying the knowledge to the work that we are busy with or in mathematics” (Teacher D)

From the remarks above, it was apparent that teachers have negative beliefs and attitudes toward problem-solving assessment tasks. Teacher B and Teacher C stressed that the problem-solving assessment tasks are challenging and difficult for the learners. This shows that these teachers had already decided that problem-solving tasks would not be easy for their learners to tackle, which led them to avoid including these types of tasks. However, a different view was shared by Teacher D, who believes that problem-solving assessment tasks help in boosting learners' confidence. Therefore, Teacher D is likely to engage learners in problem-solving mathematical tasks, whereas Teacher B and Teacher C might avoid them due to their beliefs and attitudes. According to the model of task progression by Henningsen and Stein (1997), the omission of problem-solving mathematical assessments is not good practice, since it prevents the development of critical-thinking and mathematical reasoning skills. This therefore means that Teachers B and C's tendencies to avoid the problem-solving tasks, rendered their learners undeveloped in terms of critical-thinking and reasoning.

The empirical findings from the semi-structured interviews indicate those teachers' beliefs about, and attitude towards, problem-solving are both positive and negative. The findings align with Armah and Robson's (2018) findings from their study conducted in Ghana, which indicated that teachers view mathematics problem-solving both positively and negatively. Furthermore, teachers' beliefs and attitudes strongly influence teaching and assessment practices.

Theme 3: Reliance of textbook as a source of assessment tasks

The reliance on the use of textbooks as a source of assessment tasks was also pointed out during the interviews as one of the factors that contribute towards the exclusion of problem-solving tasks. During the semi-structured interviews, teachers indicated that they use textbooks as their source of assessment tasks. Their comments were as follows:

"...The sources of my assessment tasks, I mostly pinpoint examples from resources, such as our DBE books, textbooks that I use Spot-on premier mathematics, Today mathematics....." **(Teacher D)**

"The source of my assessment task is the textbook that I use of which is Platinum. It's Spot-on, as well as DBE book from the Department of Education..." **(Teacher E)**

However, in some cases, innovative teachers use a range of sources to design their assessment tasks:

"... I use various sources, such as when I can go check my study guides, or my textbooks, or I can go check on Google..." **(Teacher A)**

Teacher D and Teacher E indicated that they rely largely on different mathematics textbooks at their disposal as sources of the assessment tasks they use, as well as the Internet. From the remarks made above, it was obvious that many teachers lacked the knowledge and skills of balancing their own assessment tasks. They used textbooks as their primary sources for mathematics assessment tasks. This means that the assessment tasks that they gave to learners were not informed by their knowledge and skills of assessment, but according to the textbook

knowledge. The model of task progressions by Henningsen and Stein (1997) discourages this form of assessment, because there is often a modification between the suggested curriculum endorsed through textbooks and decreed curriculum (CAPS document or ATP) and the way in which the assessment tasks in the textbook are formulated. Additionally, the implemented curriculum at the learner level, describes the tasks set by the teacher, taking into account the nature of the task and the cognitive demands, which are implemented by the learner in the classroom. The model of tasks progression by Henningsen and Stein (1997) requires teachers to possess the skills and knowledge of balancing assessment tasks according to the cognitive demands or levels and according to the nature of tasks, as well as beyond the mere use of textbooks. This means that the teachers should have the capacity to formulate mathematics assessment tasks that include the content/topics covered in a way that learners are capable to exhibit via examination, homework, classroom or assignments, their ability to solve problem-solving tasks. According to Zulkifli et al. (2021), teachers who lack knowledge and skills to balance the assessment, prevent learners from engaging in meaningful learning and thus deprive them of the opportunity to develop problem-solving skills. The findings above thus indicate the need to carefully utilise the textbooks by selecting the appropriate mathematical examples (including the problem-solving ones). More so, the findings indicate the need consider the additional sources wherein the problem-solving tasks can be drawn.

Theme 4: Unproductive teaching strategies

The observation of the lessons of the Grade 7 teachers, indicates that most of the teachers lack the skill of incorporating problem-solving assessment tasks in their teaching during their lessons. The lessons went as follows:

TEACHER A

Teacher: Grade 7s, today we are going to do exponential notations.

Teacher: (writes on the board) Let us see an exponential notation using a base of 10.

Teacher: (writes an example on the board)

Teacher: Ok Grade 7, let me do another one, remember if you have any questions, you are free to ask.

Learners : Yes.

Teacher: (writes another example)

Teacher: So please when you get home keep on practicing.

Learners: Yes.

Teacher: There is a hand out that I'm going to give to you and you have to do those at home.

Teacher: For now, turn on your DBEs page 32.

Teacher: Please do it fast so that we can do corrections.

(Teacher B)

Teacher: Today we will be adding fractions with a same denominator.

Teacher: (writes an example on the board)

Teacher: It is very simple, you just add the two numerators, then bring the denominator as it is.

Teacher: Do you see how I do it?

Learners: Yes.

Teacher: Let me do another example.

Teacher: (writes an example on the board)

Teacher: Grade 7s can you do this on your own?

Learners: Yes.

Teacher: So I will give your 15 minutes to do your classwork on your own after that we will do corrections.

(Teacher D)

Teacher: Good morning class.

Teacher: (wipes the board)

Teacher: Today you will be learning how to simplify a mixed fractions.

Teacher: I will write down some few examples on the board, please wait do not write anything down as yet.

Learners: Ok.

Teacher : (writes an example on the board) this is very easy

$$(a) 2\frac{1}{4} + 1\frac{1}{2}$$

Teacher: Who can tell me how we can simplify this?

Learners: Keep quiet.

Teacher: Let's do it together (the teacher explained that learners should first multiply 4 by 2 then add 1 equals 9 then bring the denominator as it is. Teacher encourages learners to do the same thing with the other fraction).

Teacher: So I will give your 2 minutes to do your simplify this on your own, after that we will do it together.

Teacher A and Teacher B used a teacher-centred approach. They both employed direct teaching as a strategy of teaching. Their teaching strategy was planned, meticulously scheduled, and controlled by the teacher. This strategy can be beneficial if the learners are participating;

nevertheless, both teachers stood in front of the class and showed mathematical concept to the learners, using the board. Teacher A gave learners the opportunity to ask questions, but learners did not have any queries. Learners only listened, took notes, and answered short questions like 'yes' and 'no' only. Furthermore, there was no opportunity for the learners to actively participate in the lesson, which is something that could have been achieved through assessment. The observation extracts indicate that teachers did not use an effective teaching strategy. Teacher D on the other hand, tried to engage learners in the lesson by allowing learners to work on their own to simplify fractions. According to Hasan and Fraser (2015), when learners actively participate in the lesson, they share ideas, discuss and logically acquire an answer to the problem.

Direct teaching, which is a strategy used by Teacher A and Teacher B, prevented learners from actively participating in the lesson and deprived them of the opportunity to creatively solve problems, share mathematics ideas, reason logically and explain their answers. These practices are against the principles of Henningsen and Stein's (1997) task progression model, which requires the teachers to use teaching strategies that allow a high-level of engagement from learners, because it influences the expansion of problem-solving skills required to learn mathematics effectively. Therefore, teachers should make use of a variety of innovative teaching strategies that engage learners in meaningful problem-solving so that they can be competent in mathematical process skills, for example problem-solving, reasoning, connecting, reflecting and communication.

Theme 5: Teachers' inability to engage learners in problem-solving activities

The following extracts from lesson observations indicate that teachers do not engage learners in problem-solving activities during their lessons. The dialogues were as follows:

(Teacher A – lesson on exponents)

Teacher: (After writing on a board) What is this? (Referring to an exponent)

Learner: Exponents.

Teacher: Good answer.

Teacher: And this? (Referring to the base)

Learner: (No response)

Teacher: 13 is a base and 2 is an exponent. So, we say 13 to the power of 2 or 13 square.

Teacher: Do you know what does the exponent tells us?

Learners: (No response)

Teacher: The exponent tells us how many times the base is multiplied by itself. (Teacher starts reading from the textbook)

Learners: (Quiet while taking notes).

(Teacher E - lesson on fractions)

Teacher: Today we are simplifying fractions (writes on the board an example)

Teacher: How can we simplify this fraction?

$$\frac{18}{24}$$

Learner: (quiet)

Teacher: Ok class listen carefully, we make equivalent fractions by multiplying the numerator and denominator by the same number.

Teacher: Do you understand?

Learner: Yes.

Teacher: Or, we divide the numerator and the denominator by the same number. (writes the steps of how to simplify the given fractions alone and without engaging the learners)

Teacher: Right! Can you see what I did? And can you do it on your own?

Learners: Yes.

Teacher: So, I'm gonna give people 2 minutes to solve this one with their friends.

Teacher A involved learners in the lesson by giving questions that “pushed” them to give quick responses rather than ones that allowed them to use reasoning and develop their mathematics terminology. Teacher A asked closed questions like, “Are we together?”; “Am I right?”; “Correct?”; “Is that clear?” and at that juncture learners responded, “Yes”. The use of this style when questioning learners implies that in Teacher A’s mathematics classroom deep learning was not prioritised, because there was no shared responsibility between the teacher and learner for developing relevant real-world applications. Learners had a more passive, rather than active role during the lessons as a result of the strategy used by the teacher. After discussing, explaining, and showing, Teacher E asked closed questions throughout the lesson that were neither hard nor interesting to the learners. Teacher E posed closed questions, such as, “*Do you understand?*” and “*Is my explanation of this length correct?*” and at that juncture learners responded, “Yes.” Teacher E’s questioning style also made the learners primarily passive listeners. What emerged from the observations is that teachers asked low-order questions. They did not engage learners with the high-order questions. These findings concur with Baskoro and Retnawati’s (2019) findings, who found that a small percentage of teachers concentrated on higher-order questions that required the learners to think individualistically or to give more than one answer. In using low-order questions, teachers deprived learners of the opportunity to engage productively in the lesson, solve problems that allow them to think and have a conversation with a teacher, which, according to Henningsen and Stein’s (1997) task progression model is significant in enabling learners to develop critical-thinking, as it builds learners’ mathematical knowledge via conversation and the search for common ground. The

above discussion clearly shows that there is a need for teachers to formulate effective questions that include problem-solving, which encourages active learning and collaboration.

Limitations of study

The study reported on the contributing factors that affect the inclusion of problem-solving tasks in Grade 7 classes. Through the study, factors that contribute towards the exclusion of problem-solving tasks were identified. Although the study has outlined the contributing factors for teachers to exclude problem-solving, the study does not provide the solutions to these factors identified, since the study's focus was on exploring the contributing factors that led to the exclusion of problem-solving by the teachers. This is because learners' perspectives based on problem-solving were excluded in this study. Based on this therefore, I recommend that further studies be conducted with the inclusion of the learners so that their perspectives could be gathered.

Another limitation is that the study was conducted in five schools and therefore, the findings cannot be generalised as they might have come out differently if a similar study was conducted in a different context. However, in contexts that are similar to that of this study, the findings of the current study would be applicable. Therefore, further studies have to be conducted in various contexts in order for more issues to be discovered and for more strategies to be identified to address the inclusion of problem-solving tasks.

Recommendations

Based on the findings of this study, we recommend that participatory action research studies should be conducted in order to show how issues pertaining to exclusion of problem-solving tasks can be addressed through collaborative endeavours. Such collaborative endeavours would be useful in terms of providing diverse perspectives, which would significantly contribute rich information. These forms of studies can also provide frameworks in terms of how issues can be addressed in different phases/stages.

Conclusion

This present study addresses one of the most important issue in mathematics, namely "the exclusion of problem-solving in mathematics assessments", which is one of the components of mathematics education that is regarded as challenging, yet very significant. The research question in this study, "*What are the contributing factors towards the exclusion of problem-solving tasks in Grade 7?*" is responded to through this article. The following were found to be the contributing factors towards the exclusion of problem-solving in Grade 7 mathematical tasks: learners' inability to make sense of English, teachers' inability to engage learners in problem-solving activities, unproductive teaching strategies, the use of textbook as a source of assessment tasks and negative beliefs towards problem-solving assessment tasks.

Based on the findings of this study, the study proposes that the assessment of mathematics in Grade 7 be approached differently to enable learners to engage more in meaningful assessment tasks that involve problem-solving. Teachers need to be capacitated on the model of tasks progression by Henningsen and Stein (1997) in order to develop an understanding that will aid

in the application of the said model. Lesson studies are highly recommended to facilitate an understanding of problem-solving tasks. Furthermore, frequent training for teachers is needed in order to address the complexities that exist within mathematics assessment tasks to include problem-solving.

Lessons learned:

-  The mere fact that a problem is written in words does not signify that the mathematical problem is a “problem-solving” task. There are distinctive requirements that qualify a mathematical problem to be regarded as a problem-solving task. This will be unpacked in-depth in the second related paper (i.e. Phase 2 of the project).
-  The teachers’ beliefs and attitudes towards problem-solving can influence how and what they are teaching, including how they perform the assessment.
-  Problem-solving as a higher-order thinking skill must be developed during teaching and learning and not only when assessments are being prepared. Incorporating problem-solving in teaching and learning serves as a vital scaffolding tool for developing learners’ critical thinking skills.
-  Teachers’ knowledge of their students can influence how they teach and assess their learners. For instance, when teachers know that their learners participate well when given low-order questions, they tend to develop a tendency to set questions of that standard. However, these questions can limit the learners’ critical-thinking skills in the long run.
-  A powerful and a big lesson out of this study is that, “Teachers cannot give what they do not have”. This means, if teachers have limited knowledge of problem-solving and how it can be integrated into mathematical assessment tasks, then their assessments will always be limiting and restrictive of the learners’ development of critical and thinking skills.

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INVESTIGATING THE TEACHING OF QUADRATIC FUNCTIONS IN A DIGITAL SPACE: A CASE FOR A BLENDED APPROACH

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The teaching of quadratic functions using the blended learning approach represents an innovative and effective approach at high schools. Therefore, using this approach, this study aims to investigate the teaching and learning of quadratic functions in a digital space. The study is foregrounded in the cognitive load theory, which was supplemented by the didactic transposition theory. An ethnographic approach was adopted to study the narrations of Grade 12 learners learning quadratic functions. Qualitative data techniques were used to collect and analyse data from a target sample of twenty learners. The main findings of the study showed that: (1) it is possible to teach quadratic functions using the blended approach; (2) the learning of quadratic functions using the approach enhances learner engagement and robust classroom discussions; and (3) the blended approach to teaching quadratic functions helps to address the learners' misconceptions and errors. As part of future research, the study can be repeated using a bigger sample.

Keywords: Blended approach; Cognitive load theory; Didactic transposition theory; Digital space; Ethnographic approach; Quadratic functions.

Introduction

The teaching of quadratic functions in a digital space represents an innovative and effective approach to enhancing learners' understanding of this fundamental mathematical concept (Fatahillah et al., 2020; Igbinosa, 2023; Pihlap, 2017). In the context of this study, *digital space* refers to the use of the GeoGebra online platform where the teaching and learning of functions occurs (Hoyles, 2018). This study limits itself to quadratic functions when studying functions. Quadratic function posts to a type of a function of the form $f(x) = ax^2 + bx + c$, where a , b , and c are constants of the standard form when $a \neq 0$ (Ubah & Bansilal, 2018). Quadratic functions are most defined in standard form. However, they can also be expressed in the factored and vertex forms (Hlangwani et al., 2023; Parent, 2015), and play a pivotal role in various mathematical domains, including calculus (Hlangwani et al., 2023; Mutambara et al., 2019).

In South Africa, quadratic functions are taught in high schools from Grade 10 to Grade 12. However, according to the Department of Basic Education [DBE] diagnostic reports (2021, 2022, 2023), the teaching and learning of quadratic functions posits various challenges. For example, the DBE diagnostic report revealed that some learners confuse the range of a quadratic function with the domain; posed conceptual obstacles with the y -intercept and the vertex; and others failed to determine the y -intercept of the vertex. This led to failure in representing the function; and few learners struggled to interpret notation of inequalities while dealing with quadratics (DBE, 2021, 2022, 2023). In addition, literature reveals that learners often encounter conceptual obstacles when trying to comprehend the parameters of quadratic functions (Childers & Vidakovic, 2014; Hlangwani et al., 2023; Ubah & Bansilal, 2018).

Moreover, Childers and Vidakovic (2014) found that some learners mistakenly used the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ instead of the axis of symmetry formula $x = -\frac{b}{2a}$ to determine the x -intercept of the vertex, indicating a lack of comprehension of the vertex concept. Similarly, Hlangwani et al. (2023) found that learners posed an obstacle on the conceptual meaning of function notation, and others demonstrated a limitation in understanding the parameter a on a quadratic function. Thus, these conceptual obstacles result in producing learners with a fragmented conceptualisation of quadratic function knowledge.

Premised on these obstacles, this study proposes a change in the paradigm of teaching and learning quadratic functions. Literature shows that previous studies have primarily concentrated on the traditional teaching of quadratic functions (Astuti & Hidayat, 2020; Hlangwani et al., 2023; Ruli et al., 2018), while others delved into technology-aided approaches (Fatahillah et al., 2020; Igbinosa, 2023; Pihlap, 2017; Widodo, 2017). However, these studies fail to provide a comprehensive guidance for using a combination of traditional and technology-aided approaches. Thus, it is worth noting that a complete shift towards technology-aided approaches is not yet feasible in the South African curriculum due to the absence of necessary tools and the current pen and paper assessment practices (Graham et al., 2020), which are mainly high-stakes in nature in high schools (Jacobs et al., 2014).

This study aims to bridge the existing gap in the literature by investigating the integration of traditional and technology-based approaches through the blended approach in teaching quadratic functions. This approach, which has not been extensively studied in South Africa, provides a unique opportunity to enhance the understanding of effective teaching methods in this context. Therefore, this study seeks to answer the following research question:

- What are the teaching and learning affordances experienced by learners when a blended approach is used to teach the concepts of quadratic functions in a digital space?

Theoretical underpinnings

The study is underpinned by the *cognitive load theory* (CLT) as a lens to investigate the teaching and learning of quadratic functions using the blended approach (Cooper, 1990). CLT is an instructional theory that focuses on the limitation of learners' cognitive system when being taught quadratic functions (Paas et al., 2010). Moreover, CLT provides a framework for understanding how individuals process and store information in their working memory during the learning process (Paas et al., 2010). In terms of the CLT, learners have limited working memory capacity, and when this capacity is overwhelmed, it can impede teaching and learning. For example, in the realm of quadratic functions, learners may encounter challenges in managing their cognitive loads when navigating intricate algebraic expressions or comprehending diverse mathematical representations simultaneously. This cognitive complexity may surpass their working memory capacity, hindering their ability to effectively internalise the underlying concepts, thereby compromising the efficacy of teaching and learning endeavours. Therefore, instructional materials and strategies should aim to effectively manage the cognitive load by presenting information in a way that optimises learning (Paas et al., 2003).

The CLT has been extensively operationalised in many studies (Aditomo, 2009; Cooper, 1990; Paas et al., 2010; Sweller, 1994). Cognitive loads in this study are categorised into two

components: *intrinsic load*, which refers to the inherent mental structures and prior knowledge necessary for understanding quadratic functions; and *germane load*, which involves relevant schemas that are formed and stored. To supplement the CLT, the *didactic transposition theory* (DTT) was adopted to explicate the cognitive demands of each load. The DTT has been operationalised as a research tool (Bosch & Gascon, 2006) for analysing the human practice of mathematics knowledge as it unfolds between institutions. Frejd (2014) asserts that DTT refers to the process where knowledge is transposed from one institution to another through a non-recursive process. Moreover, DTT refers to the deconstruction and reconstruction of knowledge to make it teachable (Frejd, 2014). Subsequently, the didactic transposer (teacher) helps to identify the specific cognitive loads associated with each component, as suggested by the proponents of the CLT (Cooper, 1990; Paas et al., 2003). Consequently, by adopting the CLT and the DTT, this study reimagines the teaching and learning of quadratic functions in a digital space by showcasing the affordances of the blended approach. The model of the theoretical framework of this current study is captured and summarised in Figure 1 below.

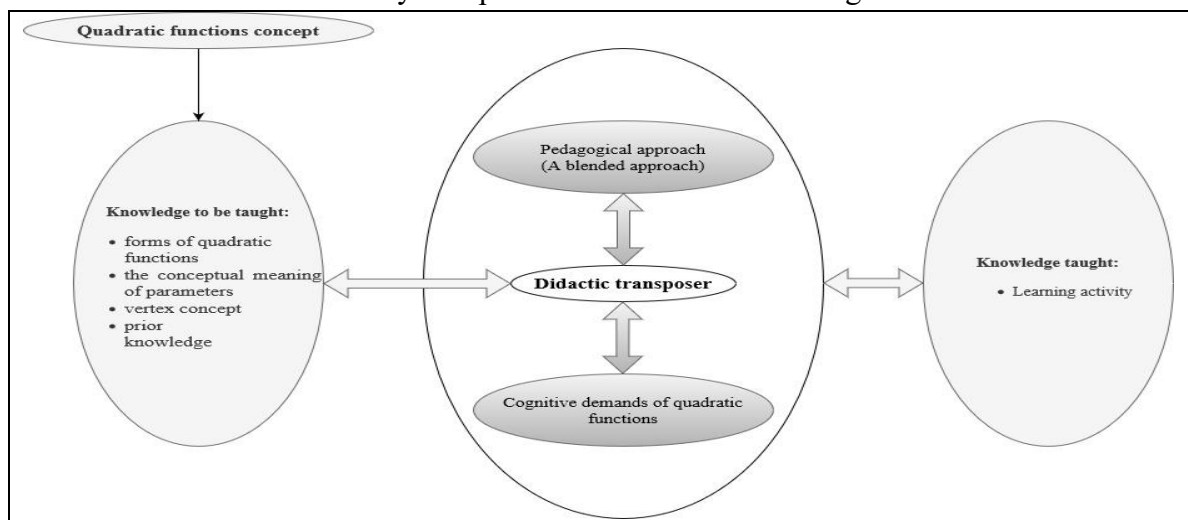


Figure 2: A model of the didactic transposition process of quadratic functions. Adapted from Bosch and Gascon (2006)

In this model (see Figure 1), the didactic transposer sifts the knowledge of quadratic functions necessary for teaching and learning. In the context of this study, a didactic transposer is the teacher who facilitates the lesson. Such knowledge is termed *knowledge to be taught* and is made available only to the didactic transposer. Knowledge to be taught refers to the forms of quadratic functions, the conceptual meaning of parameters, vertex concepts and prior knowledge of quadratic equations. The didactic transposer (teacher) engages in a synchronised interactive process through which the didactic transposer selects a pedagogical approach, specifically the blended learning approach. This choice is made possible by considering various cognitive loads. The consideration involves consulting the cognitive demands posed by the questions in the learning activity related to quadratic functions. Subsequently, after the synchronised interactive process, the knowledge to be taught is transposed to learners using a learning activity and is termed *knowledge taught*. Therefore, it is this knowledge that is responsible for nurturing the comprehension of quadratic functions.

Literature review

Literature shows that previous studies have primarily concentrated on the traditional teaching of quadratic functions (Astuti & Hidayat, 2020; Hlangwani et al., 2023; Ruli et al., 2018), while others delved into technology-aided approaches (Fatahillah et al., 2020; Igbinosa, 2023; Pihlap, 2017; Widodo, 2017). Moreover, limited studies have been done in South Africa on the use of the blended learning approach, as such international perspectives are reviewed to provide a comprehensive overview and establish a broad context. Quadratic functions play a pivotal role in various mathematical domains, including calculus (Hlangwani et al., 2023; Mutambara et al., 2019). However, according to the Department of Basic Education [DBE] diagnostic reports (2021, 2022, 2023), the teaching and learning of quadratic functions posits various challenges. For example, the DBE report revealed that some learners confused the range of a quadratic function with the domain, posed conceptual obstacles with the y-intercept and the vertex, and others failed to determine the y-intercept of the vertex. This led to failure in representing the function. Few learners struggled to interpret the notation of inequalities when dealing with quadratics (DBE, 2021, 2022, 2023).

In addition, the literature reveals that learners often encounter conceptual obstacles when trying to comprehend the parameters of quadratic functions (Childers & Vidakovic, 2014; Hlangwani et al., 2023; Parent, 2015; Ubah & Bansilal, 2018). For example, Childers and Vidakovic (2014) found that most learners had many misconceptions about the vertex concept. For example, some learners lacked the conceptual meaning of parameter “ a ”, the y-intercept of the function, and the y-co-ordinate of the vertex. Hlangwani et al. (2023) found that learners posed conceptual obstacles of a straight line and parabolas, and that some of them could not understand the meaning of $h(x)$ and $h(1)$, which impeded the full achievement of quadratic function concepts. However, such obstacles can be remedied through the integration of the blended learning approach since Childers and Vidakovic (2014), and Hlangwani et al. (2023) did not integrate the digital space while teaching the participants.

The integration of technology in pedagogy has become increasingly important (Fatahillah et al., 2020; Igbinosa, 2023; Kew, 2021; Pihlap, 2017; Widodo, 2017; Wijaya et al., 2020; Zukhrufurrohman, 2018). One area where technology can greatly enhance the learning experience is in the teaching and learning of quadratic functions. By utilising the blended learning approach, which combines traditional face-to-face instruction with digital space, teachers can create a dynamic and engaging learning environment for learners (Fitri & Zahari, 2019). Literature shows a positive impact in the use of digital space in teaching quadratic functions. For example, a study conducted by Igbinosa (2023) comparatively examined computer-aided and traditional teaching methods. The results indicated that learners who used the computer-aided methods showed significant improvement in their understanding of quadratic functions compared to those who received traditional instruction alone.

Similarly, Zukhrufurrohman (2018) used technology to find the characteristics of quadratic functions and found that it provides an affordance to demonstrate to learners the differences between the vertex and the y-intercept. Digital space in mathematics has been found to enhance student engagement and motivation in the learning of quadratic functions, and that learners who used technology-aided tools performed better than those in the traditional mathematics classroom while learning quadratic functions (Pihlap, 2017). However, it is worth noting that

a complete shift towards technology-aided approaches is not yet feasible in the South African curriculum due to the absence of necessary tools and the current assessment practices in high schools. The latter forms a gap which has not been extensively studied in the country, which this current study seeks to address using the blended learning approach.

Methodology

This paper presents an ethnographic study to investigate the integration of the blended learning approach in the teaching and learning of quadratic functions. Ethnographic approach posts to the study of learners within their communities of practice (Hammersley & Atkinson, 2019). By adopting the ethnographic approach, the study was able to gather insights into learners' understanding of the integration of blended learning approach and their learning experiences of quadratic functions (Chuene, 2021). The study conveniently selected a sample of 20 Grade 12 learners as participants. Qualitative data were collected using text analysis of learners' responses to a learning activity as well as classroom discussions during the lesson in real time. The learning activity was sifted by the didactic transposer (teacher) through the considerations of the cognitive demands as per the question. Qualitative data collected during the study were analysed using Polkinghorne's (1995) narrative analysis of eventful data. Narrative analysis is a method that explores human experience through storytelling. In the context of this study, narrations were carried out to conceptualise the learners' experiences and to present them in a way that enables the reader to comprehend the affordances of the blended learning approach (Chuene, 2021).

Gaining new insights into the blended approach

In this section, insights into the learners' experiences on blended learning approach during the study are presented. The section is divided into two phases: the first phase focuses on the integration of the digital space to remedy conceptual obstacles of quadratic function parameters; and the second analyses learners' solutions from a given learning activity. In addition, in the second part, the study examines the teaching and learning affordances offered by the blended learning approach, and experienced by learners during their engagement with the quadratic functions.

Phase 1: The digital space as a remedy to the conceptual obstacle of parameters

In this phase, the study investigated the teaching and learning of quadratic functions in a digital space to remedy learning obstacles posed by parameters of quadratic functions. Firstly, the study interrogates learners about their conceptual understanding of the parameters a , b and c , of the standard quadratic function. At the beginning of the activity, learners were presented with the quadratic function $f(x) = x^2$ with $a = 1$. They were shown different graphs on the projector using GeoGebra as the value of “ a ” increased and decreased. For example, learners were asked to explain what happens to the graph when $a = 1$, $a = 2$, $a = -1$ and $a = -2$ as seen in Figure 2 below.

$y = ax^2$

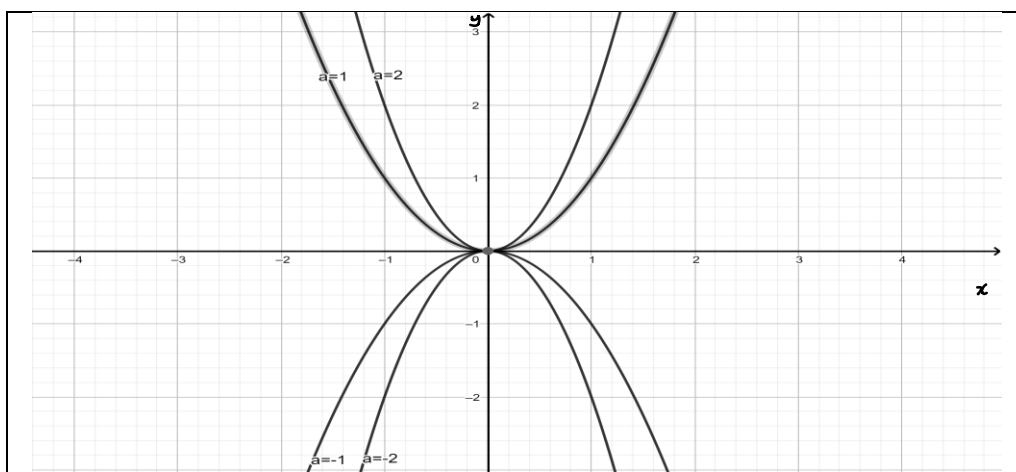


Figure 3: The effects of parameter "a"

The interesting part about this activity is that in a traditional approached classroom, such explorations are avoided due to being tedious and time-wasting. Such was observed by Igbiosa (2023), who asserted that technology integration in the teaching and learning of mathematics unnecessarily minimises traditional routines that take time to achieve. Therefore, through the integration of the blended approach, the study was able to show the learners the kind of graphical transformations related to the variations of parameter “a”, who were able to tell how “a” affects the width and narrowness of the graphical representation as seen in Figure 2. The integration of a digital space fostered learner participation in the lesson. For example, Learner A1 stated that “...it is not the narrowness or the wideness of the function but even the shape on whether it will open up or down ...”. Consistent with Zukhrufurrohmah’s (2018) findings, learners were able to depict the effects of parameters through technology integration. This is what Learner A1 stated above. Therefore, the digital space provides affordance to learners to make sense of the parameter “a” and to enhance classroom discussions.

Secondly, the study investigated the effects of parameter “b” on the function by substituting negative and positive values for “b” and keeping the value of $a = 1$ constant. For example, by substituting $b = 2$, $b = 3$, $b = -2$ and $b = -3$ into $f(x) = x^2 + bx$ as shown in Figure 3. While interrogating what learners see from the transformations of graphs while changing the value of “b”, Learner B1 stated that “...parameter “b” shifts the graph along the x-axis... moreover, when “b” is positive the graph shifts to the left and when it is negative it shifts to the right...”. Such thoughts are expressed as learners concretely see the graph transformation on the projector while using GeoGebra (Patsiomitou, 2008).

$$y = x^2 + bx$$

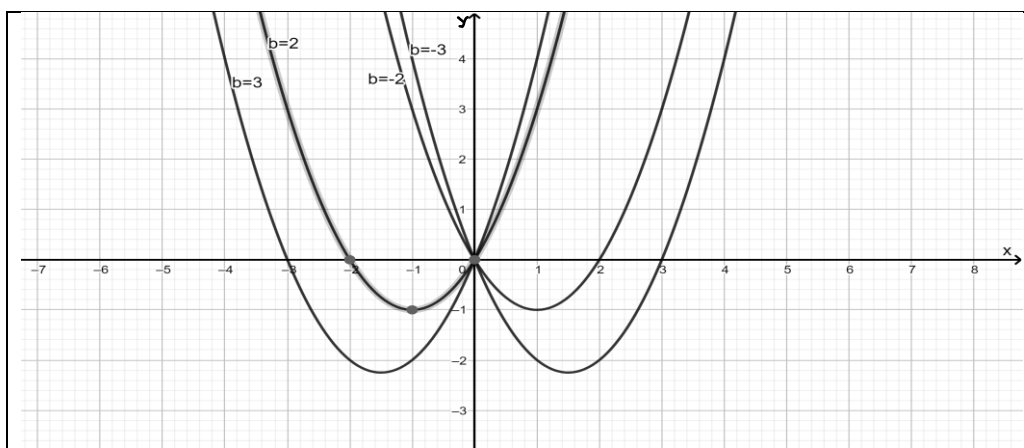


Figure 4: Effects of parameter "b"

Thirdly, learners were shown the effects on the transformation of the graph as parameter “c” was assigned different values. They were given the function $f(x) = x^2 + x + c$, and required to alter the value of “c” only. Figure 4 below shows the corresponding transformations when “c” takes values of 2, 3, -2, and -3.

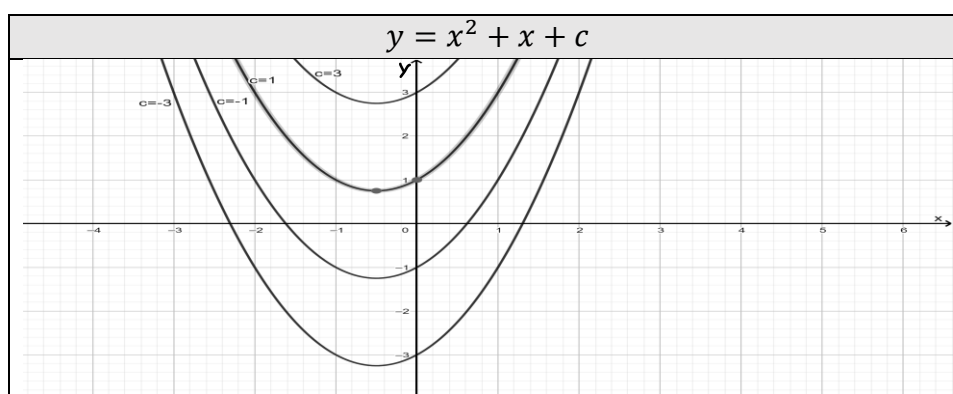


Figure 5: Effects of parameter "y"

This approach also allowed learner participation by giving their thoughts on the effect of “c”. For example, Learner C1 stated that “...look... the turning point is not changing but the graph is only moving up when “c” is positive and down when “c” is negative ... This means that Learner A1 was telling the truth that the vertex and y-intercepts are not the same...”. From this interaction, it can be noted that the digital space provided and affordance cause a change in perspective to the vertex and y-intercept. This is in contrast with Childers and Vidakovic’s (2014) findings, where learners posed various personal meanings of the vertex. In this study, such misconceptions were remedied by the integration of the blended approach.

Phase 2: Learners’ interaction with the learning activity

After taking learners through the effects of the parameters, they were divided into groups of four. Within each group, they were given a learning activity to do (see Figure 5 below). As a result of being accustomed to the traditional approach of teaching and learning mathematics, learners assumed that they needed to write and submit their solutions only. However, the objectives of the activity necessitated that they use both the traditional approach and technology to answer the questions in the learning activity.

Learning Activity

Consider the following functions that are $f(x) = x^2 - x - 2$ and $g(x) = -x^2 - x$.

- Write down the coordinates of the y -intercept of f and g .
- Determine the coordinates of the x -intercepts of f and g .
- Determine the vertex of f and g .
- On the same set of axes plot $f(x)$ and $g(x)$ showing all critical values.
- Determine the values for:
 - $f(x) > g(x)$
 - $f(x) < g(x)$

Figure 6: Learning task question

The narrations within this phase are informed by the learning process of phase 1. Thus, in question a. of the learning activity, learners were required to write the y -intercept of both functions, that is, f and g . This question did not pose much cognitive load as it required them to read the answer straightway from the function and plot it. However, most groups started by substituting $x = 0$ into both functions to solve for y as seen in Figure 6.

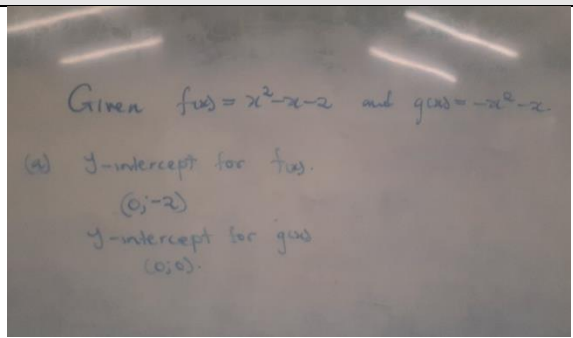
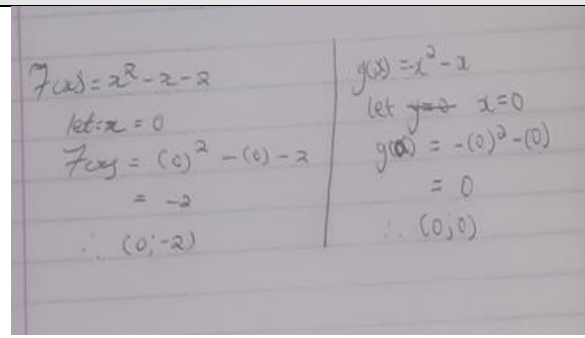
Part A	Part B
 Given $f(x) = x^2 - x - 2$ and $g(x) = -x^2 - x$. (a) y-intercept for $f(x)$. $(0, -2)$ y-intercept for $g(x)$ $(0, 0)$.	 $f(x) = x^2 - x - 2$ $\text{let } x = 0$ $f(0) = (0)^2 - (0) - 2$ $= -2$ $\therefore (0, -2)$ $g(x) = x^2 - x$ $\text{let } x = 0$ $g(0) = -(0)^2 - (0)$ $= 0$ $\therefore (0, 0)$

Figure 7: Learner's presentation on the board versus their solution as a group

However, such practice meant that the learners disregarded the essence of the question to simply write the coordinates. This accustomed practice fails in the frames proposed by Groves (2012) based on mathematical understanding. As Groves (2012) posts, the right procedure at the right time without having the corresponding comprehensive knowledge of the procedure does not mean that they fully understand the concept taught. This was seen from the groups working as they started to substitute the zero into the function without reading from the function itself. However, in this question, the demand placed on learners was to demonstrate their conceptual meaning of parameters by only reading from the function which they failed to demonstrate.

To address this, the teacher had to reconfigure the teaching and learning of the y -intercept using a digital space. The group had to look at the effects of " c " as presented in Phase 1, Figure 4. For example, learner B1 stated that "...oh I see now... c is the y -intercept...". This was done to give them a clue on why calculations were not necessary in this question. Subsequently, this led to a change in the perspective of the learner. Looking back on what the learners wrote in

their group (see Figure 6, Part B) versus what they wrote after re-visiting the y -intercept concept through the inclusion of technology (see Figure 6, Part A), the teacher could trace that there was indeed a change in perspective, which was afforded by the integration of technology to demonstrate the effects of “ c ” as seen in Figure 4. Therefore, it can be noted that technology integration in our teaching and learning provides affordances to the learning of quadratic functions as it permits learners to engage with their solutions, unlike the ordinary traditional approach where the teacher is the centre of the learning process. This was also noted in a study by Benning and Agyei (2016), who found that blended approaches employed unlike traditional methods prompted affordances to learners’ learning experiences of quadratic functions. Consequently, through the blended approach, the teacher has less authority and does not have to answer all the questions from learners.

Question b. demanded learners to determine the x -intercepts for functions f and g . This question did not pose difficulties amongst most of them in their groups apart from the question of when to use the two words “and/or” when determining the intercepts. The interesting part of the blended approach is that the learning process is drawn from learners’ voices rather than from the didactic transposer (the teacher). For example, Learner E1 managed to give guidance on the two distinct words often used synonymously, that is, “and/or” in determining the x -intercepts of function. Learner E1 stated that “...*The roots of a quadratic function are the values of x that make the function equal to zero. In other words, they are the solutions to the equation $f(x) = 0$. On the other hand, the x -intercepts of a quadratic function are the points where the graph of the function crosses the x -axis...*”. Consequently, such an explanation coming from learners depicts the mastery of prior knowledge of quadratic equations. Additionally, Learner F1 pointed out that “...*to be more specific sir... "Or" is used when referring to the solutions of a quadratic equation and "And" is used when referring to the x -intercepts of a quadratic function...*”. Therefore, drawing from these discussions, it can be noted that the two distinct words “and/or” are logical operators used in different contexts when discussing solutions to a quadratic equation or determining x -intercepts.

Question c. required learners to determine the vertex of the function. This was a challenging question for most groups as learners equated the vertex to the axis of symmetry. For example, one group argued that the vertex for $f(x)$ is $(\frac{1}{2}; 0)$. This misconception was also noted by Childers and Vidakovic (2014), who argued that learners posit various conceptions with the vertex concept. This is why this group gave their answer like this because of their understanding meaning of the vertex concept. When the researcher interrogated this group, Learner A1 argued that “...*we have let $y = 0$, to find the value of $x = \dots$* .” Hence, this was a clear meaning of doing procedures without thorough understanding as put by Tanudjaya and Doorman (2020). Learning from learners’ misconceptions rather than singling them out to either correct or not correct posts affordance to understanding (Smith et al., 1994). From the whole class discussion, it was noted that learners have some knowledge of the concept that needs to be strengthened. For example, Learner G1 stated that “...*no sir we cannot let $y = 0$ in the case of a vertex since we are not calculating the intercepts...*”. From this discussion, it was evident that they posit some traits of understanding the vertex concept, but it is not entirely developed. Consequently, we interacted with this vertex concept again. Now I had to scaffold learners on what the word “vertex” means. Learner H1 stated that “...*vertex is the turning point of the quadratic*

function....”, while Learner F1 looked surprised by the answer given by Learner H1. Straightway, the teacher questioned learner H1. Then, learner H1 stated that “...no sir... he just used another name to refer to vertex...”. Moreover, learner H1 was asked to give his thoughts on the vertex concept. Learner H1 stated that “...this is the minimum or the maximum point of a quadratic function where it turns...”. From this response, the teacher questioned them on what the relationship between the vertex and the axis of symmetry is. Learner J1 stated that “...an axis of symmetry is a line...”. From the response, if a learner states that the axis of symmetry is a line, the problem with this response is that it is not specific enough. However, we admit that the axis of symmetry is indeed a line, but it is a vertical line specifically. Therefore, this process was done to support Smith et al. (1994), who argue that a constructive knowledge transition through re-learning from learners’ misconceptions. Hence, the study followed this route to achieve this constructive knowledge construction. Subsequently, this can be remedied through the integration of a digital space during classroom discussions. This is what participants do during the classroom discussion. They had to determine the axis of symmetry traditional (*procedurally*) and graph it in a digital space (see Figure 7).

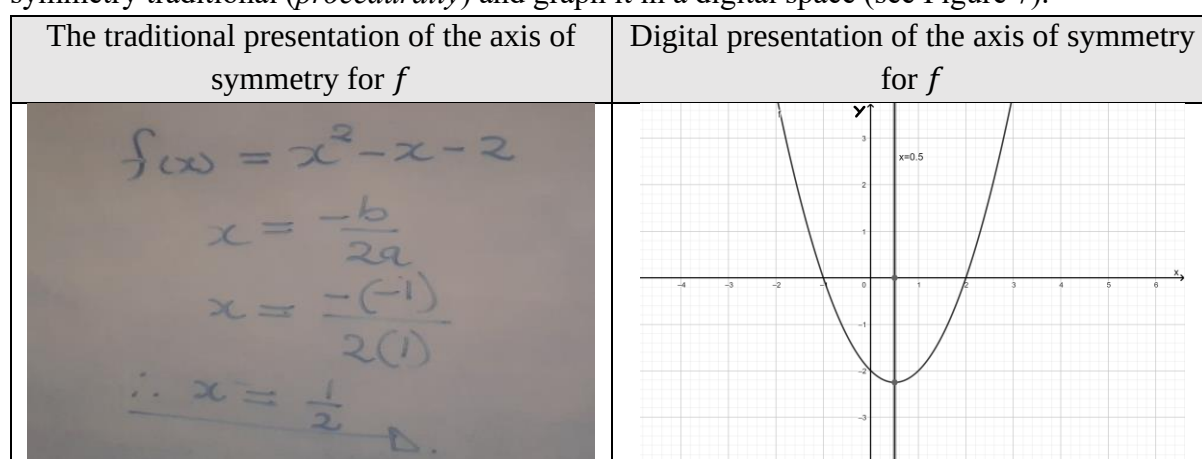


Figure 8: A blended way of teaching and learning the axis of symmetry

From this way of teaching, it could be noted that this approach provides an affordance to learners to see the orientation of the axis of symmetry not as just a line, but rather as a vertical line. Moreover, this approach expands the conceptual meaning of the axis of symmetry from just doing procedures without the essence of why learners do these procedures.

Question d. posted cognitive demands of graphical representations f and g . First, learners worked in groups to draw the graph traditionally and then presented their graph solutions to the entire class. Then after the learners’ presentations, we migrated to a digital space to draw the graphs of f and g on the same set of axes as shown in Figure 8 below.

Graphical representation of f and g

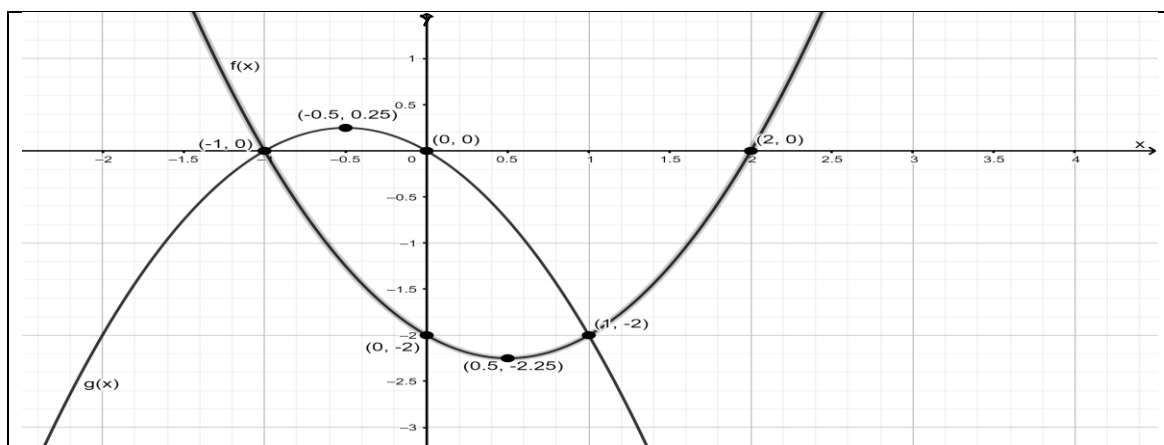


Figure 9: Digital solution for question c of the learning task

From the digital space, learners were shown all the critical values using a cursor. Consequently, this approach provided an affordance to learners as they could see the function being displayed digitally, unlike in ordinary classroom settings where we would often state that “...mine is not drawn to scale...”. For example, Learner A1 stated that “...look at the x -intercepts of f . Oh, that is where $y = 0$...”, while Learner F1 derived the reason that “...that is why we let $y = 0$ when finding x -intercepts it is because it crosses on the x -axis...”. Consequently, drawing from these narrations, it can be noted that the digital space made it possible for learners to envision the quadratic function concept concretely.

Question e. demanded learners to determine the values for which $f(x) > g(x)$ and $f(x) < g(x)$. The sub-questions in question e. required learners to determine the values for which (i) $f(x) > g(x)$ and (ii) $f(x) < g(x)$ seem to be a challenging question as most learners were just accustomed to finding the solutions without a conceptual meaning being nurtured. In this question, most learners were not able to give the conceptual meaning of the solutions to the inequalities. In most instances, the learners replaced the inequality sign that “ $</>$ ” with an equal notation, subsequently, reducing the degree of difficulty of the question. However, delving deeper into this last question required them to thoroughly demonstrate their understanding of equations and inequalities. Drawing back on their prior knowledge, most learners were able to determine the critical values but could not test if the values in the interval could work or not. In this question, learners started by determining the critical points for both cases, that is, $f(x) > g(x)$ and $f(x) < g(x)$. For case 1, they found their solution as $x < -1$ or $x > 1$. When they were asked to explicate the conceptual meaning of the values, they failed. Moreover, they could not represent the solution on the graph. While in case 2, learners found their solution as $-1 < x < 1$. In both cases, they could not unpack the meaning of the critical values. Therefore, the digital space was used to remedy their cognitive obstacles. Figure 8 below depicts two cases demonstrating how the solution can be represented graphically through the inclusion of a digital space.

Case 1: $f(x) > g(x)$	Case 2: $f(x) < g(x)$

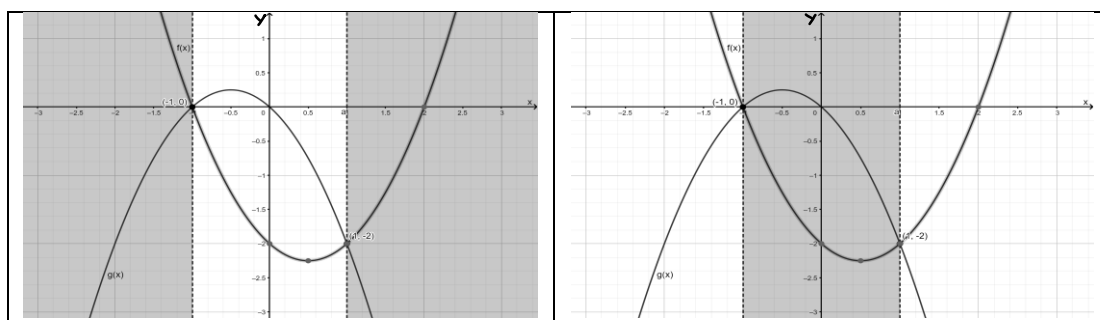


Figure 10: Case 1 versus Case 2 graphical representations

Conclusion

The study investigated the teaching and learning affordances experienced by learners when the blended approach is used to teach the concepts of the quadratic functions using GeoGebra. The study pursued the following research question:

- What are the teaching and learning affordances experienced by learners when a blended approach is used to teach the concepts of quadratic functions in a digital space?

The main findings of the study were that: (1) it was possible to teach quadratic functions using the blended approach; (2) the learning of quadratic functions in the blended approach enhances learner engagement and robust classroom discussions; and (3) the blended approach to teaching quadratic function concepts helps to address learners' misconceptions and errors. This study contributes to the knowledge of the use of blended learning to teach quadratic function concepts in high school mathematics classrooms. In-service teachers can use examples and illustrations presented in this study in their mathematics classrooms. In addition, the findings of the study can be used as a resource in in-service teachers development programs. The limitation of this study is that a small sample was used. Recommendations for future research include repeating the same study using bigger.

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TEACHING GEOMETRY USING ETHNOMATHEMATICS PEDAGOGICAL APPROACHES

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Students' spatial skills and comprehension have improved when geometry is taught utilising ethnomathematics pedagogical methodologies. The teaching and learning of geometry incorporate cultural practices, artifacts, and concepts through the use of the ethnomathematics approach. This document analysis study aims to better comprehend ethnomathematics approaches to geometry teaching and learning by presenting reasons that connect mathematics and culture. The theoretical underpinnings of ethnomathematics, which promote geometry instruction in mathematics classrooms through cultural perspectives, as well as the creation of a comprehensive literature survey for the years 2012-2023 using standards of preferred Systematic Review and Meta-Analysis (RAYYAN). The design is document analysis and this review found 50 journal articles from top indexing databases such as Scopus, EBSCOhost and Digital Dissertations. Most of the teachers supported the integration of Ethnomathematics in teaching geometry, however, there is still limited knowledge on how to incorporate or link mathematics to culture due to cultural diversity and resistance to change. Another key finding in this study is learners' geometric thinking skills are still low based on different grade levels. To effectively teach and learn geometry one needs to plan Ethnomathematics learning activities based on content-specific, grade-level, and curriculum standards supporting the appropriateness and necessity of linking Ethnomathematics to geometry. Here we introduce ideas from research on ethnomathematics pedagogical approaches in geometry education in Africa and beyond, an ethnomathematical approach assisted teachers and learners in gaining a practical, social, and cultural grasp of mathematics.

Keywords: Teaching, Geometry, Ethnomathematics, Pedagogical Approaches

Introduction

A key component of education is ethnomathematics, which helps learners and teachers build mathematical ideas and fosters a positive, enjoyable learning environment (Purwanto et al., 2022). As a fundamental skill that learners must master, geometry plays a critical role in mathematics. Many learners still struggle with geometry, and research in Junior High schools suggests that learners' understanding of the subject has influenced their deductions (Purwanto et al. 2022). Educating modern learners in mathematics is a challenge because many of them prefer experiential, visual, and contextual learning to abstract theoretical knowledge (Verner et al., 2019). The primary driving force behind the advancement in these directions is now the Ethnomathematics approach. Ethnomathematics is the fusion of traditional, formal academic mathematics with mathematical ideas and practices from the learners' culture (Verner et al. 2019). This method is used in ethnomathematics-based teaching to address a variety of important aspects of the educational process and enhance the mathematics curriculum in multiple ways. The mathematics teacher can enhance learners' understanding of both their own and other cultures through the cultural context of Ethnomathematics activities. As a result, it is

necessary to expand our knowledge of integrated learning and create pedagogical strategies for meeting curriculum objectives. According to Verner et al. (2019), the success of learning with Ethnomathematics depends on the teaching strategy selected.

In Indonesia, education is emphasised through the use of both local and national wisdom in a holistic, interactive, contextual, integrative, scientific, effective, collaborative, thematic, and learner-centered manner (Hendriyanto et al., 2021). Numerous cultural products and activities, including mathematics, are influenced by the nation's diverse ethnicities, cultures, and religions. Almost every aspect of daily life is connected to mathematics, which is regarded as a universal science. However, learners frequently struggle with mathematics, which has a negative effect on their learning outcomes. To solve this, learners' mathematical comprehension can be improved by incorporating mathematics with local culture through activities like classic games. The majority of studies that examined the Ethnomathematics approach to geometry employed van-Hiele theory to create learning resources that were organised according to its phases, which include integration, explication, guided orientation, information, and free orientation (Hendriyanto et al. 2021; Purwanto et al. 2022). Social diversity should be embraced in geometry education, and Ethnomathematics methods should be applied in geometry teaching (Sunzuma & Maharaj, 2020). In Zimbabwe, traditional methods of teaching geometry have drawn criticism for being overly theoretical and devoid of hands-on learning opportunities and real-world exposure. The use of Ethnomathematics approaches is encouraged and the necessity of relating cultural practices to the mathematics curriculum is emphasised in Zimbabwe's new mathematics curriculum (Sunzuma & Maharaj, 2020). Ethnomathematics approaches are thought to be beneficial in raising learners' cognitive levels, helping them learn mathematics in the classroom, and helping them draw connections between everyday activities and geometry lessons. Ethnomathematics is a branch of mathematics that helps learners better understand mathematical concepts in everyday situations by fusing formal concepts with cultural practices (Fouze & Amit, 2023). To improve student skills, comprehension, and mathematics achievement, researchers have suggested the Ethnomathematics approach as a solution (Fouze & Amit, 2023). This approach emphasises the significance of integrating cultural elements and values in mathematics instruction. The paper describes how Bedouin women traditionally used slide symmetry in embroidery, emphasising the precise use of geometric shapes. It highlights geometry, specifically slide symmetry, as a complex area to teach.

Aim and Objectives

The specific aim of this document analysis is to review the literature on teaching geometry through Ethnomathematics approaches. The objectives to achieve this include:

1. To outline the concepts of Ethnomathematics approaches in mathematics education.
2. To assess the relationship of Ethnomathematics approaches in teaching geometry

Outline of the Concepts of Ethnomathematics in Mathematics Education

Ethnomathematics, which can be defined as the study of mathematical concepts present in any culture, was first used by D'Ambrosio (1985) to characterise the mathematical practices of distinct cultural groups. According to D'Ambrosio (1990), the term "Ethnomathematics"

(prefix ethno) refers to the social-cultural context, which encompasses language, jargon, codes of conduct, myths, and symbols. Although mathematics is hard to derive, it usually refers to the ability to know, comprehend, and perform tasks like modelling, ciphering, measuring, classifying, and inferring. The suffix tics has the same root as technique and comes from the word *techo* (p.81). Put differently, "ethno" describes people who belong to a group and are recognised by their cultural customs, norms, and symbols. Tics refers to methods like counting, ordering, measuring, weighing, deciphering, classifying, inferring, and modelling. Mathematics is the study of the world to transcend, manage, and cope with reality so that individuals of cultural groups can survive and thrive. According to Borba (1997), Ethnomathematics is the study of how members of a specific culture apply mathematical principles and ideas to solve spatial, relational, and quantitative problems in their daily lives. The claim that "the paradigm that diverse cultures use or work within evolves out of unique interactions between their language, culture, and environment" (p. 248) is supported by additional evidence provided by Orey (2000). According to D'Ambrosio (2006), mathematical thinking is developed in various cultures based on shared problems that are encountered within a cultural context. This is consistent with Ethnomathematical perspectives. Learners can take a more active role in problem-solving through the application of Ethnomathematics, which is connected to human life (D'Ambrosio, 1987). Gilmer (1990) argues that learners' diminishing accomplishments and failures are caused by the teaching of mathematics without regard for cultural context under the guise that it is flawless, abstract, and universal. Participation in an ethnomathematics program has been linked to improved test scores for learners from a variety of ethnic and cultural backgrounds (Fouze & Amit, 2023). Furthermore, (Verner et al., 2013) developed and implemented a theoretical framework to analyse why learners did not want to participate in a cultural course for teachers. Participants included pre-service and in-service teachers, as well as Arabs, Jews, and both religious and secular groups. They studied geometry by examining geometric ornaments from various cultures and gained expertise in multicultural education. According to Rosa and Orey (2019), an Ethnomathematics programme now provides a more comprehensive understanding of mathematics by incorporating implicit knowledge, concepts, theories, procedures, methods, and practices associated with various cultural contexts. The cognitive processes, learning capacities, and attitudes that may influence the learning processes taking place in many mathematics classrooms are more strongly demonstrated by this feature.

Ethnomathematics Approaches in Teaching and Learning Geometry

The Ethnomathematics approach is a highly effective and prominent methodology in the field of mathematics education. It involves integrating cultural educational elements into teaching to convey mathematical concepts that are relevant to learners' everyday experiences (Fouze & Amit, 2023). The Ethnomathematical approach has been the subject of numerous international studies that demonstrate its beneficial effects on successful learner learning and help to create engaging and effective teaching and learning strategies. One method to help learners understand the connection between mathematics and their culture is to incorporate local cultural scenarios into the classroom. Then, they can use this connection to help with geometry teaching (Fouze & Amit, 2023). To scaffold learning, culturally relevant pedagogy emphasises the connections that learners have to their education. According to Rosa and Orey (2019), the

integration of learners' ethnic history, cultural backgrounds, and current societal interests into daily pedagogical teaching practices is validated and incorporated. All learners' academic achievement is defined and encouraged by a culturally relevant pedagogy because effective teaching occurs in a learner-centered, culturally supportive environment where learners' strengths are recognised, developed, and used to advance their academic success (Rosa & Orey 2019). Other key ideas in mathematics are geometry, both Euclidean and non-Euclidean. The review covers both because of the implementation at different grade levels of the curriculum but with more emphasis on Euclidean geometry. Nonetheless, a large number of learners studying mathematics education in schools still struggle with the ideas and theories of geometry, particularly non-Euclidean geometry. Thus, to make it more understandable, an Ethnomathematics approach is required (Sukestiyarno et al., 2023). For non-Euclidean geometry, these learners require new spatial skills. A culturally relevant pedagogy's central tenet is that "teachers should teach using philosophies and methods that respect, value, and positively use the strengths of students' home culture, context, and language" (Rosa & Orey 2019). This review also raises the possibility that some ethnic groups have unintentionally incorporated mathematical ideas involving geometry into their cultures (Monilisa et al., 2022). From an educational standpoint, Ethnomathematics makes it possible to view mathematics in the classroom as a "process of inducting young learners into mathematical aspects of their own culture" (Gilmer 1990, p. 4). Due to the requirement that a learner's community's practices and issues be more fully represented, an Ethnomathematics perspective has positively reshaped cultural identity (Sharma & Orey, 2017). Restoring a sense of enjoyment or engagement and fostering mathematical creativity are two benefits of applying Ethnomathematics as a pedagogical strategy (Rosa & Orey 2015). The application of an Ethnomathematics perspective in mathematics education allows teachers to reconsider the nature of mathematics and recognise that a wide range of individuals, regardless of their formal educational background, can measure, classify, order, organise, infer, model, and reason using algebra, geometry, and visuospatial skills. Through the identification of learners' everyday mathematical components, this pedagogical approach can enhance learners' motivation for mathematics in general and geometry in particular (Sharma & Rose 2017).

Method

The researchers used a systematic review of RAYYAN. A website or application called RAYYAN is available for free to help researchers with systemic review and meta-analysis. Users can upload full-text articles and citations together into a single review (Natasha & Margaret, 2018). To address the stated research objectives, the RAYYAN procedure is responsible for reviewing and choosing all relevant empirical evidence. Journal articles, conference proceedings, and research reports representing scientific research findings serve as the empirical evidence inferred in this case. This approach reduces bias and offers a framework and set of guidelines for reporting. Title, author, year, research goal, cultural context, methodology, and results/findings were the main criteria used to score or rate the articles for the systematic review. First and foremost, items that complemented further analysis and were pertinent to the study goals were chosen. Journal articles (both domestic and foreign), proceedings with Scopus indexing, theses, and dissertations were the target documents that were located. To locate articles that were not found on a specific search portal, document

searches were then conducted on South Africa's national ETD portal/UWC Library database, publish and perish (Semantic Scholar, Google Scholar, Scopus, web of Science, PubMed), and other search portals. A number of proceedings with Scopus index and articles from international journals were acquired. The search terms for articles include "Ethnomathematics," teaching geometry, local and traditional mathematics, and "Ethnomathematics approaches." In the meantime, searches for "Ethnomathematics approaches to teaching geometry" were conducted in Scopus proceedings and international journals.

This is a comprehensive explanation of the steps involved in choosing research sources: 1) finding journal articles and learner research findings in theses and dissertations; 2) using Mendeley to help with multiple document screening; 3) eliminating documents that are not complete; 4) choosing documents pertinent to teaching geometry and Ethnomathematics; 5) evaluating the quality of documents pertinent to the research objective; 6) summarising the pertinent documents for further analysis.

The extracted and tabulated data were presented using descriptive statistics. Next, a narrative summarising the research findings with the objectives was provided. As a source for the articles reviewed in compliance with the RAYYAN systematic review, the sample selection flowchart is provided below (see Figure 2).

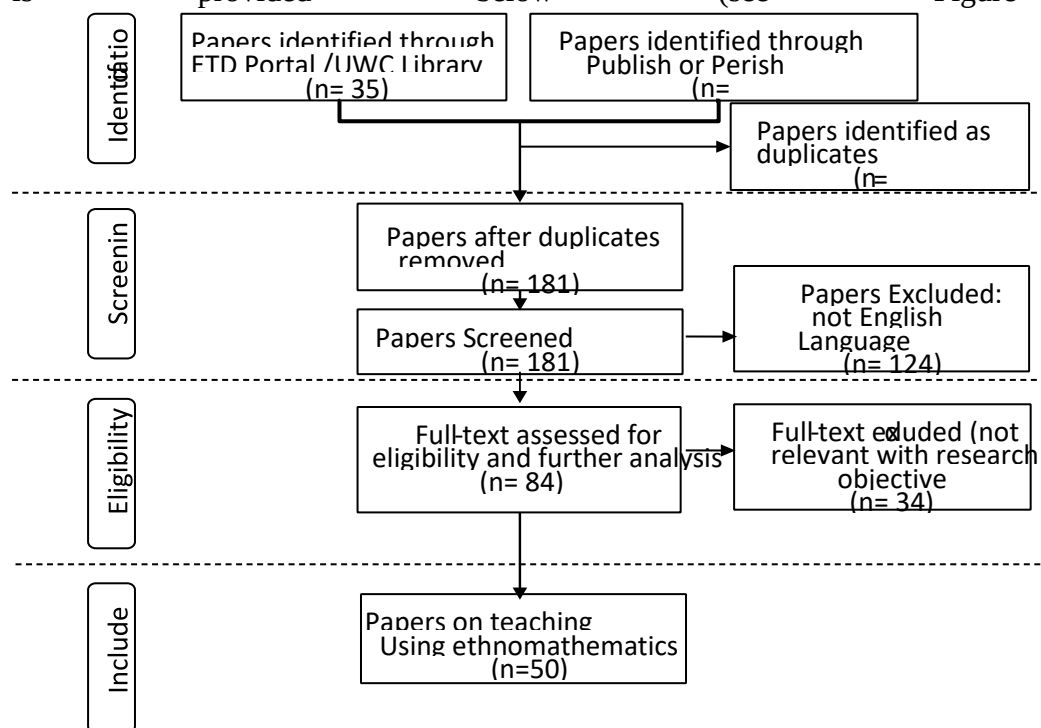


Figure 2

Flowchart of RAYYAN adapted (Lidinillah et al., 2022)

The number of sources generated at each RAYYAN stage by the selection process is provided below to provide an in-depth example of the document selection process.

Table 1. Selection of Documents for Research

Types of Document	Search Engine	Number of Documents			
		Identification	Screening	Eligibility	Included
Local/domestic Journals	Publish or Perish	64	64	31	10
International/foreign Journals	Publish or Perish	57	57	39	25
Scopus-Indexed Proceedings	Publish or Perish	35	35	15	10
Theses/Dissertations	National ETD Portal South Africa/UWC Library Database	35	25	14	5
	TOTAL	191	181	84	50

The scope of the Ethnomathematics approaches to teaching geometry study in these sources' documents was described in source documents acquired before the final selection phase. Research objectives were specifically the focus of the data presentation and discussion. Table 1 displays the identification phase. The first search phase involved employing the keywords to gather 191 articles and scientific papers. Even with this collection of documents, there are still duplicates with Ethnomathematics studies used in geometry teaching. Identification of Ethnomathematics in geometry teaching requires more than just reading the article's title; it also requires studying the abstract and full text of the articles. Following the multiple and incomplete document screening process, a total of 181 documents were left. A review of the documents to determine which ones included the study on teaching geometry using Ethnomathematics approaches, and as a result, 84 documents met the eligibility stage requirements. Papers or articles written in languages other than English were not included in the review. Finally, 50 documents were included and arranged in a table according to the following criteria: publication year, method, title, author, and findings/results to be used in the next analysis.

Results and Discussion

The field of study known as Ethnomathematics, as it is commonly used in English-language articles, is based on pedagogical approaches to teaching geometry using cultural backgrounds, customs, and artifacts in the local setting, both in Africa and throughout the rest of the world. The study excludes papers written in other languages.

Scope of the Ethnomathematics Study

The scope of using Ethnomathematics approaches to teach geometry was described in 84 journal articles, proceedings, and student dissertations. Data that were not subjected to a

selection process during the eligibility stage were used in this selection. Right now, it still has source materials that aren't about teaching geometry but are instead about studies of Ethnomathematics approaches—speaking of the four general areas of interest in Ethnomathematics described by the international study group in Ethnomathematics (Rose & Shirley, 2016). The current document sources generally cover two topics of interest: 1) Ethnomathematics pedagogical approaches and 2) using Ethnomathematics approaches to teach geometry.

Table 2. Ethnomathematics area of interest

Types of Document	Number of Documents	
	Ethnomathematics approaches	Teaching Geometry
Local Journals	21	10
International Journal	14	25
Scopus-indexed proceedings	5	10
Theses/Dissertations	9	5
TOTAL	49	50

Despite some differences in specifics, it is evident from the source document data in Table 2 that the number of Ethnomathematics approaches to geometry teaching is fairly balanced. Only one author has examined ethnomathematics in mathematics education generally out of all the source documents. The majority of these writers examined Ethnomathematics methods in light of diverse indigenous communities' cultural contexts and customs. The Ethnomathematics that were discussed connected mathematics to African and global cultural viewpoints.

Some research findings that are summarised from source materials that are thought to present investigations on the relationship between teaching geometry and using strategies/approaches in Ethnomathematics are included in Table 3.

Table 3. Findings about the relationship between teaching geometry and using ethnomathematics approaches.

Author(s)/Year	Research Focus	Approaches Used	Findings
Sunzuma, (2018)	This study investigated the readiness of in-service Zimbabwean math teachers to apply ethnomathematics methods in geometry instruction.	Social constructivism of ethnomathematics approach.	Positive views on integrating ethnomathematics approaches in geometry teaching.
Sunzuma & Maharaj (2020)	The study concentrated on how in-service secondary teachers approached and	Both teacher-centered and learner-centered approaches involve an	Teachers supported the integration of ethnomathematics

	perceived the incorporation of ethnomathematics into geometry instruction.	ethnomathematics approach.	approaches into geometry teaching.
(Hendriyanto et al., 2021)	The goal of the project was to create educational resources using the Van-Hiele learning phases and incorporate them with the ethnomathematics methodology.	Ethnomathematics approach	learners' geometric thinking skills are still low based on levels.
Mogari (2014)	The author introduced the ethnomathematics approach in South Africa through an in-service mathematics program.	Ethnomathematics approach	Teachers aligned practice with curriculum demands and improved effectiveness.
Yudianto et al. (2020)	This study addressed the use of ethnomathematics as a teaching tool for geometry in Gintanga Village, Banyuwangi, through the production of woven bamboo handicrafts.	Ethnographic approach	Geometry teaching material made from woven bamboo handicrafts for counting and designing enhanced learning.
Fouze & Amit (2023)	Bedouin embroidery samples from traditional clothing produced by Bedouin women from the Negev region in southern Israel were subjected to an ethnomathematical analysis in this study.	Ethnomathematics approach	Integrating values and traditions enhances learners' learning experiences and academic achievement in mathematics education.
(Supriyadi et al. 2022)	This study focuses on implementing Ethnomathematics in geometry teaching and its benefits, along with challenges using Bibliometric analysis emphasising the evolution of publications' most cited and current	Ethnomathematics approach	This paper revealed a lack of understanding about the Ethnomathematics approach and its integration into teaching geometry and also insufficient knowledge about geometry content and resistance to change.

	research trends on geometry.		
Massarwe et al. (2012)	This study, known as "joymetry," used building and analysis of decorations as a pleasurable way to teach geometry and culture. It was carried out in Israel.	Ethnomathematics approach	Teaching geometry in a sociocultural context significantly enhanced pre-service and in-service teachers' content and pedagogical content knowledge. It emphasises the importance of Ethnomathematics learning activities.
Owusu-Darko et al., (2023)	This study has shown that the Akan ethnic group in Ghana uses Ethnomathematics to teach geometry.	Exploratory and demonstration approach.	The study demonstrated the pedagogical action of Akan Ethnomathematics in teaching geometry.
Rosa & Orey (2019)	The embeddedness of linguistic and cultural forms of knowledge on the overall quality of students' educational experiences was investigated in social justice-focused research.	Culturally relevant pedagogy	Focuses on promoting social justice and enhancing students' educational experiences.
Verner et al. (2019)	This study concentrated on the particular skills educators require to teach geometry in a cultural context and identify opportunities for their own growth through an Ethnomathematics investigation of geometric ornamentation.	Ethnomathematics approach	Teachers developed competencies for teaching geometry in cultural contexts through professional development courses.
(Sunzuma & Maharaj, 2022)	This study examined learner variables that make it difficult to incorporate ethnomathematics methods into geometry instruction.	Social constructivist approach.	Learners' attitudes and lack of knowledge impede Ethnomathematics integration. Cultural diversity and resistance to change hinder geometry teaching.

(Sunzuma & Maharaj, 2019)	This study investigated the difficulties faced by teachers when incorporating an Ethnomathematics approach into geometry instruction.	Ethnomathematics approach.	Teachers' views on roles affect the integration of Ethnomathematics into geometry teaching and also challenges include lack of knowledge, resistance to change and teaching competence.
(Fouze & Amit, 2018)	This study highlights cultural values, traditions, and symbols manifested in the life of every society including geometry.	Ethnomathematics approaches	Learning difficulties in mathematics due to lack of meaning.
(Suharta et al., 2021)	This study determines the effectiveness of Ethnomathematics integration in learning geometric transformations (symmetry, reflections, and shifts).	Ethnomathematics approach	Ethnomathematics-based learning enhances learners' interest and problem-solving skills.
(Sevgi & Erduran, 2020)	This study investigated how students in a Senior grade in a High School participated in and interpreted the transformational geometry activities integrated with the cultural context in the Aydin district of Turkey.	Ethnomathematics approach	Learners recognised hidden mathematics in daily life and cultural games.

The results can be categorised using the framework of seven (7) universal aspects: art system, religion system, art forms, social elements, ethnic groupings (linguistic elements), and culture (Lidinillah et al., 2022). The relationship between teaching geometry and using Ethnomathematical techniques through mathematics and culture can be found in research findings, which can be used as instructional materials in classrooms and educational settings generally. With a few notable exceptions, the majority of approaches used in this study involved Ethnomathematics, or mathematics drawn from cultural contexts or experiences. These approaches included, culturally relevant pedagogy, a social constructivist approach, and the use of Van-Hiele phases as learning resources for Ethnomathematics. Thus, the Ethnomathematics approach addresses curriculum content, classroom procedures, teacher

expectations, professional development, and relationships between teachers, learners, and the community. By using this approach, learners can draw connections between the historical development of mathematics and the contributions made by various groups and individuals (Aikpitanyi & Eraikhuemen, 2017). Geometry education did not employ some of the innovative pedagogical approaches of Ethnomathematics, such as ethnopedagogy, which examines particular facets of culture, such as structure, function, and relationships with people (Klara et al., 2015). An awareness of local educational discourse has grown to such an extent that it should establish an educational approach to local culture and values of ethnopedagogy (Sulistiyani & Rahrdjo, 2018). Ethnomodelling is another important pedagogical approach appropriate for an Ethnomathematics program because it contextualises mathematical knowledge developed locally. Ethnotransdisciplinarity is an ethnomathematics approach to indigenous education, professional context, game playing, and the urban and rural context in mathematics (Rosa & Orey in 2023). According to Rosa and Orey (2023), ethnomodelling is the study of mathematical phenomena that arise in various cultural contexts.

Findings on research objective two (2) on the relationship of Ethnomathematics approaches to teaching geometry, the findings indicated the difficult nature of applying Ethnomathematics approaches to learners with different cultural diversity and experiences (Sunzuma & Maharaj, 2022). Another key finding to consider is that learners' geometric thinking skills are still low based on different grade levels (Hendriyanto et al., 2021).

As mathematics educators ensuring professional development courses for both learners and teachers enhanced their understanding of integrating Ethnomathematics in teaching geometry (Massarwe et al., 2012; Verner et al., 2019). This motivated us to help our colleagues in teacher education institutions to consider the following as pedagogical theories and their implications in the teaching and learning of mathematics especially in geometric teaching.

Table 4: A detailed description of some pedagogical theories and their implications in teaching and learning in the mathematics classroom.

Pedagogical Theory	Implications in the Teaching and Learning of Ethnomathematics in the Classroom
Reflective Thinking Pedagogy	Reflective thinking pedagogy with the aid of projects, group presentations, and lessons learned, students can improve their strengths and weaknesses, eliminate threats related to abstract concepts, and seize opportunities to enhance their mathematical performance (SWOT Analysis). The reflections extend beyond the classroom to include cultural artifacts from their homes that can deepen their understanding of mathematics.
Culturally Responsive Pedagogy	When it comes to aiding students in comprehending geometric concepts, this pedagogy especially respects diversity and addresses the need to understand diverse cultures in the classroom.

	This aims to provide fair and inclusive access to student contributions and questions from all cultural backgrounds.
Integrative Pedagogy	This pedagogy is the process of making connections between concepts and experiences through the transformation of skills applied to novel and complex problems.
Inquiry-Based Learning Pedagogy	Students are encouraged to ask questions and use a variety of concepts to find answers to this theory. Using this pedagogy, students are encouraged to critique their peers' work constructively and exhibit their own knowledge and skills, becoming independent learners. When students present their research or solve mathematical problems in small groups, this pedagogy works effectively.
Problem-Based Learning	Using real-world problems related to their culture, this pedagogy encourages students to investigate, review, and explore mathematical concepts. Through knowledge acquisition, communication, and teamwork, they achieve this. Students are divided into groups of three to five, given topics or problems, and asked to present their research and ideas for improvement.

Conclusion

The studies on Ethnomathematics approaches to geometry teaching in this study demonstrated how teachers supported the integration of Ethnomathematics in geometry teaching by linking mathematics to cultural practices, artifacts, and activities in both the learners' and the communities' surrounding contexts. However, there is a lack of understanding of Ethnomathematics approaches and how to incorporate them into their lesson delivery. Integrating Ethnomathematics is very paramount in the curriculum but some of the teachers still resist changes to 21st century approaches but rather prefer their conventional way of teaching. This is a result of insufficient content, and pedagogical knowledge, and teaching competence outlined in the study (see Table 3). The research's conclusions demonstrate how pedagogical approaches to teaching geometry using Ethnomathematics are still evolving, adapting to different conceptual approaches created in Africa and other parts of the world. The use of Ethnomathematics in the classroom was identified as the third, or number three, of the four general areas of interest by the Instructional Study Group on Ethnomathematics. In implementing Ethnomathematics approaches to geometry teaching, teacher educators should plan Ethnomathematics learning activities based on content-specific, curriculum and grade level to enhance the comprehension of geometric concepts.

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USING EVALUATIVE ASSESSMENT TO ASSESS THE KNOWLEDGE ACQUIRED BY GRADE 3 IN NUMBERS, OPERATIONS AND RELATIONSHIPS

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Continuous assessment in the Foundation Phase is vital to enable the teachers to identify the learning gap so that they are able to modify their teaching to accommodate the learning needs of all the learners. In mainstream schools it is possible that most learners do not perform well in mathematics due to the less knowledge and skills acquired by learners in the previous grades. It is likely that most learners are unable to construct new knowledge from their prior knowledge due to learning gap that they experienced. In semi-rural of Limpopo where most schools admit learners who speak different languages and are taught mathematics in vernacular in the Foundation Phase, evaluative assessment is important to assist learners in acquiring more knowledge and skills in numbers, operations and relationships. The non-equivalent comparison group quasi-experimental design was implemented to measure the difference between the pre-test and post-test between the experimental group and comparison group and subsequently decide on the significance. In this design, the experimental group (53) and the comparison (54) undertook the pre-test and post-test, and only the experimental group received the treatment. The study intended to test the hypothesis that evaluative assessment in assessing Grade 3 learners has significant effect on the improvement of performance in the NOR. In summary, evaluative assessment in assessing Grade 3 learners has significant effect on the improvement of performance in the NOR.

Keywords: Evaluative assessment, Mathematics Education, language proficiency, basic operations

Introduction

Mathematics plays an important role in people's lives in different ways whether educated or not. In the Foundation Phase (FP), it involves activities that need the application of addition, subtraction, multiplication, division, sorting, creating patterns or solving problems involving different operations (DBE, 2011). These skills and knowledge should be acquired by learners by the end of Grade R – 3. Learners are expected to have acquired skills and mathematical knowledge that would enable them to be ready to learn and understand mathematical content that will be learnt in the next grade. If learners have acquired the basic skills required to perform basic operations during early grades, they will be able to master other difficult skills required to solve challenging activities. Building mathematical skills and knowledge as early as in the Foundation Phase is important for the achievement of the learners later in the following grades. Evaluating early mathematics skills of learners offer a valuable foundation for later mathematical learning and prevent a vicious circle of poor basic skills which lead to poor mathematical learning and poor numerical limitations (Mejias, Muller & Schiltz, 2019).

Fenwick and Parsons (2009) explain evaluation as the procedure that is used to view progress from learning perspectives in order to identify the knowledge and skills gained and to identify

areas that require further effort. It is a continuous process that is used that helps to form value judgement and learners' achievement so that teaching and learning is improved (Ifeoma, 2022). Huba and Freed (2000) define assessment as the process of gathering and discussing information from learners in order to develop a deep understanding of what they know, understand, and can do with their knowledge as a result of their educational experiences in order to improve successive learning. Amua-Sekyi (2016) define assessment as all activities that teachers and learners undertake to get information that can be used to modify teaching and learning. Drawing from all of the above, we can define evaluative assessment as a process of identifying the knowledge and skills acquired by learners in order to adjust subsequent teaching and learning. Evaluative assessment in the study was used to accumulate information about the Grade 3 learners in order to get an idea of the level of knowledge and skills (Ifeoma, 2022), they have acquired in numbers, operations and relationships (NOR).

It is important to evaluate the learners' mathematical knowledge and skills in Grade 3 which is the exit grade in the FP in order to determine the appropriate support that will be provided to learners before exiting the phase. According to Van De Walle et al., (2022) mathematics presently necessitates not only computational skills but also the capability to think and reason mathematically to solve new problems and learn new ideas that learners will face in the future. Hence, early mathematics skills serve as the basic foundation of future studies of mathematics in higher grades and at tertiary level. Celemin (2023) also support that one of the key skills that the learners should acquire in order to solve the problems in academic and outside school life is problem solving skill.

There is a major concern that most learners exit the PF without fully necessarily grasping mathematical curriculum content required as the basic content of grade 4 curriculum. South African learners in the FP are expected to developed number vocabulary, number concept and calculation, application skills, reason logically and represent and interpret information (DBE, 2011) among other essential mathematics skills before exiting the FP. Learners knowledge is built from the prior knowledge; hence it is difficult to build from the missing foundation. According to Machaba (n.d), the learners should be able to explain what he has learnt or use the gained knowledge in new content. Onyishi and Sefotho (2021) support that it becomes a challenge for a learner who lacks prior knowledge and resources for a particular content to development with other learners in class when the levels of the tasks become more challenging. Most learners do not perform well in mathematics in Grade 3 and lack the skills to interpret mathematical information or link mathematical ideas to their prior knowledge.

Most learners are unable to link the prior knowledge to the new knowledge due to lack of mathematical skills. Siemon (2022) argues that learners develop mathematical understanding when they link ideas, represent the concepts in different ways, when recognising similarities and variances between features of content, when describing their thinking mathematically and when interpreting mathematical information. If learners are not able to recognise similarities and differences between the features of the mathematical content, a gap might be created that will prevent learners from understanding more advanced content related to what they could not recognise earlier.

In SA, studies have been conducted in assessing learners in FP on numbers, operations and relations. However, there is dearth of research focusing on longitudinal study in semi-rural schools using evaluative assessment providing support to learners from the beginning of the year to end of the year. This study is unique in that quazi-experimental design is used as this longitudinal evaluative research appear to be effective in the teaching and learning of mathematics. In this study, the researchers worked with Grade 3 teachers in implementing evaluative assessment providing support to learners who lacked skills in NOR to ensure that they gain those skills in NOR to avoid the recurring of lack of skills to Intermediate Phase as this can cause learners to struggle in learn subsequent mathematics in higher grades. This study enabled the researchers to measure the effect of evaluative assessment in Grade 3 when teaching and learning NOR with the teacher in the experimental school. The purpose of this study was to ascertain the effectiveness of evaluative assessment in assessing learners' skills in NOR throughout the teaching and learning of these topics. For this study to achieve this objective, the researchers gave tested the following hypotheses that, the null hypothesis was tested against the alternative hypothesis.

H₍₀₎ Evaluative assessment in assessing Grade 3 learners has no significant effect on the improvement of performance in the NOR

H₍₁₎ Evaluative assessment in assessing Grade 3 learners has significant effect on the improvement of performance in the NOR

The study has supported the alternative hypothesis that evaluative assessment in assessing Grade 3 learners had significant effect on improving learner performance in NOR. The study also should that the participation of the teacher in evaluative assessment to assess learners throughout the teaching and learning of NOR has improve the teacher's pedagogical practices which had positively impacted on the learner performance.

In this study, related literature pertaining to evaluative assessment is presented. Constructivist learning theory was used as a framework for this study. Research methodology is discussed responding to the hypothesis tested in this study. finally, the study provides the suggestions and recommendations to mathematics teachers across the phases (FP, IP, SP and FET) on evaluative assessment assessing learners' mathematical skills and knowledge to provide support that can have positive effects on learner academic achievement, curriculum advisors and developers are made.

Limitations

The study focused on Grade 3 mathematics learners in Limpopo province of SA from the beginning of teaching NOR to the end. Including other research sites may yield different results. In addition, the results analysed in this study had no significant covariates. Future studies may incorporate these additional features in their research designs.

Theoretical lens

This study espoused the constructivists framework which emphasises that knowledge is actively constructed by an individual from the existing knowledge. The study focuses on the conceptual framework of Piaget (1953) which indicates that knowledge is not passively received but is actively constructed. Bruner (1966) showed that knowledge is not merely

“taken”, but as the individual develops to higher levels of understanding, he/she actively and continuously constructed and reconstructed knowledge. Similarly, Maija and Erkki (1994) stated that knowledge cannot be transformed, but it can be sent and the receiver has to construct his/her own knowledge. They further emphasize that for knowledge to have wide-range self-determining use in our living world, it has to have many references and meanings, then knowledge which cannot be used for a particular intention remains futile. Learning of new content is mainly influenced by the existing knowledge that an individual learner might poses. This article focuses on employing evaluative assessment to assess the numbers, operations and relationships mathematical knowledge the grade 3 learners have acquired prior exiting the FP. According to constructivists point of view, learning occurs when an individual integrate new knowledge with the existing knowledge (Tracey & Marrow, 2017). Thus, integrating new knowledge without building from some structure developed from prior knowledge is impossible, hence, teaching attempt should be linked and be established on the learner’s prior knowledge (Hein, 1991).

Review of literature

The third grade is regarded as an important moment in a learner’s educational experience because the mathematical skills and reading outcomes predict the future challenges including dropping out of school or predict good outcomes in adulthood (Mattera et al., 2021). Contrary to this, Watts et al. (2018) proclaim that early learning does not appear to influence later achievement gains, hence, they think that the answer could simply be that it is nearly impossible to control for all of the potential confounds between early and later test scores. On the other hand, Mejias, et al. (2019), believe that thorough knowledge concerning learners’ early abilities allows ideal adjustment of learning and instruction to their individual needs.

Psycho linguistics agree that learners struggle to learn the basics of language usage during the age of 6 to 10 years and they might not have developed the language needed to express logical thinking (Machaba, 2021). Plana and Pimm (2023) proclaim that sometimes the language can be used in ways that are difficult to understand in teaching and learning of mathematics. If the language is presented in a confusing manner, it will be difficult for learners to understand what the problem is about. The principle of communication emphasises that mathematical should be clear and credible in both written and spoken (Cumhur & Tezer, 2020). Omidire (2019) states that learning in a language in which one lacks adequate academic proficiency has undesirable consequences that have been well documented and learners are often labelled because of their lack of proficiency in the language of teaching and learning. This means that mathematical concepts should be well presented or documental in order to promote understanding and development of mathematical knowledge to the learners.

According to Machaba (n.d), most South Africa Grade 3 learners from previously disadvantaged schools lack the ability to perform basic operations in mathematics. Likewise, Bussi and Sun (2018), state that almost everywhere in the world, most learners lack basic mathematical skills and they are unable to answer simple addition and subtraction challenges. Maghfirah and Mahmudi (2018) proclaim that learners should be taught the number sense which is a holistic notion of the capability to comprehend amounts and the measure of the numbers, in numbers, operations and relationship.

Karah (2022) found that challenges experienced by learners is comprehending what they have read which affects their mathematical learning. Hence, they dislike reading word problem because they have a challenge of identifying mathematical operation to be used in solving the problem (Mingke & Alegre, 2019). In other words, learners might be lacking the ability to translate the mathematical word statement into numerical expression. Reading skills is essential in mathematics in order to understand the mathematical terminology, explain and solve the problems. Most South African learners by the end of Grade 3 cannot read with fluency, accuracy, rhythmic pattern either in their home language or in English (Pretorius et al., 2016)

Karah (2022) further indicates that learning process is affected by the learner's level of development, age, cognitive level, prior knowledge and the learning speed of learners. Betts et al. (2020) state that deficits in mathematics understanding occur due to the gap between learners who start school after their more ready peers. Hachey (2021) asserts that in terms of mathematical pedagogy, there remain gaps in the knowledge about how best to systematically help young children to learn mathematical concepts.

Ramorola and Nyaunwe's (n.d) findings revealed that the teachers experience difficulties in teaching both gifted and less gifted learners at the same time because the gifted ones understand the content very fast whereas the less gifted take time to understand the content. Some teachers have challenges in planning their mathematical teachings according to individual learning differences so that they accommodate every individual. Vodičková et al. (2023) pointed out that most mainstream teachers do not believe they have the skills and knowledge to work with learners with diverse needs. They believe that there is an army of 'experts' who deal with specific groups of learners on an individual basis or in small, more manageable groups.

Methodology

The purpose of this study is to ascertain the whether the implementation of evaluative assessment would improve Grade 3 learner performance in NOR. The research design employed in this study can be described as non-equivalent comparison group quazi-experimental design. The aim of this research design is to measure the difference between the pre-test and post-test between the experimental group and comparison group and subsequently decide on the significance. In this design, the experimental group and the comparison undertake the pre-test and post-test, and only the experimental group received the treatment.

This study was conducted between February and September 2022 in one of the semi-rural primary school underperforming in mathematics in the Waterberg district of the Limpopo province. The school consisted of one principal, two deputy principal, four departmental heads (DHs) including the one in mathematics department, and twenty four teachers including four mathematics teachers. The school has an enrolment of 1200 learners. This school was chosen as was involved in the first researcher's doctoral study that performed differently in mathematics. The mathematics teacher and the researchers have implemented evaluative assessment in assessing learners' skills and knowledge in the teaching and learning of NOR to enhance learner learning and their academic achievement. This study is found to be notable as

the effect of evaluative assessment in assessing Grade 3 learners in the teaching and learning of NOR is somewhat under-evaluated.

The study comprised Grade 3 teachers and learners as participants. Two teachers who are females from the selected schools and 54 learners (30 females and 24 males between the ages 8 and 9) from the experimental group and 53 learners (29 females and 24 males between the ages 8 and 9), from the comparison group were purposively selected to participate in this study. Purposive sampling, which is at times described as purposeful, judgmental, or selective sampling, the researcher intentionally chooses individual participants who experienced the concept being studied (Gray et al., 2017; Creswell & Plano Clark, 2018). This study has used a pre-test prior the inception of the evaluative assessment and the post-test was administered after the treatment to measure the significant effect between the pre-test and post-test and between the two study groups. The question items were adapted from the previous Grade 3 mathematics examination question papers on the NOR.

The researchers sought permission to conduct the study from the Ethics Committee Chairperson in the College of Education, ethical clearance number 2020/11/11/31640036/07/AM. Permission was also sought from the two teachers, the teacher from the experimental group and the one from the comparison group before the inception of the study. The purpose of the study was explained with the two teachers and participation was declared voluntary and ensured that they may withdraw from the study anytime without any punitive measures. The teacher from the comparison school has only participated in invigilating the pre-test and post-test sessions. Table 1 shows the teachers' biographical information. Their names and schools were used pseudonyms to protect their identities and ensuring their anonymity.

Table 1: Teacher biographical information

Pseudonym	Gender	School	Mathematics Education qualification	Mathematics teaching experience
Sewela	Female	Experimental group	• Secondary Teachers Diploma (STD) with Sepedi and Mathematics. • Bed Degree in Education	• 34 years of teaching experience • Taught all FP subjects: Sepedi HL, Maths in both English and then in Sepedi, English First Additional Language (EFAL) and Life Skills. • Taught multigrades from (Grade 1 – 3) for 22 years • Has taught Grade 3 for 12 years

Maxaka	Female	Comparison group	• Senior Primary Teachers Diploma (SPTD) with Sepedi as the other specialization subject.	• 19 years of teaching experience. • She taught Sepedi in Grade 4 and Social Sciences in Grade for 10 years. • She has been teaching Sepedi HL, Maths in both English and Sepedi, EFAL and Life Skills for 9 years now.

Data collection procedures

A pre-test was administered in the two study groups on the same day at the same time prior the implementation of evaluative assessment in the experimental group. The two teachers assisted in invigilating learners during the administration of the pre-test and post-test. The tests took place immediately after school avoided to interrupt the smooth running of the school programmes. The researchers and the teacher introduced the evaluative assessment in the experimental school, they critiqued every lesson using questions and answers to assess learners' skills and knowledge. Furthermore, activities were given to assess the learners to plan a support plan in the next lesson to ensure that all learners develop skills and knowledge of NOR prior promoted to IP. Then, the teacher will apply the strategies suggested in the de-briefing sessions. The treatment took eight months, and the lessons were conducted twice a week with the first researcher visited the school to do lesson observations and conducting de-briefing to suggest the possible strategies to support the learners who lacked knowledge and skills of NOR. After the treatment, the researchers then administered the post-test in the two study groups to measure the effect of the treatment between the pre-test and post-test if the hypothesis is supported when using evaluative assessment has significant effect in teaching NOR.

The researchers have established the validity of the learner assessment results (Creswell & Creswell, 2018). After marking of the scripts, the researchers gave the two teachers the marked scripts for verification purposes. The data were organised and were given to a professional statistician to analyse the data using Stata Release computer software. The statistical analysis data from the statistician was given to the three teachers for confirmation and approval.

Data analysis procedures

The researchers used Microsoft Excel to capture, organise and manage the data from the pre-test and post-test scripts (Microsoft, 2015). The mean scores were used to determine the significant difference between the pre-test and post-test, and then the two study groups. A t-test was also used to compare the mean scores and to determine the significant difference using the p-value. So, the mean score was used to determine the results of the treatment determined

at a 95% confidence limit. note, the results are declared significant if the probability value (*p-value*) is less than 0.05, that is, there is a 0.05 chance that the result could have been attained by chance.

Results

This study intended to ascertain whether the evaluative assessment assessing Grade 3 learners' skills and knowledge in the teaching and learning of NOR to enhance academic achievements. If the $p\text{-value} > 0.05$, then the outcomes are not significant, conversely, if the $p\text{-value} < 0.05$, then the outcomes are significant.

The mean scores used to test the significant difference between the two groups, is determined per questions (Q1-Q3, Q4-Q6, Q7-Q10 and the summary results Q1-Q10). Table 2 presents data collected from Q1-Q3, assessing learners' understanding on numbers.

Table 2: Q1-Q3 tests results

Setting	Group	obs	Mean	Std.Dev	p-value	Conclusion
Pre-tes	Comparison	54	2.1621622	2.5874598	1.5212	No significant
	Experiment	53	2.6315789	2.7843968		
Post-test	Comparison	54	3.2352947	2.726856	1.6917	No significant
	Experiment	53	3.5454544	3.0135425		

Table 2 above shows that there was no significant difference in the pre-test results between the experimental ($x=2.6315789$) and the comparison ($x=2.1621622$) groups with the $p\text{-value}=1.5212$ which was more than 0.05 at 95% interval. The outcomes displayed that the experimental group documented the mean score greater than the comparison group in the pre-test for questions 1 to 3 which assessed learners about writing the number represented by based ten blocks and arranging of numbers in order.

The findings from the post-test revealed that the comparison and experimental groups' performance had no significant difference ($p\text{-value}=1.6917$) higher than 0.05. However, the experimental group achieved a higher score $x=3.0135425$ more than the comparison group $x=2.726856$. The results displayed that some learners' knowledge in linking the based ten blocks to the numbers, recognising similarities of the features of content, when describing their thinking mathematically and when interpreting mathematical information (Siemon, 2022) improved in the post-test.

Table 3: Q4 – Q6 tests results

Setting	Group	Obs	Mean	Std.Dev	p-value	Conclusion
Pre-test	Comparison	54	2.9594597	2.7009263	1.5918	No significant
	Experiment	53	2.7543864	2.9519486		
Post-test	Comparison	54	3.1029409	2.7080021	0.3887	No significant
	Experiment	53	4.0000007	2.8192359		

Table 3 above reveals that the comparison group performed better with the mean score $x=2.9594597$ as compared to the experimental group's $x=2.7543864$ when comparing, identifying and doubling the numbers in the pre-test results. The results showed no statistically significant difference between the two study groups with the $p\text{-value}=1.5918$ that is more than 0.05 at 95% interval. Even though the experimental group achieved the higher mean than the comparison group, the results demonstrated that most learners lack basic mathematics skills and are unable to answer simple addition and subtraction challenges as indicated by (Bussi & Sun, 2018) since the mean score was low. However, in the post-test, the mean score for the experimental group improved to $x=4.0000007$ as associated to $x=3.1029409$ of the comparison group. The enhancement demonstrated that some learners who lacked the knowledge of performing basic mathematics operations (Machaba, n.d) from the pre-test, had gain knowledge.

Table 4: Q7 - Q10 tests results

Setting	Group	Obs	Mean	Std.Dev	p-value	Conclusion
Pre-test	Comparison	54	2.7908182	3.2671937	0.8938	No significant
	Experiment	53	4.7894742	4.235377		
Post-test	Comparison	54	5.2352939	3.5834368	0.5383	No significant
	Experiment	53	6.2181818	3.9458159		

The outcomes from table 4 above displays some improvement for both experimental and comparison groups for Q7 to Q10. The results displayed that the mean score for experimental group increased from the pre-test ($x=4.7894742$) to the post test ($x=6.2181818$) and the comparison group from the pre-test ($x=2.7908182$) to post test ($x= 5.2352939$) when assessing learners in word problems. Nonetheless, the outcomes demonstrated that there was no statistically significant difference between the two groups from both tests with $p\text{-value}=0.8938$ and $p\text{-value}=0.5383$, respectively for pre-test and post-test which were greater than the 0.05 at 95% interval. The outcomes from the pre-test displayed that most learners experienced the challenges of comprehending what they have read from word problems which affected their performance (Karah, 2022). This support what (Mingke & Alegre, 2019) postulated that most learners dislike reading word problem because they have a challenge of identifying mathematical operation to be used in solving the problem. Nevertheless, the development in the post-test revealed some learners from the experimental group could build new knowledge from the previously learnt information that helped them to be effective (Cumhur & Tezer, 2020) and solve some word problems.

Table 5: Summary of performance of control and experimental groups in pre- and post-tests

Setting	Group	Obs	Mean	Std.Dev	p-value	Conclusion
Pre-test	Comparison	54	31.62163	21.26074	0.0255	Significant
	Experiment	53	40.70176	24.63589		
Post-test	Comparison	54	46.29413	20.91993	0.0268	Significant
	Experiment	53	55.05456	22.32373		

The results for summary of learners' performance from the table above disclosed that the comparison and experimental groups performed significantly differently (p -value = 0.0268), less than 0.05 from the post-test, as compared to the pre-test (p -value = 0.0255). The results demonstrated that the experimental group exhibited improvement in the post-test with the mean score of ($x=55.05456$), as compared to the pre-test ($x =40,70176$). In particular, the experimental groups scored a higher mean or average ($x=55.05456$) than their control counterparts ($x=46.29413$) in the post-test. The low results of the pre-test might have been what (Vodičková et al., 2023) pointed out that most mainstream teachers do not believe they have the skills and knowledge to work with learners with diverse needs. The experimental teacher prior the evaluative assessment planned her mathematical teachings according to individual learning differences so that she accommodate every individual. The learners appeared to have developed knowledge and skills when answering NOR questions. This concur with Siemon (2022) that learners develop mathematical understanding when they link ideas, represent the concepts in different ways, when recognising similarities and variances between features of content, when describing their thinking mathematically and when interpreting mathematical information. For learners to effectively gain more knowledge, skills and understanding, they should be taught the number sense which is a holistic notion of the capability to comprehend amounts and the measure of the numbers, in numbers, operations and relationship (Maghfirah & Mahmudi, 2018).

Discussion

The results for Q1 to Q3 in the pre-test showed that both experimental and comparison groups did not perform well in identifying numbers presented with base ten blocks. the results displayed that there was no significant difference between the two groups with the p -value= 1.5212 which was higher than 0.05 at 95% confident level. The learners could not link the based ten blocks to the numbers, recognising similarities of the features of content, when describing their thinking mathematically and when interpreting mathematical information (Siemon, 2022). However, Van De Walle et al., (2022) highlighted that mathematics presently demands not only computational skills but also the capability to think and reason mathematically to solve new problems and learn new ideas that learners will face in the future. The post-test outcomes for Q1 to Q3 revealed that the was no significant difference between the performance of the two groups with the p -value=1.6917 higher than 0.05. This might have been caused by lack of basic mathematical skills in numbers.

Similar to Q1 to Q3, the performance of learners from Q4 to Q6, the results displayed that the was no significant difference from the performance of the two groups with the p -value=1.5918 that is more than 0.05 at 95% confident level. The findings revealed that most learners lack basic skills of identifying numbers represented by base ten blocks. This might be that most learner did not acquire much knowledge and skills from numbers in their previous grades or were not yet ready to learn the concept. Karah (2022) indicated that learning process is affected by the learner's level of development, age, cognitive level, prior knowledge and the learning speed of learners. Betts et al. (2020) noted that shortfalls in mathematics understanding occur due to the gap between learners who start school after their more ready peers. However, the post-test results reported some improvement for both groups, in particular mean score for the

experimental group improved to $x=4.0000007$ as associated to pre-test $x=2.7543864$. The evaluative assessment might have assisted the experimental teacher to teach learners the number sense which is a holistic notion of the capability to comprehend amounts and the measure of the numbers, in numbers, operations and relationship (Maghfirah & Mahmudi, 2018).

The outcomes Q7 – Q10 showed that the experimental and comparison groups did not perform differently from the pre-test. The findings showed that there was no significant difference between the two groups from their mean score with the p-value of 0.8938 above 0.05 at 95% confident level. Similarly, from the post-test the performance of the two groups displayed no significant difference with the p-value 0.05383. However, the experimental group improved significantly from the post-test $x=6.2181818$ than the pre-test $x=4.7894742$. with the mean score. The performance might have been due to some of the learners being able to comprehend what they have read from word problems (Karah, 2022). Some learners might have gain knowledge of identifying mathematical operation to be used in solving the problem when reading the problem (Mingke & Alegre, 2019).

The results for summary reported that displayed that the experimental group obtained better mean score from the post-test $x=55.05456$ as compared to the pre-test $x=40.70176$. This might be that the evaluative assessment assisted the experimental teacher to help learners not to passively receive knowledge but to actively construct their knowledge (Piaget, 1953) during teaching and learning. Learners should not be left to be knowledge absorbers but be encouraged to continuously construct and reconstruct knowledge in order to develop to higher level of understanding (Bruner, 1966).

Constructivists emphasized learning occurs when an individual integrate new knowledge with the existing knowledge (Tracey & Marrow, 2017). Subsequently, integrating new knowledge without constructing from previous knowledge is impossible, hence, teaching attempt should be linked and be established on the learner's prior knowledge (Hein, 1991). Linking ideas, representing concepts in different ways, recognising similarities and variances between contents in interpreting mathematical information enable the learners to develop mathematical understanding (Siemon, 2022) and to construct their own knowledge.

Since the performance of the comparison group also improved during the post-test, and did not take part in the intervention, it provides a challenge as what contributed to the improvement of their mean score. Mejias, et al. (2019) highlighted that thorough knowledge concerning learners' early abilities allows ideal adjustment of learning and instruction to their individual needs. Maija and Erkki (1994) postulated that for knowledge to have wide-range self-determining use in our living world, it has to have many references and meanings, then knowledge which cannot be used for a particular intention remains futile. Which provide the conclusion that learners from the comparison group might have used their prior knowledge in a fruitful way.

Conclusion

The study concluded that learners should be assisted in using their prior knowledge in order to construct new knowledge when learning more advanced skills. By continuously constructing

and reconstructing knowledge, the individual learner develops to higher levels of understanding since knowledge which cannot be used for a particular intention remains futile. The study showed that evaluative assessment allowed the teacher to encourage learners in constructing their own knowledge rather than simple absorbing knowledge without understanding the concept being taught.

For learners to perform better, it depends on the ability of the individual being able to use the prior knowledge to construct and reconstruct new knowledge. Teachers should continuously evaluate learners with the purpose of identifying the strategies they can use to help them to close mathematical gaps experienced during teaching and learning of a particular concept. Learners should be encouraged to be actively involved during teaching and learning so that they do not become passive knowledge absorbers. Assessments that will assist teachers to assess for active learning should be identified and actively implemented so that when learners exit the phase they do not experience any learning gap in mathematics.

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EXPLORING OUTDOOR MATHEMATICS LEARNING FOR CONCEPTUAL UNDERSTANDING THROUGH SMARTPHONE.

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The majority of learners in South Africa face challenges in understanding trigonometry concepts. The challenges could be ascribed to interrelatedness of trigonometry topics and other mathematics topics. There are various mathematical programmes and applications that teachers and learners may utilise for educational benefit. One such application is the MathCityMap application that uses mathematics trails (outdoor tasks). This application nudges learners to make appropriate mathematical connections during the trailblazing (walking of the mathematics trail). This study focuses on outdoor mathematics learning for conceptual understanding, using a smartphone. The twelve mathematics learners of grade 11 at secondary school-from Lejweleputswa district in the Free State Province- were purposefully selected to take part in this mixed method of exploratory case study, premised on Realistic Mathematics Education as a lens. Data was collected through questionnaire (survey), observations and focus group interviews. The findings of the study revealed that outdoor mathematics learning through a smartphone has potential to enhance conceptual understanding. Therefore, the study recommends the use and integration of targeted outdoor activities in the mathematics teacher's annual teaching plan.

Keywords: Outdoor tasks, conceptual understanding, realistic mathematics education, smartphones

BACKGROUND AND INTRODUCTION

The learning of mathematics in South African schools has been and is still met with challenges (UNESCO, 2023; TIMSS, 2019). Many different strategies, initiatives and resources have been invested in the learning of mathematics yet there seems to be little improvement (Taylor, 2021). There are many factors ascribed to the mathematics education status quo in South Africa. One could be how learners negatively perceive mathematics in South Africa and compounded by how it is presented to them (Jojo, 2019). Research has shown that many learners perceive mathematics as an abstract subject that is confined within four walls of the classroom without any relevance to the real world (Sunzuma et al, 2021; Mutambara, 2020; Mutodi 2014).

Commenting on observations made from the South African context, Malatjie and Machaba (2019) posit that ‘... a learner has conceptual understanding when he or she is able to explain, describe and apply the same concept in different ways and in different situations’ (p. 2). Thus, learning approaches that help learners to see the integration of concepts and that mathematical concepts are not isolated but connected are necessary. Learners exposed to such approaches may better understand why a mathematical idea is important and it is useful in particular contexts. When learners have conceptual understanding, they can organise their knowledge coherently, enabling them to learn new ideas by connecting them to what they already know (Wilson & Heid, 2015). Teaching for conceptual understanding presents several advantages to

learners during and after learning. As argued by some researchers (Gilmore & Bryant, 2008; Hallet et al, 2012; Wiggins, 2014), learners with conceptual understanding can identify and apply mathematical ideas that are key to a particular problem and explain why they are important.

According to Rojano and Sutherland (2018), current mathematics curricula in many countries emphasize the integration of technology into mathematics learning. Using technology has potential to improve mathematics learning (TIMSS, 2019, Mullis, Martin, Goh and Cotter (2016)). Thomas (2023) and Pieterse (2016) state that smartphone usage among teenagers is so rife that when taken away, withdrawal symptoms are similar to those experienced in substance abuse. Studies have since shown that the academic usage of smartphones is now an integral part of the learners' modern lives in general (Donga, 2018; Johnston & Ophoff, 2014). Teenagers in South Africa account for 40% of social media users (Lama, 2020). The COVID-19 (Corona Virus) pandemic posed a lot of challenges that we had to face and solve in education, in particular mathematics learning. Overcrowding in classrooms has resulted in some learners having to alternate days of school attendance. As a result, the incorporation of technological devices such as smartphones in the learning of mathematics could assist in meeting these challenges.

The Department of Basic Education National's Curriculum Statement (NCS) (2011) grades R–12 advocates for a learner-centred approach to learning. It subscribes to an active and critical approach rather than rote and uncritical learning of given truths. The mathematics curriculum is based on the constructivist approach, which is crucial in this study. The Department of Basic Education (2011) aims to produce learners who can use science and technology effectively and critically demonstrate an understanding of the world as a set of related systems by recognising that problem-solving contexts do not exist in isolation. However, most schools in South Africa barely use smartphones and outdoor mathematics tasks because smartphone use on school premises is prohibited and schools still use traditional methods of teaching and learning (Venkat, 2021; Mkhize & Nair, 2022). Mpungose (2020) states that learning in South Africa is traditional and does not consider learners' experiences. He argues that learning happens in the classroom where a teacher dispenses knowledge and learners are mere recipients (teacher-centred). It is further indicated that a classroom is characterised by traditional resources like textbooks, chats and chalkboards amongst others. Gontyeleni (2019) participated in the Global Education and Skill Forum (GESF) in Dubai (2019), where more than 2,000 delegates from 144 countries attended. She realised that SA needs to shy away from traditional methods of teaching and learning, where learners are only the receivers of content and teachers are the deliverers. She further states that teachers need to create opportunities where learners gain autonomy, take charge of their learning and can dissect and engage with the content that enhance their conceptual understanding. The GESF forum emphasises the use of technology in learning. Even the work schedules or pace setters from DBE omit the use of smartphone and outdoor mathematics tasks. In my opinion DBE does not give enough or the same energy as other ideas stipulated in the Curriculum Assessment and Policy Statement (CAPS), For example, in the DBE (2011) NCS, it is stated that grades R-12 real-life problems should be

incorporated into all sections ‘whenever’ is appropriate and that they should be realistic. This may be interpreted as optional.

This study employed innovative learning of mathematics through outdoor tasks in the form of mathematics trails. Mathematics trails are mathematics tasks tied to real objects; learners need to locate them through the navigator on their smartphones. Therefore, in this study learners walked mathematics trails using smartphones with the MathCityMap (MCM) app to enhance conceptual understanding. This paper reports on how the MCM app, among other things, assisted learners in locating real-world problems on the school premises and solving them using trigonometry concepts they learned in grade 11. This study asserts that mathematics trails on trigonometry helped develop learners’ conceptual understanding of trigonometry concepts and acknowledges the relevance and utility of trigonometry in real life. The study was thus guided by the following research question: How does the use of smartphones to learn mathematics through outdoor tasks enhance conceptual understanding of trigonometry concepts in a selected grade 11 class?

Mathematics learning in South Africa

Learning, especially of mathematics, is much deeper than memorization and information recall. Deep and long-lasting learning involves understanding, relating ideas and making connections between prior and new knowledge (Wokoma 2020). In this paper we argue that learning of mathematics is not something done to learners, but rather something learners do themselves (Strickey, 2021). Learning is thus an ongoing, active processes of inquiry, engagement and participation in the world around us (Paul Main, 2023; Bransford, Brown & Cocking, 2000). In constructivist terms, learners construct their own knowledge and understanding using situations presented to them. As opposed to passively receiving information, learners reflect on their experiences, generate mental representations and incorporate new knowledge into their schemas. This promotes deeper learning and understanding (Mcleod, 2024). Through outdoor activities, this study focussed on creating an active learning space where learners discussed and debated ideas as they walked mathematics trails.

Researchers in education argue that quality mathematics learning in South Africa will remain a phantom unless teachers make an effort to understand how their learners think, learn and adapt ways of teaching (Moloi, 2012; Karrim 2023). Learning trigonometry prepares learners for advanced and challenging topics such as complex numbers, limits, derivatives and integrals (Nabie et al, 2018). However, it is perceived to be one of the challenging topics in the South African Further Education and Training (FET) phase (Ngcobo et al, 2019). A study by Rankweteke (2020) in Limpopo Province found that many learners were passive, listened to or watched the teacher without being actively involved in the learning process of trigonometry. The teacher-centred approach was and still is the dominant teaching strategy used in South Africa (Maile & Makofane, 2019). In addition to this, learning is seen as a set of predetermined content to be learned by rote without any critical thinking and engagement with or exploration of the information (Jojo, 2019). Learners were simply expected to give the correct answer without any meaning or explanation for the answer being provided or discussed (Taylor, 2021). This practice is still prevalent in many schools even today. Our involvement in the Association

for Mathematics Education in South Africa (AMESA) activities for many years and in many grade 12 intervention projects in the Free State province attest to this.

Therefore, there is a need to invest in an alternative learning strategy for developing the learning of mathematics. Thus an important work of a teacher is to plan for diversity or different learner abilities. One way is the use of outdoor mathematics trails. The mathematics trail use of a smartphone is characterized by problem-based learning approach and is a teaching strategy in which teachers present content using real-world tasks (learner-centred). It requires reasoning, collaboration and develops learner understanding of mathematics concepts (Braintrust Tutors; Van de Walle et al., 2007; Rezio, Andrade & Teodoro, 2022). Traditionally, mathematics is often seen as something abstract, something that can only be learnt through textbooks and predominantly indoors (Cahyono, 2017). The MCM presents an alternative way of learning mathematics with the help of interesting and authentic tasks designed from real outdoor objects and arranged in a sequential trail of well-chosen authentic tasks that learners can follow and do. Following the trail, learners have a platform to discuss, share ideas among themselves and come up with different solutions. Hence, every participant involved in this study had an opportunity to learn from each other (Jablonski et al., 2018). Thus, this study employed outdoor mathematics tasks, where learners walked mathematics trails using smartphones with the MathCityMap (MCM) application. This happened without the teacher standing in front of them, directly presenting facts and concepts to them (learners).

Conceptual understanding

Conceptual understanding is part of the five interwoven mathematical proficiency strands identified by Kilpatrick, Swafford and Findell (2001). Conceptual understanding is an integrated and functional grasp of mathematical ideas and with this understanding learners know more than isolated facts and methods. Conceptual knowledge enables them (learners) to learn new ideas by connecting these ideas to what they already know (Kilpatrick et al., 2001). Such learners understand why a mathematical idea is important and the kind of contexts in which they are useful. When learners have conceptual understanding, they can organize their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know (Wilson & Heid, 2015).

Teaching for conceptual understanding presents several advantages to learners during and after learning. Kilpatrick et al (2001) assert that competence in an area of investigation relies upon knowledge that is not only stored but linked and structured in ways that facilitate suitable retrieval and application. Thus, learning with understanding is more beneficial than just memorizing because the organization enhances retention, promotes fluency and facilitates learning related material. Because concepts and procedures learned with understanding are interrelated, they are easier to recall and apply, and they can be reconstructed when forgotten. (p.118).

The following are the key conceptual understanding indicators adapted from Kilpatrick et al. (2001):

Connecting concepts and ideas in mathematics: Learners who have a conceptual understanding of trigonometry would be able to apply the knowledge in another knowledge domain, such as geometry. Secondly, a learner conceptually understands trigonometry when he or she is able to represent the same concepts presented in a particular mathematics trail in diverse ways (*representing mathematical concepts in multiple ways*). Learners would be able to provide explanations that are consistent and connected and not in isolation to one another. Thirdly, conceptual understanding would help the learners to think and reason inductively and deductively when solving outdoor mathematics tasks using a smartphone (*Extending mathematical concepts to real-life contexts*). Lastly, learners bring with them an existing amount of knowledge which they use to build on or access new knowledge which they would make sense of (*Building on prior knowledge*). This suggests that a learner's prior knowledge will give meaning to the new knowledge, after it has been organised, structured and restructured. Learners have to grapple with new ideas to incorporate them into an existing schema, in order to enhance their mathematical concepts and those of others.

Theoretical framework

This study is underpinned by the Realistic Mathematics Education (RME) theory, considered as a teaching and learning theory in mathematics education (de Lange, 1996). RME has its roots in Hans Freudenthal's interpretation of mathematics as a human activity (Freudenthal, 1973; Gravemeijer, 1994). According to Freudenthal (1977), mathematics must be connected to reality, stay close to learners and be relevant to society in order for it to remain valuable to humanity. In the CAPS document mathematics is outlined as a human activity (DBE, 2011). The principles of RME are implied in the Curriculum and Assessment Policy Statement (CAPS). The curriculum (CAPS) aims to ensure that learners actively construct knowledge, apply it, and procedures in ways that are meaningful to their own lives. Rather than rote and uncritical learning. In this manner, the curriculum encourages knowledge in local contexts (mathematics trails), while being sensitive to global imperatives. Hence this study employed outdoor mathematics tasks around school premises.

While learning, learners go through stages referred to in RME as horizontal and then vertical mathematisation (Rasmussen & King, 2000). Developing from Freudenthal (1977)'s work, Treffers (1987) formulated the idea of two types of mathematisation, by distinguishing 'horizontal' and 'vertical' mathematisation. Jupri and Drijvers's (2016) define horizontal mathematisation as the process of transforming a realistic problem (mathematics trail tasks) into a mathematical problem, by representing the task in a model (emergent or self-developed model principle) and solve the outdoor task; and thereafter interpret the solution (that is, vertical mathematisation). Models are used to yield learners' growth in understanding of mathematics concepts. (Kant, 2020). The latter refers to RME principles such as learners' use of interaction, contributions and use of topic relatedness. The RME argues that learners can reinvent mathematical ideas, insights, knowledge, procedures, and concepts through mathematisation (guided reinvention principle using a smartphone). Further, allows exploration of real-world problems and not as a place to merely transfer knowledge of mathematics from a teacher to learners (Wahyudi, 2017). Models (a mathematical tool replicating real-world situation) in mathematics are used to enhance understanding of

mathematical concepts. (Van den Heuvel-Panhuizen, 2003; Kant, 2020 & Da, 2023). Learners should, therefore, learn mathematics by mathematising reality. Therefore, RME principles required learners to be active participants and assisted to enhance their conceptual understanding all without the direct involvement of the teacher, but the smartphone played adult guidance role.

The traditional formal and authoritarian approach to teaching mathematics that has dominated South African classrooms for a number of years has not afforded learners many opportunities to make use of horizontal mathematisation. (Harber & Serf, 2006; Stols et al, 2015 & Mbhiza 2021). The danger in this is that when learners have entered that process without first having gone through a process of horizontal mathematisation, a strong possibility exists that if they forget the algorithms they were taught, they do not have a strategy in place to assist them in solving the problem (Barnes, 2005; Tshesane, 2014; Kant, 2020). Learners should therefore learn mathematics by mathematizing subject matter from real contexts, as a source of development, application area and their own mathematical activity rather than from the traditional view of presenting mathematics to them as a ready-made system with general applicability (Gravemeijer, 1994; Da, 2023).

This study concurred with Freudenthal in exposing learners to mathematics trails (real-world tasks). Learners discussed ideas, interpreted information, and responded to questions or represented information using mathematical models (a mathematical tool replicating real-world situation) and symbols (mathematisation). RME is one of the learning approaches that remedies problems caused by traditional and abstract mathematical learning (Bray & Tangney, 2015). Misunderstandings and misconceptions related to learning mathematical concepts can be fixed or addressed by employing RME. (Laurens et al., 2018). The use of RME in the mathematics trails provides tasks tied to real-world objects, which assisted learners in solving the problems, and improved their understanding (Laurens et al., 2018). The study documents different experiences and opinions as reflected by data that were extracted from survey/questionnaire, observed mathematics trails and focus group interviews.

Research methodology

The study employed a case study design as the main research method to explore, whether (or not) the use of smartphones to learn mathematics through outdoor activities can enhance conceptual understanding of trigonometry concepts in a selected grade 11 class? The study participants were purposively selected. Purposive sampling intentionally selects participants based on their knowledge, characteristics, experience or other appropriate criteria (NCSC, 2023). Twelve purposively selected grade 11 mathematics learners participated in this study. The criteria were that learners must be in grade 11 during the time of the study, they must be doing mathematics at school and are willing and available to participate in the study.

We designed two mathematics trails, comprising five tasks each. The mathematics trails were designed using objects in the school grounds and premises. The tasks were also aligned to grade 11 trigonometry concepts (heights and distance, angles, trigonometry ratios, sine, cosine and area rule). The designed tasks were uploaded on to the MCM application. The tasks, three hints

per task and a sample solution for each task were developed and uploaded on the app as well. It must be noted that these are the requirements stipulated for the tasks to be accepted on the app. The MCM technical team, who are themselves expert teachers of mathematics, assessed and approved our tasks. This is one way in which our tasks were validated.

The first trail had five tasks that looked at the following:

- School flagpole (calculate the height),
- School billboard (The size of the angle that the supporting pole in the middle makes with the ground),
- Roof and wall (the magnitude of the angle that the roof makes with the horizontal),
- Pipes at angle (calculate the acute angle between the water pipe and the drainage pipe), and
- Drainage pipes (calculate the acute angle formed where the two drainage pipes meet).



Task A: School flagpole:

Calculate the height of the school flagpole in metres.

The second mathematics trail also comprised five trigonometry tasks that focused on:

- School grandstand (calculate the slope of the grandstand in degrees),
- Grandstand stairs (Find the angle of depression of the grandstand stairs),
- School flagpole (Calculate an angle of elevation of the school flagpole, from the observer standing 680 cm away from the flagpole),
- Drainage pipes (Determine the size of the area bound by drainage pipes), and
- Roof verge boards (calculate the magnitude of the angle that the two roof verge boards make).

When learners got stuck or did not know where to start, they would press the hints button and they would also get the sample solution to compare answers. They did not need a teacher but a smartphone to enhance trigonometry conceptual understanding. To locate the task, learners had to use the navigator on their smartphone.

Data was collected through observations while learners were walking the trails and the interviews after learners had walked the trails. The observations and interviews were video recorded.

The mathematics trails were located within the school premises. The learners were divided into 4 groups of three members each, with each member having alternate roles to play. Series of 40 mathematics trail observations sessions were conducted and enabled the researchers to identified learners' experiences with MCM app and to give meaning to them by highlighting conceptual understanding indicators; and RME principles to analyse if conceptual understanding was enhanced. In this study a variety of data collection and analysis techniques were employed to ensure the credibility of data and data analysis. Through engagement with the participants, with survey, mathematical trails and interviews (focus group), an effective vehicle for obtaining and processing reliable information was established and maintained. Triangulation was another means of strengthening data collection and analysis, to seek multiple and comparative opinions about the same issue.

During trailblazing, the mathematics trails based on grade 11 trigonometry concepts were located through navigation system on the smartphones. Please note that: G stand for girl and B for boy, subscript number (e.g. 4) stand for group number, subscript letter/alphabet (e.g.) it means there is a gender that repeats (we differentiate by saying the first boy will be B_{2a} and the second one will be B_{2b} , meaning they are both in group two).

Findings and recommendations

This section reports on the findings and discussions. To organise our analysis, we used conceptual understanding indicators adopted from Kilpatrick et al (2001) and RME principles developing from Freudenthal (1977)'s work. With these we explore the learners' enhanced conceptual understanding guided by mathematics trails using MCM app (smartphone), its components: trigonometry tasks tied to real objects, hints, sample solution and navigator.

Outdoor tasks and learners' responses

Theme 1: Multiple representations and mathematical connections

Task 1: **Grandstand stairs:** Find the angle of depression of the grandstand stairs. (Assume that the ground is horizontal).

Excerpt 1: Group 2's discussions while walking trail (school grand stairs)

B_{2a}: In simple terms angle of depression is the angle below x-axis, so when we talk about the angle of depression, we are referring to this (drawing a horizontal line on the page for fellow group members to see), that's the angle of the depression (pointing to the drawing on a page). Refer to figure 1 below.

B_{2b}: Ok let's go (heading the grand stairs). Angle of depression is this one (pointing at the angle between the stairs and the stick that he used as a learning aid). Refer to figure 2 below.

*B_{2a}: Look, let's say ... (drawing on the page the sketch to represent the angle of depression and elevation in such a way that they form a Z-shape), angle of depression is below the x-axis akere (**right?**)? Refer to figure 3 below.*

B_{2b}: Yes

*B_{2a}: Angle of elevation ke (**is**) this angle (pointing at the drawing on the page) that is above the x-axis, because angle of elevation ya (**it elevates**) eleveita eya hodimo (**it goes up**), ena e tsang? Depression e ya fatshe (**goes down**), you see?*

B_{2b}: Ya (yes)

B_{2b}: So that the line you are talking about is parallel to the horizontal surface.

B_{2b}: Employ the angle of elevation since angle of elevation and depression are alternate angles.

B_{2a}: So, our beta is 35,7 degrees.

B_{2a}: checking the answer on the MCM app. They were happy to get the positive feedback they also checked the sample solution despite the fact that they got it correct

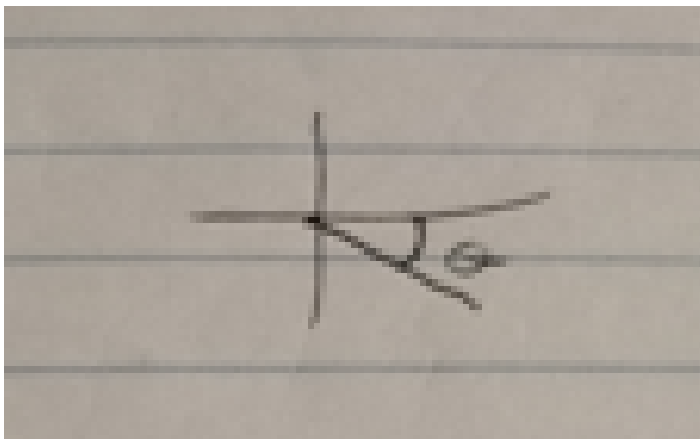


Figure 1: Angle of depression drawing (Group 2)



Figure 2: Group 2 trying to identify angle of depression

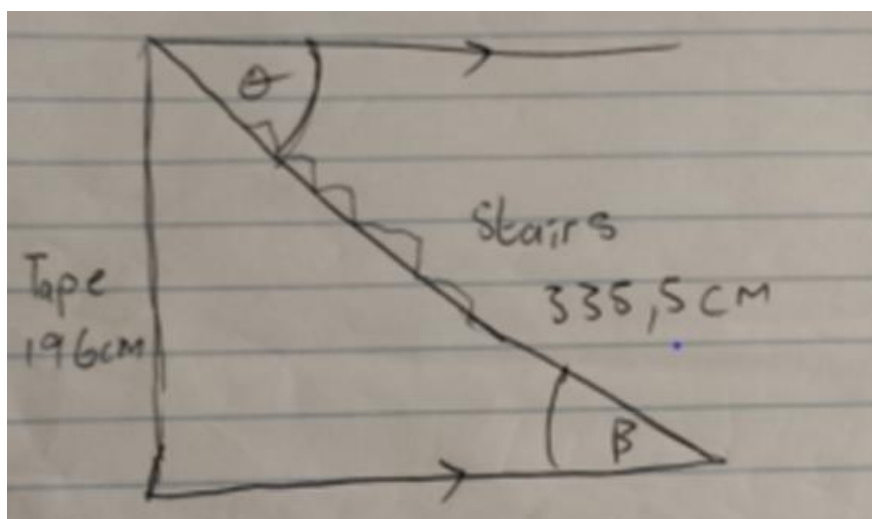


Figure 3: Angle of elevation and depression (Group 2)

The data presented in Excerpt 1 and in Figures 1, 2 and 3 above showed that learners used different representations, ranging from verbal representation, as B_{2a} said: *'In simple terms angle of depression is the angle below x-axis'*, to diagrams (external representation) as shown in figure 1, 2 and 3 learners illuminated the concept depression as an angle in diverse ways and the relation between it and angle of elevation. Thus, learners seemed to have gained a better understanding and successfully solved the mathematics trail tasks because they debated and used different representations, which is one of the crucial elements of conceptual understanding (CU). B_{2b} demonstrated *'Ok let's go (heading the grand stairs). Angle of depression is this one (pointing at the angle between the stairs and the stick that he used as a learning aid)'*. (that was after the illustration of the same angle on paper and already verbalised, as indicated in excerpt 1) Refer to figure 1 and 2. In excerpt 1, B_{2a} added by saying *'Look, let's say ... (drawing on the page the sketch to represent the angle of depression and elevation in such a way that they form a Z-shape), angle of depression is below the x-axis akere (right?)'* (Refer to figure 3). The latter suggested that learners indicated the connection between geometry and trigonometry and B_{2b} confirmed *'employ the angle of elevation since angle of elevation and depression are alternate angles'* (excerpt 1). Malatjie and Machaba (2019) posit that *'... a learner has conceptual understanding when he or she is able to explain, describe and apply the same concept in different ways and in different situations'* (p. 2). Eckert (2022) states that mathematical ideas can be communicated in different ways. For example, a learner can use verbal, written, symbolic, iconic and visual representations to illustrate understanding on mathematical concepts. Learners used their smartphones to get the feedback and checked the sample solution on the MCM app. B_{2a} said, *'So, our beta is 35,7 degrees'*. B_{2b}: *(checking the answer in the MCM app. They were happy to get the positive feedback and also checked the sample solution).'* Thus, smartphones came in handy in this learning situation of trigonometry concepts, as learners compared their solution with the sample solution.

Theme 2: Prior knowledge and links to real-world

Task 2: School flagpole: Calculate the height of the school flagpole in metres.

Excerpt 2: Group 4's discussions while walking the trail. (School flagpole)

G₄: Calculate the height of the school flagpole in metres (reading from the MCM app)

B_{4b}: So, is it proportionality or similarity of triangles first?

B_{4a}: Ke (*it's*) similarity ebe o tloba (and then) proportionality (writing down on the page)

B_{4b}: So, we have calculated the height and the breadth of the stick so now we are going to for the breadth of the flagpole.

G₄: We are cross multiplying... then we divide both sides, the reason we divide both sides, we want to make AB subject of the formula.

B_{4a}: $AB =$ (pressing the calculator) ... = 830, 91 cm

B_{4b}: So, we will convert it to metres, using King Harry Died Miserable Death Called Measles (one, zero, zero, showing group mates on the page) so we are going to divide by 100.

B_{4a}: check (now checking the MCM app)

B_{4b}: We are right.

B_{4a}: Now let's check the sample solution.

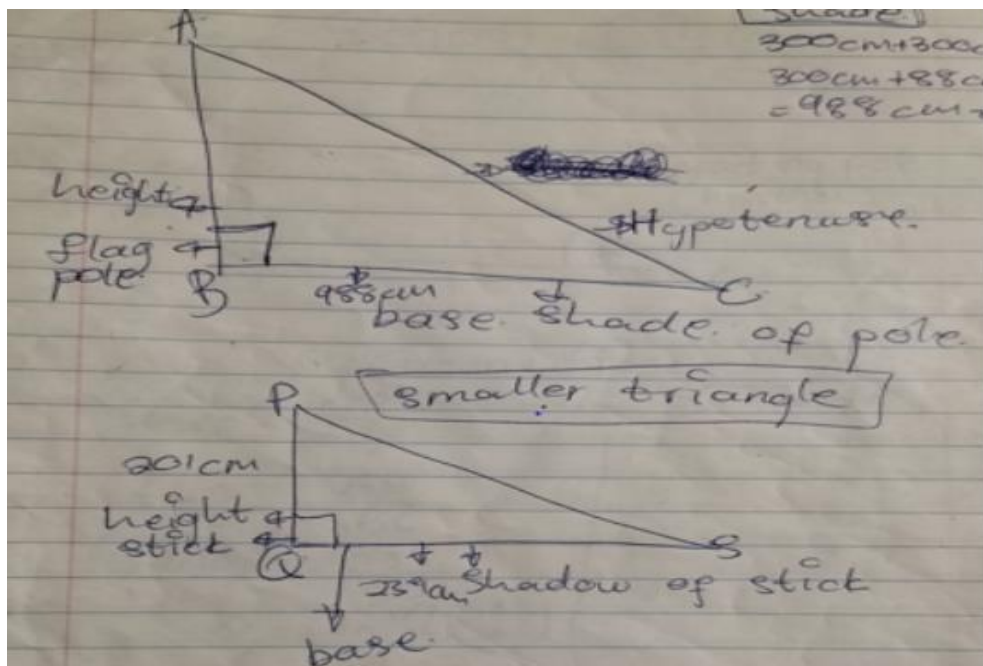


Figure 4: Group 4 using idea of similar right-angled triangle.

The image shows three lines of handwritten work on lined paper. The first line is a proportion: $\frac{AB}{PQ} = \frac{BC}{QS}$. The second line shows the same proportion with numerical values: $\frac{AB}{201} = \frac{988}{239}$. The third line shows the final result: $AB = 830,91 \text{ cm.}$

Figure 5: Group 4 using proportionality and algebraic equation.

According to the data in Excerpt 2, learners recognised (prior knowledge) and identified a link between similar right-angled triangles. B_{4b} asked fellow group members, ‘So, is it proportionality or similarity of triangles first?’ and B_{4a} replied, ‘Ke (it’s) similarity ebe o tloba (and then) proportionality’ (writing down on the page). The implication was that the learners understood that two pairs of corresponding angles of the two triangles are equal, and the third pair is through the sum of interior angles of a triangle (Refer to figure 4). Therefore, mathematics trails afforded learners an opportunity to connect concepts and ideas in mathematics; and to apply knowledge to real world or object (extending mathematical concepts to real-life contexts). In Excerpt 2, G₄ said ‘We are cross multiplying... then we divide both sides, we want to make AB subject of the formula’. (see Figure 5). The participants employed proportionality theorem and algebra (see Figure 5). The latter showed that participants’ prior knowledge played a crucial role and also implied that learners made suitable links within mathematics domains. Learners also made use of conversions, as noted by B_{4b}: ‘So, we will convert it to metres, using King Harry Died Miserable Death Called Measles (one, zero, zero, showing group mates on the page) so we are going to divide by 100.’ (Refer to excerpt 2). Conversions are a topic on their own in mathematics. According to Kilpatrick et al., (2001), Prior knowledge acts as a hinge between the learner’s previous experiences and new information. Learners can use knowledge they understand to generate new knowledge and solve unfamiliar problems. When learners have conceptual understanding, they can organise their knowledge coherently, enabling them to learn new ideas by connecting them to what they already know (Wilson & Heid, 2015). Alreshidi, (2023) note that meaningful learning maximises the use of prior knowledge. The knowledge from their previous experiences came in handy hence the mnemonic device and they managed to effectively solve the outdoor task. (refer to excerpt 2). They used the smartphone to get feedback and check the sample solution; they got feedback that they successfully solved the outdoor task (see B_{4b} in Excerpt 2). This implied that the outdoor task (mathematics trail) persuaded learners to recognise and identify

appropriate connections within mathematics and apply appropriate procedures to solve the task effectively.

Theme 3: Learners' emergent or self-developed models

Task 3: School billboard: What is the size of the angle that the supporting pole in the middle makes with the ground? (Assume the ground is horizontal). Give your answer to one decimal place).

Excerpt 3: Group 1's discussion during Billboard Task

B_{1a}: Re tlo sebedisang? (What are we going to use?).

G₁: Angle e mona yona mara keng? (What is the size of this angle here(pointing)?)

B_{1a}: Ke 90 (is 90 degrees).

B_{1b}: Yes, is 90 degrees because the base is horizontal.

G₁: So, guys re tlo etsang? (So, guys what are we going to do? Meaning how are we going to solve this task?) Hint number 1 ya rona ere re batle angle of elevation (hint number one says we need to calculate angle of elevation)

B_{1b}: Angle of elevation. (B_{1b}: drew right-angled triangle on the page).

G₁: trigonometry ratios or maybe we use sides first?

B_{1b}:, looking for an angle of a triangle we have to go for sides first before we can have the angle, which is tangent to the power negative one.

The learners at first did not know what to do (see *B_{1a}* excerpt 3). Learners ascertained the 90-degree angle between the vertical pole and the horizontal ground. *B_{1b}* said, 'Yes, is 90 degrees because the base is horizontal' and drew a right-angled triangle to employ a suitable trigonometry ratio to find the size of the angle of elevation (see *B_{1b}*, *B_{1a}* and *G₁* Excerpt 3). They were guided by smartphones through hints, as noted by *G₁*, 'Hint number 1 ya rona ere re batle angle of elevation (hint number one says we need to calculate angle of elevation).' The mathematics trail task indicated that participants should assume that the ground is horizontal (Excerpt 3). This meant that mathematics trails were the source of mathematical concepts to be reinvented (guided re-invention). Participants were stuck at first, but the outdoor task and smartphone assisted with a hint, a role that would normally be that of the teacher in a classroom. The hint unlocked suitable mathematical ideas or concepts to be utilised. *G₁* said, 'trigonometry ratios or maybe we use sides first?', and then *B_{1b}* replied, 'looking for an angle of a triangle we have to go for sides first before we can have the angle, which is tangent to the power negative one'. Barnes (2005), Wahyudi (2017) and Sumirattana et al. (2017) postulate that, to realise the RME principle, learners need to be given opportunities to use and develop their own models when solving mathematics problems. Learners demonstrated their responsibility by deciding which trigonometry ratios and procedures to apply without the teacher's help. The data implied that the mathematics trail allowed learners to identify relevant mathematical concepts and ideas. It can be presumed that an integrated and functional grasp of mathematical concepts was enhanced.

Task 4: **Roof and wall:** Determine the magnitude of the angle that the roof makes with the horizontal. (leave your answer to one decimal place).

Excerpt 4: Group 3's discussions while walking the trail. (Roof and the wall)

B_{3b}: Droya this one (draw this one) (meaning a triangle)

B_{3a}: (taking a page and start drawing a right-angled triangle after depicting the wall and the roof)

G₃: So, height ya rona ho tlameha re constructe (so we need to construct our height)

G₃: E tloba (H is going to be our height) height (for height) ya rona (feeding the drawing) akere (am I right)

B_{3a}: So where do we start? (checking hints)

B_{3b}: Ho chong re tlo qala ka base yane (that means we are going to start with that base) pointing at the part of the wall (meaning the top length of the imagined rectangle) mejara (take measurements) (pointing to the horizontal wall) kapo? eya, base ena e le kana le yane base e mona e le kana le yane (yes, this base is equal to that other base) (pointing at two opposite parallel sides of a rectangular wall saying those sides are equal)

G₃: Tangent theta = opposite over adjacent. AB over BC and then AB is 126... (doing calculations)

B_{3a}: 19.17

G₃: Check (MCM app)

B_{3b}: Well done (answer from the MCM app).

G₃: Ha re shebe (let us look at the) sample solution (they checked the sample solution against their answer and trying to make sense of the answers from the app)

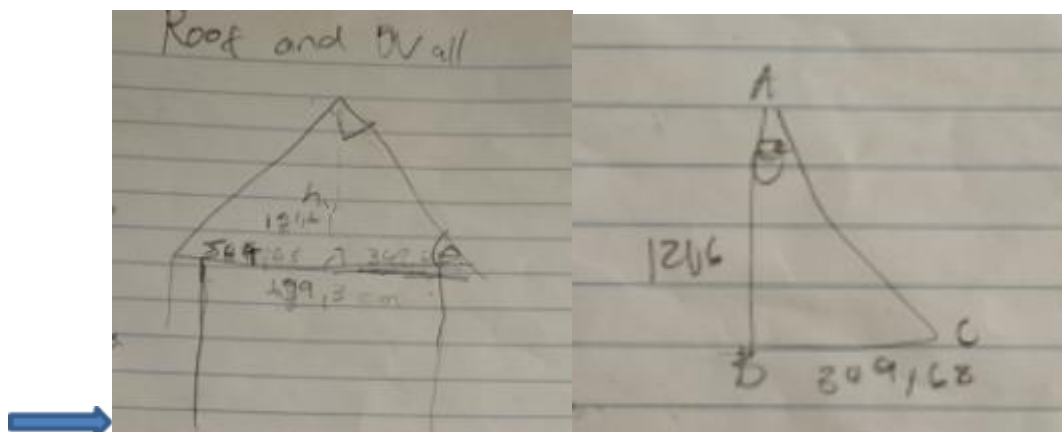


Figure 6: Group 3 identifying the right-angled triangle

$$\begin{aligned} \tan \theta &= \frac{\text{opp}}{\text{adj}} \\ \tan \theta &= \frac{BC}{AB} \\ \tan \theta &= \frac{349.68}{121.6} \\ \theta &= \tan^{-1} \left(\frac{349.68}{121.6} \right) \end{aligned}$$

Figure 7: Group 3 solving the trigonometry equation

In Excerpt 4 above, G₃ asked B_{3a} to draw the right-angled triangle, i.e., transforming the real-world task into a mathematical model and they subsequently solved the real-world task using another appropriate mathematical model, trigonometry ratio or equation (see B_{3b}), and G₃ noted, '*tangent theta = opposite over adjacent. AB over BC, and then AB is...*' (doing calculations). Figures 6 and 7 show the right-angled triangle derived from the diagram of the (roof and wall); then the equation of the employed trigonometry ratio. Peters (2016) postulates that learners should be able to experience a process similar to the one mathematicians underwent, during the learning process, learners should be able to build their own mathematical knowledge. RME principles improve understanding as learners identify relevant, integrated mathematical concepts (Yuwono, 2007; Laurens et al., 2018). Mathematics trails allow a variety of solutions, reflecting potential learning routes since learners can access sample solutions on their smartphones. Mathematics trails were activities for constructing knowledge instead of transmitting the existing knowledge. Therefore, mathematics trail assisted participants in developing proper mathematical models that enabled them to solve the real-world task, implying an improved functional and integrated grasp of mathematical concepts. There was no triangle on the wall, but he imagined it, shared the idea with his group members and asked B_{3a} to draw the right-angled triangle on the page (see B_{3a} in Excerpt 4 and Figure 6). They accessed the MCM app for feedback and samples solution to explore other appropriate models. (refer to Excerpt 4, G₃). G₃ said, '*Check MCM app*' and B_{3b}: '*Well done (answer from the MCM app)*'.

The process of connecting trigonometry concepts to everyday life situations through the MCM app could be inferred as a success while improving conceptual understanding from the analysed data. Because the participants indicated an integrated and functional grasp of mathematical concepts, they responded to the outdoor tasks and received positive responses from the MCM app. The data indicated that learners used their knowledge to complete the tasks and implied that learners had the competence to recognise and make the necessary links in the mathematics trail. It further demonstrated that learners were able to extend mathematical concepts to real-life. The participants had informative discussions and debates that informed appropriate decisions. By inference, the mathematics trail using a smartphone dismissed the notion that mathematics is an abstract subject, confined within the four walls of the classroom, without any relevance to real-life situations. The learners' work demonstrated horizontal and vertical mathematization because all the tasks were attached to real objects outside the classroom but

had to be solved using trigonometry concepts. Learners managed to re-invent appropriate mathematical models for transformation and proper trigonometry concepts to solve the outdoor tasks mathematically.

Conclusion

This study was focused on how the use of smartphones to learn mathematics through outdoor tasks enhances conceptual understanding of grade 11 trigonometry concepts. The findings indicate that outdoor mathematics tasks using smartphones for conceptual understanding is beneficial to learning grade 11 trigonometry. The study also found that using the outdoor tasks and the MCM app prompted learners to make some mathematical connections, they were able to show some multiple representations of some of the concepts they were working with. The learners were seen and heard drawing from prior knowledge as well as everyday life as they were solving tasks in groups during trailblazing. While it was not very evident, some elements of reinventing mathematical ideas were fostered as they needed to be innovative during trailblazing. To some considerable extent, learners were able to see the practical value of the trigonometric concepts they usually work on in class on mostly theoretical levels. They thus were able to gain from the unconfinedness of the mathematics learning process.

The study recommends regular incorporation of outdoor mathematics learning activities to break the monotony of confining learning to the classroom. We also recommend that teachers be encouraged to design context relevant tasks of mathematics concepts. With sufficient caution being exercised, the study encourages mathematics teachers of grade 11 learners and other Further and Education and training (FET) grades to deliberately incorporate smartphones in the learning of mathematics. Purposefully and cautiously we encourage teachers and schools especially in rural and township areas to start viewing smartphones and outdoor mathematics trails as tools that can be effectively and correctly used for learning purposes.

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THE EFFECTIVENESS OF LAPTOPS ON TVET LEVEL 4 STUDENTS' PERFORMANCE IN SOLVING COMPLEX NUMBERS.

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Level 4 Technical Vocational Education and Training (TVET) students experience uncertainties with the representation and performance of operations on complex numbers, especially in relation to the identification of the imaginary numbers (i), and operations to convert complex numbers. The purpose of this study was to explore the effectiveness of laptops on TVET level 4 students' performance in solving complex numbers. To address the question, 'what is the effect of laptop devices on TVET Level 4 students' performance in solving complex numbers?', this paper reports on a mixed methods study that followed a quasi-experimental research design in which 150 students were grouped together by the college admission team to create heterogeneous groupings of 30 students per lecture room. Data was collected using the pre- and post-test, semi-structured interviews with three students. Quantitative data was analysed statistically with the Statistical Package for the Social Sciences (SPSS), while qualitative data was analysed thematically. Findings indicated improved students' performance in learning complex numbers when using laptops. This study recommends the integration of laptops when students are learning mathematics.

Keywords: Mathematics, Laptops, APOS theory, Complex Numbers, Imaginary Numbers, Standard form, Polar form, performance.

Introduction and background of the study

As part of a strategy towards improving performance, the Department of Higher Education and Training (DHET) gradually introduced the use of laptops into institutions of higher learning including TVET colleges. The introduction of laptops to mathematics teaching and learning at the institution where this study was conducted was effected in 2022 with the level 4 (L4) students as a pilot study to explore if learning through the use of a laptop could improve performance in mathematics at the college. Those laptops allowed students and lecturers at the college to access internet only when loaded with data or connected to Wi-Fi, to practice typing skills, or view and research educational materials. These devices also influenced students' self-concept towards mathematics teaching and learning through their performance after learning using laptop devices. Self-concept refers to student ratings of their skills, ability, enjoyment, and interest in mathematics (Trigueros, 2019). Laptops provides additional opportunities for students to see, explore and interact with mathematical concepts as well as make discoveries with games, simulations and digital tools.

Moreover, Preciado-Babb (2012) adds that laptops assisted students to increase new relationships with complex numbers when they were able to learn and practice content related tasks using the device. In addition, Maharaj (2015) adds that students liked their classes better when instructions and question papers were delivered on integrated laptop mode (Kulik, 2014). L4 paper one question paper was structured into five questions where complex numbers weighed the most with 30 marks question and the rest of the questions with totals less than 30

marks. In addition, complex numbers always appear as question one in the question paper requiring responses based on properties and application of imaginary numbers. Furthermore, requiring conversion of $\sqrt{-1}$ and i^2 , writing expressions in rectangular form $a + bi$ and conjugate expressions. From the researchers' observation, students experienced challenges in simplifying such problems.

Recently, according to the researcher's findings, students at TVET Colleges were motivated by the management to use laptop devices to learn and improve performance in different subjects. Pursuant to the above, the researcher wished to explore the effectiveness of laptops on TVET level 4 students' performance in solving complex numbers. Those laptop devices enabled students to explore and learn things through websites, videos, apps, and games. Furthermore, students appeared to be fascinated to learn and had fun at the same time, and that helped them stay engaged with the mathematics resources. This concurs with Cheng et al. (2020) who notes that today's students work well with laptop devices and find the drawing of the argand diagram fascinating when drawing using laptop devices. In addition, ChanLin (2017) further assert that: i) Students can learn quickly with laptop devices as they use it to write, draw and solve complex number sums, and ii) laptop devices have direct communication in the sense that they connect all major stakeholder, students, parents, and lecturers. The researcher will never repeat the study about the origin, discovery and history of complex numbers in education. Such evidence exists in abundance Nahin et al (2010). What is lacking in literature is how to improve students' performance in the learning of complex numbers using laptop devices.

Literature review

Complex numbers are one of the most important concepts in mathematics at TVET Colleges (Nordlander 2012). However, some researchers have considered it to be one of the most difficult concepts students encounter when learning mathematics Level 4. Anevskia et al., (2015) notes that the study of complex numbers is associated with a variety of difficulties and misconceptions that are linked to its nature and terminology. The results of that study revealed that students experienced challenges in learning the basic properties of complex numbers. According to Anton (2010), a complex number is an expression of the form $a \pm bi$, where a and b are real numbers and i is the imaginary unit. Anton (2010) further elaborates that when learning concepts on complex numbers, students will encounter new terms such as imaginary, conjugate, argand diagram, rectangular and polar form, which are difficult to comprehend.

Habre (2017) also notes that TVET College students struggle to understand concepts on polar coordinate system and its application in the real-world. Anevskia (2015) believes that TVET College students encounter difficulties because of lack of various proof techniques and make hasty generalisations. Habre (2017) asserts that the content of mathematics level 4 is highly abstract and formal for undergraduate students as compared to the computational mathematics used at high school grades. It was observed by the researcher that most students are usually asked to memorise the definitions in preparation for an exam and as a result perform badly if asked application questions during assessments. It is also important to note that the content on complex numbers is embedded in definitions, where students memorise the terms without

conceptual understanding will lead to incompetence during assessments. At the end, if students perform badly, different stakeholders, such as the Lecturers, Senior Lectures, HOD's, Campus Managers, Academic Programme Coordinators and Vocational Programmes Manager, get worried, causing tensions and uneasiness amongst themselves.

Wawro (2014) notes that research on students' understanding of polar coordinates and graphing of complex numbers was scarce. Despite the importance of complex numbers in the study of Engineering Science at a TVET College, it is disappointing to note that students' performance remained constantly poor from the researchers' experience when teaching these concepts. The facts on the difficulties as outlined above and the need to add new knowledge to existing literature motivated the researchers to use APOS theory to investigate the effect of laptops on TVET Level 4 students' performance in solving complex numbers. The researchers also noted that very few studies focused on division of complex numbers especially the idea on the use of radicals and standard form of complex numbers. This study thus fills a gap in this research area.

Theoretical framework

This study is framed in APOS theory, which outlines how mathematical concepts are learned. According to Maharaj (2015), the development of APOS theory is stimulated by Piaget's concept of reflective abstraction in child learning. Reflecting abstraction according to Arnon et al. (2014) refers to the formation, development and modelling of certain concepts as well as the explanatory power of the mathematical concepts formation. Weyer (2010) adds that APOS theory is an extension of the last stage of Piaget's theory, that is the formal operational stage. The acronym APOS stands for action, process, object, and schema. Arnon et al. (2014), highlights that APOS theory proposes that an individual has to have appropriate mental structures to make sense of a given mathematical concept. Arnon et al. (2014) further argues that in order to apply APOS theory to describe the particular constructions by students, one needs to develop a genetic decomposition.

A genetic decomposition involves hypothesised detailed descriptions of the different levels of APOS which students might need, and which gives the researcher insight of how the students might construct mathematical knowledge (Arnon, 2014). Mathematical knowledge is needed to understand a certain concept and its relationships with the concepts. Initially, a concept is first perceived as an action, externally requiring step-by-step physical or mental transformation of objects that need to be performed explicitly or from memory (Arnon et al., 2014). Furthermore, the actions are based on rules and algorithms and take place without much thought, where a rule is practiced repeatedly until it becomes a routine (Brijlall, 2015). Weyer (2010) adds that at that stage an individual can perform an action from memory without specific instruction so as to transform each object.

An action also involves multiple-step sequence of responses whereas Dubinsky and Wilson (2014) refer to it as a mental structure that performs the same operation but takes place totally in the mind of the individual. An object, according to Brijlall (2015), is constructed from a

process when the individual becomes aware of the process as a totality and realises that transformations can act on it. Over and above, a schema involves many actions, processes and objects that are connected together in the mind of an individual to form a coherent structure (Kemp & Vidakovic, 2023). The term coherence of the schema refers to an individual's ability to ascertain whether the schema can be used to solve a mathematical problem or a particular mathematical situation, or which mental structures to use to do so. For this stage, Brijlall (2015) outlined that one has to bring together various mental constructions and schemas to address the problem. An example here would be that if one needs to define the term complex number, the individual should bring together the various schemas that characterise what a complex number is, such as the real number, imaginary number, and the symbol i .

A pedagogical approach known as ACE activities, class discussion and exercises teaching cycle was followed and this guided how teaching of complex numbers was carried out. Weller et al. (2003) says that in implementing the ACE teaching cycle the topic complex numbers must be broken into subtopics which are then each exposed to an iterative approach of the cycle. The topic of complex numbers was broken into subtopics, imaginary numbers, standards and polar form as well as conjugate activities. Those sub-topics were designed in such a way that they develop mental structures as prescribed by the APOS theory. The researchers' role was to ensure that activities are aligned to mathematical concepts under consideration. The discussion among students was facilitated by the lecturers. Students were given homework as follow up exercises to allow self-reflection on the work done to consolidate understanding of problems. Learning cycle through APOS theory is displayed below.

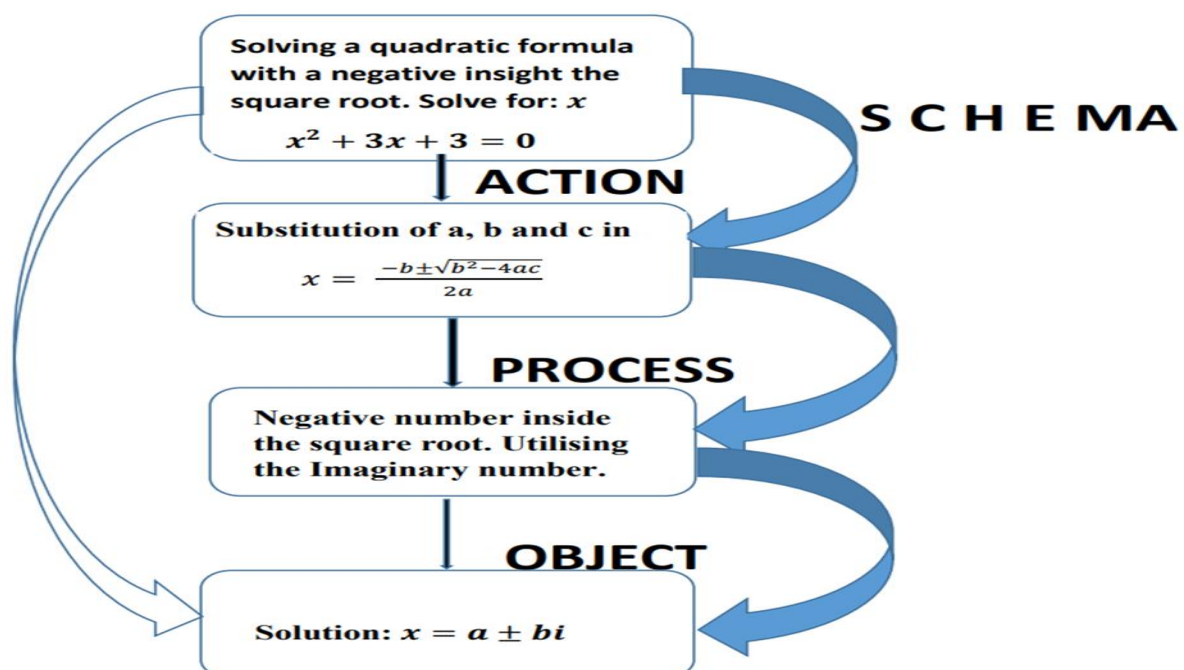


Figure 1: A model showing the connection of APOS theory and components of a learning/teaching cycle.

Figure 1 indicates the conceptual difficulties and understandings of complex numbers that level 4 students have, through APOS theory lens. APOS pre-supposes what it means to learn and

understand a concept by constructing and reconstructing mathematical actions, processes and objects as well as organising these in schemas to use in dealing with the application of those concepts. The model indicates that some of students operated in the action stage where they could not perform mental manipulation that transform objects by reacting to external cues that give precise details on what steps to take. Additionally, the model also shows evidence of a mental construction as an action that happens in the mind of students, which could be interiorised as the process on understanding complex numbers.

Methodology

From a pragmatic worldview, a mixed approach was used following a quasi-experimental research design. Quasi-experimental design according to Arnon, (2014) is identical to experimental research in the sense that there is manipulation of independent variable. Between the two categories of the one-group designs, this study followed a one-group pre-test-post-test design where the dependent variable is students' performance. Creswell (2017) asserts that the independent variables which are a traditional method and the use of laptop device are a reason a change occurs in the dependent variables. This study involved a mixed methods approach in which a population of 150 Level 4 TVET College students, three heterogeneous groups of 30 students in each class formed the sample of this study.

This study aimed to determine the effectiveness of using laptop on Level 4 students' performance in learning to calculate complex numbers. Consequently, this study sought to evaluate how students' performance in complex numbers differ when using laptop as compared to the traditional textbook method of learning. In this study the main research question was: 'what is the effectiveness of using laptop on Level 4 students' performance in learning to calculate complex numbers?' Data was collected through two different teaching approaches which were the traditional textbook approach, and tablets approach together with tests written at different intervals during intervention. The following instruments were used to collect data, pre-test to establish baseline scores for complex numbers performance, post-test were also used after analysing the pre-test to collect quantitative data. The purpose of the two methods was to weigh the interconnection of learning performance towards complex numbers of L4 students when learned using laptops and when the textbook method of learning was used.

A pre-test to establish baseline scores for complex numbers performance was administered to three classes of L4 students at a TVET College. The first question in the pre-test was used to determine if students could simplify complex numbers expressions by showing all calculations and giving the answer in rectangular form $a + bi$ after using the textbook method of learning. Question 1.1.1 for example required students to simplify $(\sqrt{-4} + 4)(\sqrt{-9} - 1)$. The second question 1.1.2 required them to simplify $\frac{(2+3i)(1-4i)}{(1-2)}$ and the last 1.1.3 expected them also to simplify $2i^3 - 4i^2 - i^5 + \sqrt{25}$ and for all of them to leave the answer in rectangular form $a + bi$." These questions assessed students' ability to convert $\sqrt{-1}$ and i^2 correctly, to check the

understanding of conjugate in complex numbers and finally writing the expression in rectangular form $a + bi$ after learning with the textbook method.

To gather qualitative data, interviews with the participating students were conducted at the end of the collection of quantitative data where three students were interviewed based on their performance. Finally, the quantitative data was analysed using statistical software's, while qualitative data was analysed thematically.

Results and discussion of findings

The researchers performed a qualitative question-by-question analysis that according to Creswell (2014) included processes of coding, categorising, and interpreting data to provide explanations concerning a single phenomenon of interest of the pre-tests and the post-tests administered. Interviews were transcribed, read several times, and coded to look for patterns and common themes that emerge from the data (McMillan et al., 2010, 98). The codes that were used to derive categories and then themes that were evident from the text. The most central themes emerging from the data were utilised to respond to the research question.

Clarification on the process of solving the questions on complex numbers after learning with textbook

When students were asked during the face-to-face interviews to briefly clarify the process of solving the questions on complex numbers after learning with textbook, some of their responses were:

Researcher: Briefly clarify the process of solving questions in the written test.

Student S1: 'The answering of complex numbers questions was challenging after learning with textbook. I could not understand that $\sqrt{-1} = i$. In my calculations of 1.1.1, I kept on using $\sqrt{-1}$ as is without converting it. When coming to 1.1.2, I only dealt with the numerator and left the denominator as is because I did not know what to do. Eish, as for 1.1.3, I only remembered that the $\sqrt{25} = 5$. For the expressions i^2 , i^3 and i^5 I did not know how to approach them. So I left the question as is.

Responding to the same question, S2 and S3 said:

Student S2: 'it was difficult to answer the activity, when looking to my attempt of 1.1.1, I only remembered that I have to write $\sqrt{-4}$ as $\sqrt{-1} \cdot \sqrt{4}$ and $\sqrt{-9}$ as $\sqrt{-1} \cdot \sqrt{9}$ but from there I could not understand what to do. For 1.1.2, I just wrote the question and left it as is and did not know what to do.' When looking at 1.1.3, I got so frustrated because of so many powers and could not even attempt the question.

Student S3: 'I attempted 1.1.1 but I realise that something is not right as I could not proceed with the answering of the question. I got cognitively impaired on the way. For 1.1.2, I multiplied the numerator correctly but did not know what to do with i and i^2 . Then I left the question to

that level. For 1.1.3, I did nothing as I kept on cancelling trying to think about the correct method, but nothing came out.

These were responses to the questions in the pre-test that provided a sense of the students' difficulties in learning complex numbers traditionally and helped the researchers in planning ways of addressing challenges. The students' results in both tests (pre- and post-test) were further compared to determine if the students' understanding had improved or not after learning complex numbers using laptop. The researchers drew preliminary conclusions from the displayed data by using tables and graphs as the project progressed while the final conclusions were drawn only after the analysis of results was completed. Results indicated that only 6% of students could remember to convert $\sqrt{-1}$ to i symbol needed to simplify the questions asked about complex numbers. It was also observed that 90% of students could not remember to convert i^2 to -1 nor to remember how to write the conjugate when not using laptop. For example, students could not attach a meaning to $i - 2$ when required to write the conjugate of the expression while 10% of students changed the expression to $i + 2$ without rearranging to $-2 + i$ in order for them to change the sign of the imaginary part. Students were supposed to rewrite the expression properly as $-2 + i$ so that they can change the sign of the imaginary part i and not of the real part -2 . This implies that about 94% of students operated in the action stage of APOS.

Extract 1 represents some of the work by one of the students before using laptops.

S1

Question 1

1.1.1 $(\sqrt{-4} + 4)(\sqrt{-9} - 1)$
 $= (\sqrt{4} \cdot \sqrt{4} + 4)(\sqrt{9} \cdot \sqrt{9} - 1)$
 $= (2\sqrt{4} + 4)(3\sqrt{9} - 1)$
 $= 6 \cdot 1 - 2\sqrt{4} + 12\sqrt{9} - 4$
 $= 6 + 10\sqrt{4} - 4$
 $= 2 + 10\sqrt{4}$

1.1.2 $\frac{(2+3i)(1-4i)}{(i-2)}$
 $= \frac{2 - 8i + 3i - 12i^2}{i-2}$

1.1.3 $2i^3 - 4i^2 - i^5 + \sqrt{25}$
 $= 2i^3 - 4i^2 - i^5 + 5$

Extract 1: S1's response

It can be noted that student S1 did not understand how to use $\sqrt{-1}$ in statement three of 1.1.1. Instead of converting $\sqrt{-1}$ to i before removing the bracket, the student just multiplied $2\sqrt{-1}$ and $3\sqrt{-1}$ to get the answer $6 \cdot 1$ which is incorrect. S1 also did not simplify 1.1.2 correctly. The student just multiplied the numerator but did not convert i^2 of $-12i^2$ to continue with the activity. It also indicates that S1 does not have knowledge about conjugate because he could not attempt that part of the question. From the script above, one could notice that S1 lack the knowledge of converting i^3 , i^2 and i^5 properly according to complex numbers properties. What the student did was just to get the square root of 25 only and the whole expression was left as is.

Researcher: Briefly explain how you can express the expression in 1.1.1 to 1.1.3 in the form $a + bi$.

S₂

Question 1

1.1.1 $(\sqrt{-4} + 4)(\sqrt{-9} - 1)$
 $= (\sqrt{-4} \cdot \sqrt{-9} + 4)(\sqrt{-4} \cdot \sqrt{-9} - 1)$
 $=$

1.1.2 $\frac{(2+3i)(1-4i)}{1-2}$
 $=$

1.1.3

Extract 2: S2's response

Student S2: Hmmm, for 1.1.1, I think you have to remove the bracket first, but I am not sure as I did not get it correct. Eish, 1.1.2, it's when you are given a fraction, the first thing is to come up with complex conjugate. Multiply the numerator as well denominator by that complex conjugate, we call $\bar{z}$.

As a follow-up question, the researcher asked the following question.

Researcher: So, what is the complex conjugate in the expression $\frac{(2+3i)(1-4i)}{(i-2)}$?

Student S2: Aah, that is about 1.1.2. The complex conjugate of -2 is 2 , and the expression will be written as $(i + 2)$, but in my answer I did not write anything as I was not sure of my calculations.

Student S3 added as follows:

Student S3: We simply write the conjugate of the expression and use it by multiplying, but I forgot which part of the expression do we multiply.

Researcher: The question requires you to express the questions in the form $a + bi$. So, what must be your final answer?

Student S3: I wrote the conjugate as $i + 2$ in question 1.1.2 and did not know what to do next. What I did also, I removed the brackets from the numerator to get the expression $2 - 5i - 12i^2$, from there I did not know what to do as I forgot how to change i^2 . I was not sure as to whether to convert it to $\sqrt{-1}$ or $-i$. That is why I was unable to write the answer in the form $a + bi$.

Below is student S3's attempt of answering 1.1.1 to 1.1.3 question.

S₃

Question 1

$$\begin{aligned} 1.1.1 & (\sqrt{-4} + 4)(\sqrt{-9} - 1) \\ &= 36 + \sqrt{4} + 4\sqrt{9} - 4 \\ &= 32 + \sqrt{4} + 4\sqrt{9} \end{aligned}$$

$$\begin{aligned} 1.1.2 & \frac{(2+3i)(1-4i)}{(i-2)} \\ &= \frac{2-8i+3i-12i^2}{i-2} \\ &= \frac{2-5i-12i^2}{i+2} \end{aligned}$$

$$\begin{aligned} 1.1.3 & \cancel{2i^3 - 4i^2} \quad 2i^3 - 4i^2 - i^5 + 25 \\ &= \cancel{2i^3 - 4i^2} \end{aligned}$$

Extract 3: S3's response

In extract 3, it can be noted that student S3's response defines what she thought the interpretation of $\sqrt{-4}$ and $\sqrt{-9}$ in question 1.1.1 was. Ironically, the student interpreted the question as $(a + b)(c + b)$ without considering the complex number roles. That is why the student wrote a wrong expression $32 + \sqrt{4} + 4\sqrt{9}$ as the final answer and is not in the form $a + bi$. This indicates lack of concepts understanding. S3 operated in the process stage of APOS. For 1.1.3, the researcher is of the opinion that student S3 just avoided leaving the question blank. She cancelled the question, wrote the question as is and finally, attempted the question by writing and cancelled again. This kind of behaviour indicate lack of concept understanding and a level of frustration when students were learning traditionally. The student did not answer question 1.1.3 properly. Meanwhile, it can be seen from all the extract, that students did not allocate the relationship amongst the important concepts like conjugate, $\sqrt{-1}$, i^2 and $a + bi$. Instead, the students just mixed up the concept or at some stage did not reach the concept at all.

Theoretical analysis of complex numbers using an APOS approach, proposed the following genetic decomposition. In the above extracts, student's actions, process, objects and schema were applied as follows.

Action

In 1.1.1 students should use the definition $\sqrt{-a} = i\sqrt{a}$ when working with negative radicals. Additionally, in 1.1.2, the student should be able to identify the complex conjugate of the denominator and multiply numerator and denominator by the complex conjugate in a step-by-step manner as well as to combine the real and the imaginary part. They should also use definitions $i^2 = -1$ for 1.1.1, 1.1.2 and 1.1.3 when there is a need to convert the complex value.

Process

Students can imagine what the product of the corresponding element is and able to combine the real and the imaginary parts without carrying out the step-by-step procedures. The multiplication of a number by its conjugate can be done in one step. At this level, students are able to predict the result in terms of the real and imaginary parts.

Object

Students can see the effect of division of a complex number as a totality. Students are also able to apply processes, for example, $i^2 = -1$ and $\sqrt{-1} = i$. Above all, converting expressions $(\sqrt{-4} + 4)(\sqrt{-9} - 1)$, $\frac{(2+3i)(1-4i)}{(1-2)}$ and $2i^3 - 4i^2 - i^5 + \sqrt{25}$ in the form $a + bi$.

Schema

Finally, a schema involves many actions, processes and objects that are connected together in the students' mind to form a coherent structure (Dubinsky, 2004). The term coherence of the schema refers to the student's ability to ascertain whether the schema can be used to solve a mathematical problem or a particular mathematical situation, or which mental structures to use to do so. For this stage, Brijlall (2015) outlined that a student has to bring together various mental constructions and schemas to address the complex number problem. An example would be that if a student needs to define the term complex number, the student should bring together the various schemas that characterise what a complex number is, such as the real number, imaginary number, and the symbol i , $i^2 = -1$, $\sqrt{-1} = i$ as well as writing expressions in rectangular form $a + bi$.

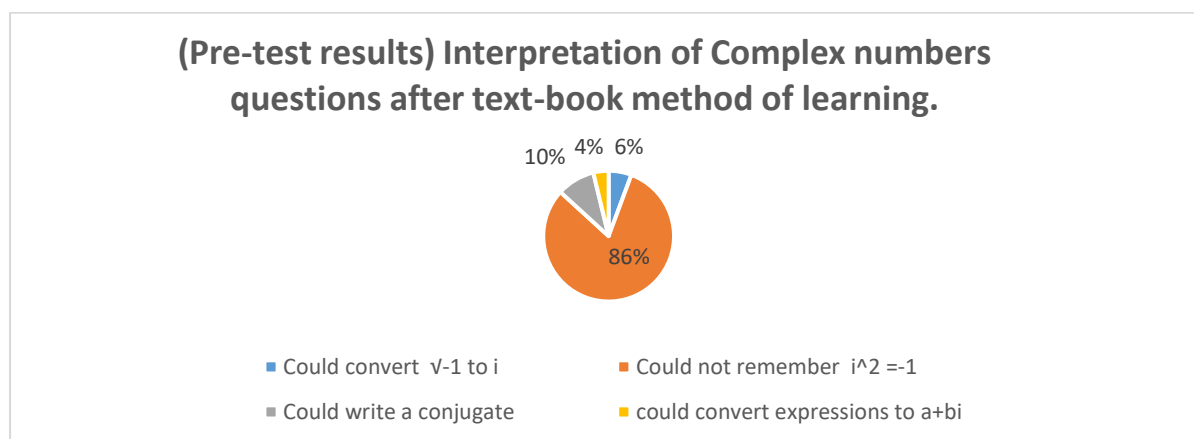


Figure 1: Pre-test average scores

Out of a sample of 90 Students, 6% could convert $\sqrt{-1}$ to -1 , 4% could convert complex expressions to rectangular form $a + bi$, 10% could write a conjugate and the rest, which is 86% could not remember $i^2 = -1$. These average scores indicate that students did not understand the learning of complex numbers when learning with text-book method and perpetuated the

researcher to think about intervention method. The intervention method the researcher came up with was the use of laptops when learning complex numbers. WITH THIS INTERVENTION, THE RESEARCHERS WANT TO EXPLORE if integrating laptop in the mathematics classroom would increase level 4 students' performance in learning and calculating complex numbers.

The results of the tests written by students, interviews conducted after using laptop devices provided evidence for improved students' performance in the learning of complex numbers and should be taken into consideration. The results from the quantitative data revealed positive changes in students' performance in learning to calculate complex numbers.

Extract 4 represents the work done by S1 after the used a laptop.

S₁

Question 1

1.1.1 $(2\sqrt{4} + 4)(\sqrt{9} - 1)$
 $= (2 \cdot \sqrt{4} + 4)(\sqrt{9} - 1)$
 $= (2i + 4)(3i - 1)$
 $= 6i^2 - 2i + 12i - 4$
 $= 6(-1) + 10i - 4$
 $= -6 - 4 + 10i$
 $= -10 + 10i$
 $= 10(-1 + i)$

1.1.2 $\frac{(2+3i)(1-4i)}{(1-2)}$
 $= \frac{2 - 8i + 3i - 12i^2}{1-2}$
 $= \frac{2 - 5i - 12(-1)}{1-2}$
 $= \frac{2 - 5i + 12}{1-2}$
 $= \frac{14 - 5i}{1-2} \cdot \frac{1+2}{1+2}$
 $= \frac{14i + 28 - 5i^2 - 10i}{i^2 + 2i - 2i - 4}$
 $= \frac{4i + 28 - 5(-1)}{(-1) + 0 - 4}$
 $= \frac{4i + 28 + 5}{-5}$
 $= \frac{33 + 4i}{-5}$
 $= -\frac{33}{5} - \frac{4}{5}i$

1.1.3 $2i^3 - 4i^2 - i^5 + \sqrt{5}$
 $= 2 \cdot i \cdot i^2 - 4(-1) - i^2 \cdot i^3 + \sqrt{5}$
 $= -2i + 4 + i^2 \cdot i + 5$
 $= -2i + 4 + -1 \cdot i + 5$
 $= -2i + 9 - i$
 $= 9 - 3i$

Extract 4: S1's response

Question 1.1.1 and 1.1.3 were well answered by Student S1 but the challenge was with 1.1.2 where the S1 forgot to rearrange $i - 2$ to become $-2 + i$ so that he could look for the correct conjugate. Student S1 has the concept of answering complex numbers after using laptop because questions 1.1.1, 1.1.2 and 1.1.3 were well answered except for rearranging the denominator of 1.1.2. When looking at the concept of writing the answer in the form $a + bi$, Student S1 managed to show understanding of the concept even though the answer was incorrect due to the conjugate which was not rearranged before it could be expressed. Additionally, in question 1.1.3 student S1 indicated understanding of the concept after learning with laptop as he managed to expand i^3 and i^5 correctly to $i \cdot i^2$ and $i^2 \cdot i^3$ respectively.

Furthermore, Student S1, managed to convert i^2 to -1 as required in the complex number rules and this indicates understanding of complex numbers after learning with laptop.

S2

Question 1

1.1.1 $(\sqrt{4} + 4)(\sqrt{9} - i)$
 $= (\sqrt{4} \cdot \sqrt{9} + 4)(\sqrt{4} \cdot \sqrt{9} - i)$
 $= (2i + 4)(3i - 1)$
 $= 6i^2 - 2i + 12i - 4$
 $= 6i^2 + 10i - 4$
 $= 16i^3 - 4$

1.1.2 $\frac{(2+3i)(1-4i)}{(i-2)}$
 $= \frac{2 - 8i + 3i - 12i^2}{i-2}$
 $= \frac{2 - 5i - 12i^2}{i-2}$
 $= \frac{-12i^2 - 5i + 2}{i-2} \cdot \frac{i+2}{i+2}$

1.1.3 $2i^3 - 4i^2 - i^5 + \sqrt{25}$
 $= 2i^2 - i^5 + 5$
 $= 2i^3 + 5$

Extract 5: S2's response

S2 demonstrated an improvement after learning complex numbers using laptop. When looking at 1.1.1, S2 multiplied the expression correctly. The challenge came when he was supposed to change i^2 to -1 so that he can be able to continue with the problem. What S2 did, was mixing-up concept, instead of converting i^2 to -1 , S2 just treated $6i^2 + 10i$ as though they are like term by adding 6 and 10 to get 16 but at the same time making the mistake of multiplying $i^2 + i$ to get i^3 . In the very same question 1.1.1, S2 remembered that $\sqrt{-1} = i$ which indicates improvement after intervention. For 1.1.2, S2 remembered what a conjugate is and tried to write it properly but forgot to rearrange the expression $i - 2$. Otherwise the student managed to remember that with a conjugate we have to change the sign of the imaginary part.

S3

Question 1

1.1.1 $(\sqrt{4} + 4)(\sqrt{9} - i)$
 $= (\sqrt{4} \cdot \sqrt{9} + 4)(\sqrt{4} \cdot \sqrt{9} - i)$
 $= (2 \cdot 2 + 4)(2 \cdot 3 - i)$
 $= 6i \cdot 2i$
 $= 12i^2$

1.1.2 $\frac{(2+3i)(1-4i)}{(i-2)}$
 $= \frac{2 - 8i + 3i - 12i^2}{(-2+i)}$
 $= \frac{2 - 5i - 12i^2}{-2+i} \cdot \frac{-2-i}{-2-i}$
 $= \frac{-4 - 10i + 12i^2}{4 + 2i - 2i - i^2}$
 $= \frac{-4 - 10i + 12i^2}{4 - (-1)}$
 $= \frac{-4 - 10i + 12i^2}{5}$
 $=$

$$\begin{array}{l}
 \text{S3} \\
 1.1.3 \quad 2i^3 - 4i^2 - i^5 + \sqrt{25} \\
 = 2i^3 \cdot i - 4(i^2 - i^5) + 5 \\
 = 2i - 4(i^7) + 5
 \end{array}$$

Extract 5: S3's response

Student S3 also shows some level of understanding of how to do the calculations after learning with the use of a laptop. The calculation of 111 was much better as S3 managed to change $\sqrt{-1}$ to i . The only noticeable mistake was to treat the insight of the brackets as though they were like terms even though they were not and this resulted in the incorrect answer. At least, S3 attempted 1.1.3 after intervention which she could not do before intervention. Correct expansion of exponents was noticed which is part of simplifying complex numbers as well as finding the correct square root of 25 which is written as 5. There is an improvement after using laptop as the question was not attempted before intervention.

Clarification of the process of solving the questions on complex numbers after learning with laptop

When students were asked to briefly clarify the process of solving the questions on complex numbers after learning with laptop, some of their responses were:

Student S1: 'I found the question manageable as I could answer most of them correctly.' The only challenge I realised after submission was that I forgot to rearrange the denominator so that I can get the correct conjugated. Above all, I managed to write the answer of 1.1.2 in the form of $a + bi$.

Student S2: 'To be honest, without these online complex number method learning, I would not have understood concepts or passed this course. I wasn't able to attend all of the lectures, so laptop allowed me to go at my own pace and learn concepts at my own convenient time.'

Student S3: "I understand the complex number lesson more when I use laptop because I use the search engine to remind myself about the concepts I forgot. In addition, all the students in the lecture room could set up live interactive video calls with experts on a wide variety of complex number definitions using sites such as skype and nepris. Nepris is where students can post a question requesting experts to share ways of solving complex numbers concepts, expressions and equations in their daily work. What I loved most, was that during the interaction with laptop students could virtually meet an expert who demonstrated how he uses complex numbers measurement, conversion, and more in his decision-making and planning."

Findings in this study revealed that there are considerable benefits with integrating laptop in learning to calculate complex numbers. This concurs with Audi & Gouia-Zarrad (2013) who found that there were higher levels of active engagement when students utilised laptops compared to students who were using traditional method of learning. Findings in this study also concur with Arnon, (2014) who demonstrates that students at a public university favoured using laptop devices to explore learning of concepts in lecturer room and listen to lectures outside of the physical lecturer room. Likewise, in this study, the students similarly

communicated their experience that the use of laptops in learning to calculate complex numbers increased their performance in the learning of mathematics.

In addition, the results presented in this paper, demonstrate that integrating laptop in complex numbers has a positive impact on struggling students. In fact, the use of laptop increases the interaction between the lecturer and the students rendering the class more dynamic and tailored in real time to the students' needs. Moreover, it has been demonstrated that students are more engaged in laptop classes and keep concentrating till the end of the class. This concurs with Audi & Gouia-Zarrad (2013) who notes that, laptops are user-friendly device creating an environment that motivates students to succeed. In agreement, Dubinsky (2014) asserts that when using laptop as a means for learning, students were more on task, even those who are usually hard to engage for long periods of time. This study discovered that laptops helped and encouraged students to work and keep them focused. It was also noticed that students' application in learning the calculation of complex numbers was interactive and afforded them to be hands-on with the practice that laptops provided. In addition, with laptops, students were able to set up live interactive video calls with experts on a wide variety of complex numbers curricular topics using sites such as skype in the lecture room and nepris. Nepris is where students can post a question requesting experts to share ways they use complex numbers concepts in their daily work and life. In so doing they were able to operate on the process level of APOS. Finally, it was possible that during the interaction with laptops, students could virtually meet a playground designer who demonstrate how he uses complex numbers measurement, concepts, conversions, and more in his decision-making and planning.

Conclusion

This study sought to explore the effectiveness of using laptop on students' performance in calculating complex numbers. Using a quasi-experimental design, the results in this study indicate that students' performance improved in the learning of complex numbers when using laptop devices. We also found that students' experience with the use of laptop devices increased their collaborative learning style as they were grouped heterogeneously and hence boosted their academic performance as depicted in their post-test performance and interviews after learning using laptop in mathematics. This study recommends that laptop devices should be integrated in the lecture room when students are learning mathematics.

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PERSONAL EXPERIENCES AND REFLECTIONS OF USING ONLINE PLATFORMS FOR TEACHING AND LEARNING MATHEMATICS IN SOUTH AFRICA

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The start of the COVID-19 pandemic in early 2020 caused a major disruption to education in South Africa and the rest of the world. For safety measures, there was a proliferation of online teaching and learning at schools and universities throughout the world. With the pandemic receding and the move to face-to-face classes, online teaching and learning still takes place under certain conditions. This paper describes the writer's involvement in the online teaching and learning of mathematics and his reflections thereof. This involvement has largely been positive and has enabled the writer to work with and train talented young mathematics learners for the South African Mathematics Team competition. It has also confirmed the writer's affinity for mathematics problem-solving with his participation in online courses. The knowledge gained from these courses may be shared with both primary and high school teachers.

Introduction

The COVID-19 pandemic in early 2020 caused a major disruption to education in South Africa and the rest of the world. On 15 March 2020, South African President, Mr Cyril Ramaphosa, addressed the nation and declared a National State of Disaster and announced the closing of schools on 18 March 2020. A week later, 26 March 2020, the entire country went into lockdown, with schools only reopening on 8 June 2020. (Government Gazette, 2020). The closure of schools in South Africa led to schools and education departments considering different strategies to ensure that learners were gainfully occupied during national lockdowns and during rotation of classes when schools reopened. These included the use of online platforms and social media to reach learners. However, online education has been in existence for several years. In the modern era, online education is defined as a form of distance education that uses computers and access to the internet as the mode of delivery, with at least 80% of the course content delivered online (Allen & Seaman, 2008; Shelton & Saltsman, 2005). The use of computers to educate arose in the corporate arena during the 1980s as companies used computer-based programs to train new employees (Rudestam & Schoenholtz-Read, 2002).

There is no doubt that the COVID-19 pandemic accelerated the use of online platforms and social media for teaching and learning. Despite the mask mandate ending in South Africa and the rest of the world from June 2022 onwards (in South Africa it was 24 June 2022), online teaching and learning in various forms still takes place in South Africa and across the world.

Problem statement

While schools and universities are back to normal in South Africa with all students attending face-to-face classes, there are times when lecturers at universities resort to online teaching of their modules for a variety of reasons. These include having other commitments which may clash with lectures or there are student protests at the campus over some issues. In this regard,

lecturers are encouraged to arrange online classes for their students at an appropriate time. There are also online short courses offered by universities, professional organisations and other training companies, both locally and internationally. These courses may be free, or a registration/course fee may be requested. These *short courses* tend to be ideal for both young and old people who would like to acquire new skills and knowledge, advance their career or even make a career change. These courses may be taken in the comfort of one's home or office. There is no need for any unnecessary travel to complete courses (UNISA, 2024)

Since 2020, the writer has learnt a quite a bit, mostly self-taught, on online teaching and learning, working with platforms such as Google Meet, Zoom and Microsoft teams. He has conducted online lessons/workshops for teachers and learners; and has participated in online courses, through different Learning Management Systems (LMS) organised both locally as well as internationally. Thus, it would be fair to say that the writer has gained valuable experience in online teaching and learning.

Research question

In the light of the discussions in the preceding section of this paper, the following research question is posed:

What are some experiences of the writer in the online teaching and learning of mathematics?

To respond to this research question, the following sub-questions are posed:

- What are the writer's experiences in teaching mathematics through an online platform?
- What are the writer's experiences in learning mathematics through an online platform?
- To what extent were these experiences successful?
- How does the writer reflect on these experiences?

Literature review

As stated earlier, the closure of schools during the COVID-19 pandemic in South Africa and the rest of the world in March 2020 resulted in a move to online learning. This short literature review focusses on experiences in online teaching and learning in different contexts. As far back as the early to mid-1990s, universities and colleges in the USA began experimenting online courses. While there was an explosion of online courses during that time, many online educational programmes did not survive. Most lecturing staff were indifferent to these online courses and there was a lack of understanding of online pedagogy and online learning styles compared to teaching and learning in face-to-face environments. Most lecturers just provided online students with lecture notes from the traditional classroom with little or no active engagement with students (Kentnor, 2015).

Pinchot and Poullet (2014) use the VAK Learning Styles (Visual, Auditory, & Kinaesthetic) as a framework for addressing the learning needs of online students. This was done to accommodate students with different learning styles. They use free tools that are available online to create audio, video, sound and hands-on activities to enhance the online classroom

experience. These tools can provide a wealth of opportunities for making the online learning experience more effective for students with each learning style. A Philippines study (Bringula, Reguyal, Tan and Ulfa, 2020) investigated the profile of online learners and its influence on their mathematics self-concept. It was found in the study that most learners had access to devices for learning. However, most learners reported an intermittent internet connection and expressed concerns about their possible results in mathematics at the end of the semester. For some learners, the lack of physical spaces made online learning inconvenient and contributed to their low self-concept in Mathematics. This study also highlighted the significant role of the teachers in improving and sustaining the mathematics self-concept of online learners.

A major concern in other online learning studies was the access to a reliable internet and a device. Learners without reliable internet and/or technology tend to struggle to participate in digital learning (OECD, 2020), thus, confirming some of the results of the Philippines study above. In this regard, it was felt that families and teachers had an important role to sustain learning during school closures. Since COVID-19 has receded into the background, and face-to-face classes have resumed at schools and universities, there have been moves by some educational jurisdictions to continue with some form of online learning (Dunn, Gottlicher-Hill & Stephens, 2022). For example, the New York City Department of Education has decided that online learning will be used on snow days and other traditional days off such as Election Day (Cramer, 2020) cited in Dunn, Gottlicher-Hill & Stephens (2022). In 2021, all secondary school students in Singapore, were provided with digital learning devices which are connected to the internet (National Centre on Education and the Economy, 2021) cited in Dunn, Gottlicher-Hill & Stephens (2022). As a result, all secondary school in Singapore will have up to two days per month of online learning, which may take place in school or at home. An extension to the primary school sector is being considered for Singapore. These online learning days can provide all teachers with further opportunities to trial and assess online pedagogies

Theoretical Framework

In coming up with a theoretical framework for this paper, four learning theories or models are considered: connectivism, online collaborative learning, experiential learning and reflection. **Connectivism** has its roots in the early twenty-first century when first George Siemens (2004), and then Stephen Downes (2005) published articles published articles addressing the role technology has to play in modern learning. Siemens, one of the early pioneers of massive open online courses (MOOC), has been the main proponent of this learning model that acknowledges the way knowledge and information flows, develops and changes across the internet. In this regard, internet technology has moved learning from internal, individual activities to group, community and large-scale activities.

Online collaborative learning (OCL) is also a new theory of learning which focuses on the facilities of the internet to provide learning environments that encourage collaboration and knowledge building (Harasim, 2012). In OCL, there exist three phases of knowledge construction through discourse in a group. These are:

- **Idea generating:** the brainstorming phase, where divergent thoughts are gathered

- **Idea organizing:** the phase where ideas are compared, analysed, and categorized, and
- **Intellectual convergence:** the phase where intellectual synthesis and consensus occurs, including agreeing to disagree, usually through an assignment, essay, or other joint piece of work (Harasim, 2012: 82).

Kolb's theory of experiential learning focuses on the student's personal development and perspective. Unlike other learning methods, the onus is on the student to guide his/her learning process in experiential learning. Experiential learning allows students to apply knowledge in real life situations and encourages active participation, critical thinking, creativity, problem solving, collaboration, and communication skills (Kolb,1984)

Dewey (1991) refers to reflection as to an active and intentional process that can begin with some discomfort with an experience and end with learning and deeper insights. According to Dewey, aspects of reflective thought include perplexity, elaboration, generation hypothesis, comparing hypothesizes, and taking action. Boud et al. (1985) define reflection in the context of learning and, in line with Dewey, emphasizes one's personal experience as the object of reflection, referring to intellectual and affective activities that individuals engage with to explore their experience, which leads to new understanding and appreciation.

In this research the writer was **“connecting”** with the team members for the South African Mathematics Team competition via the Microsoft Teams platform; the participants, including the writer, in both problem-solving courses were connecting with the course material via the Learning Management System for each course. There was some online **collaboration** between the Olympiad learners when they were paired and placed in different “rooms” to discuss team questions. There were online sessions arranged for participants where they could collaborate with the facilitators and others in clearing up any issues, seek clarity on some problems and share ideas. The writer was an active participant in both activities and shares his **experiences**. This is followed by the writer's **reflection** of his experiences in these activities. Thus, a suitable framework in which to locate this paper is: *“Connecting, collaborating and reflecting on one's experiences within an online teaching and learning context”*

Research Methodology

This research is about the writer's involvement in online teaching and learning of mathematics. The writer outlines his involvement based on his experiences. For training sessions with Mathematics Olympiads for the South African Mathematics Team competition (SAMTC), the writer developed a series of lessons which were used in 2021 to 2023. These lessons are well documented and readily available within certain folders in the writer's records for the SAMTC. The writer also has records of his involvement as a participant in both the primary school problem solving course of 2021 and the high school problem solving course of 2022. Thus, this research is a form of descriptive research. Descriptive research describes the characteristics of the phenomenon being studied (Manjunatha, 2019). This methodology focuses more on the “what” of the research subject rather than the “why” of the research subject. In this research the phenomenon being studied is the “writer's involvement in online teaching and learning”. Given that this research attempts to describe the experiences of the writer in his “own voice”,

the research is mostly qualitative in nature (Hatch, 2002). There is also some reflection by the writer on these online experiences. In summary there are two major data sources for this research:

- The writer's training of team members for the South African Mathematics Team Competition (SAMTC) for the years 2021 to 2023, using the Microsoft Teams platform
- The writer's participation in two online Mathematics problem solving courses in 2021 (primary school) and 2022 (high school).

Results

Training for the South African Mathematics Team Competition (SAMTC)

The writer has been regional coordinator of the South African Mathematics Team Competition since 2014. Teams are entered for the Junior section (grade 8 and 9) and the Senior section (grades 10, 11 and 12). The first part of the competition is a one-hour individual problem paper, consisting of 15 problems in multiple-choice format. After a break for refreshments and a discussion of team strategy, the second part of the competition takes place. The second paper consists of ten quite difficult problems, and the team must work together and must submit just one set of answers. (SAMF, 2024). Teams are selected based on their performance in round 2 of the South African Mathematics Olympiad as well as their availability. For example, due to a clash of the SAMTC with the grade 12 trial examinations, many grade 12 learners opt out of participation in the SAMTC.

The writer worked with another mathematics specialist in the SAMTC until 2020. However, due to other commitments, the other person opted out from 2021 onwards. Thus, the writer was the only local person available to train learners for the SAMTC. In these years one junior and one senior team was selected, and the training was done virtually using the Microsoft Teams platform. Due to the impact of COVID-19 in South Africa and other countries, there was a move to online learning and other media platforms from March 2020. Thus, learners were already used to learning via online platforms in 2021. Links were sent to the learners prior to each training session to enable them to login to the session.

The writer compiled PowerPoint slides showing the questions and solutions of both individual and team questions from past year papers. The solutions were discussed in a structured manner to enable learners to apply what they learnt to solve other problems. These slides were shared during each of the training sessions.

A summary of training for 2021 to 2023 is shown in table 1:

Table 1 Training sessions for the SAMTC

Year	Junior team	Senior team
2021: Team competition took place on Saturday 11 September	20 August; 27 August; 3 September	21 August; 28 August; 4 September

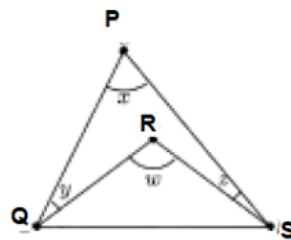
2022: Team competition took place on Saturday 10 September	26 August; 2 September	27 August; 3 September
2023: Team competition took place on Saturday 9 September	25 August; 1 September	26 August; 2 September

There were three training sessions in 2021 and two training sessions in 2022 and 2023. Attendance at these training sessions were very good with most learners participating in the training. Those who were absent were involved in other school activities on the Friday and Saturday afternoons such as school sport and prefect camps. Interestingly, no learner indicated that they had connectivity issues. The slides were emailed to all learners at the end of each training session. All training sessions followed a similar format with individual questions and team questions in the sessions. In the last session before the competition, learners were divided into different virtual “rooms” where team members, in pairs, engaged in discussions of two team questions, in an online collaborative manner (Harasim, 2012).

Some questions discussed during the training:

Junior Individual Question

In the figure shown find x in terms of y , z and w .



- A. $w - y - z$ B. $w - 2y - 2z$ C. $180^\circ - w - y - z$ D. $2w - y - z$ E. $180^\circ - x + y + z$

Solution:

PQRS is a convex quadrilateral: So we have $x + y + z + 360^\circ - w = 360^\circ$

Thus, $x = w - y - z$

Junior Team Question

Peter has 25 cards, each printed with a different integer from 1 to 25. He wishes to place N cards in a single row so that the numbers in every adjacent pair of cards have a prime factor in common. What is the largest value of N for which this is possible?

Solution

All prime numbers less than 25 are 2; 3; 5; 7; 11; 13; 17; 19; 23 (9 numbers)

The non-prime numbers 1; 4; 6; 8; 9; 10; 12; 14; 15; 16; 18; 20; 21; 22; 24; 25

Non-prime numbers in a row with prime numbers in between:

1; 2; 4; 3; 6; 5; 8; 7; 9; 11; 10; 13; 12; 17; 14; 19; 15; 23 (18 numbers)

Senior Individual Question

A farm has a certain number of black cows and a certain number of brown cows. Suppose all the cows of the same colour give a constant amount of milk everyday. Four black cows and three brown cows give as much milk in five days as three black cows and five brown cows give in four days. What is the ratio of the amount of milk given by a black cow to that of a brown cow?

- A. 5:8 B. 7:9 C. 2:1 D. 8:5 E. 5:4

Solution

Let a black cow give x litres per day: In 4 days it gives $4x$ and in 5 days it gives $5x$

Let the brown cows give y litres per day: In 4 days it gives $4y$ and in 5 days it gives $5y$

So $4(5x) + 3(5y) = 3(4x) + 5(4y)$

$$\Rightarrow 20x + 15y = 12x + 20y$$

$$\Rightarrow 8x = 5y$$

$$\text{So } \frac{x}{y} = \frac{5}{8} \quad \text{Answer: A}$$

Senior Team Question

Let a , b and c be positive integers such that $28a + 30b + 31c = 365$. Find all the solution sets (a, b, c) .

Solution

First solution: $a = 1$; $b = 4$ and $c = 7$ (days of the year)

If $a = 2$, then we have $30b + 31c = 309$

Multiples of 30: 30; 60; 90; 120; ...; 270; 300

Multiples of 31: 31; 62; 93; 124; ...; 279

Now $279 + 30 = 309$; $b = 1$; $c = 9$

Second solution: $a = 2$; $b = 1$; $c = 9$

The performances of the teams are shown in table2:

Table 2: Performances of the teams in 2021 to 2023

Year	Junior team	Senior team
2021	8 th out of 92 teams	6 th out of 87 teams

2022	15 th out of 155 teams	13 th out of 142 teams
2023	9 th out of 42 teams	6 th out of 42 teams

Reflections from the writer's online training for the SAMTC

- During the years 2021 – 2023, the writer had become experienced in online teaching and learners had become used to online learning. The sessions were well-attended and learners indicated that they had received sufficient preparation for the South African Mathematics team competition.
- Individual and team questions, as well as the solutions, were discussed during these sessions. Learners were free to ask questions during these sessions, seeking clarity on some of the questions discussed.
- The grouping of learners into pairs to work on team questions enabled learners to meet colleagues from other schools and build relationships in pursuit of mathematics excellence.
- The training helped learners in both the junior and senior teams perform very well, especially in 2021 and 2023, where both teams featured in the top 10.
- Learners have become used to participating in the SAMTC and look forward to the competition, each year. Sometimes, eager learners contact the writer about the competition if there is a delay in the announcement of team members.

Participation in online Mathematics Problem Solving courses

Primary school course

Despite being involved in mathematics competitions and Olympiads for more than 20 years, and being a certified facilitator in Mathematics Olympiad training, the writer participated in two online Mathematics Problem Solving courses. The first one was in July 2021 and targeted those involved in the training of primary school teachers in problem solving. The lead facilitator gave an introductory presentation and covered the importance of problem solving and discussed a few problems. The following key points were covered in this presentation:

- What is mathematics?
- Non-routine problem-solving
- Exploring patterns; formulating conjectures
- Mathematical proficiency
- Inductive and deductive reasoning
- Factors influencing problem solving
- Discussion of different problems such as: determining the sum $1 + 3 + 5 + 7 + \dots + 100$ numbers = ... in different ways (through inductive reasoning, geometrically and numerically); working through the sum of the angles of a polygon, examples involving discount and VAT, using the full-time score in a netball match to determine the possible half-time scores, given a number of points on the circumference of a circle work out the number of chords that can be drawn

In summary, the activities of the course involved working through the following units and completing activities in each unit.

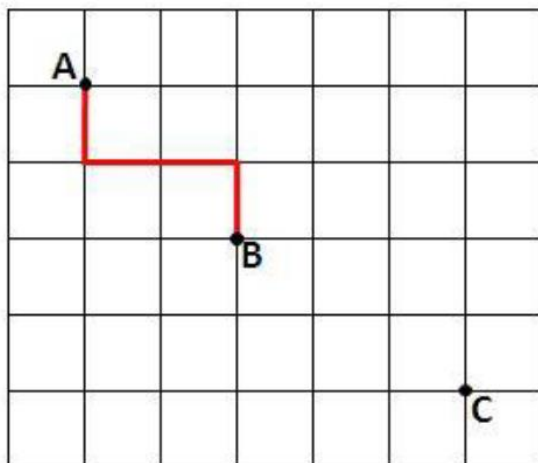
- What is a problem?
- Control the Problem-Solving process
- Focus on structure
- The power and pitfalls of induction
- Think like a Mathematician: structure, control, generalize; pose new problems
- Think like a Mathematician: be systematic

The assesment consisted of an assignment and an online test (once all the activities in each unit were completed). The online test comprised 20 multiple choice questions. The weighting of the assignment was 20% and the online test 80% when determining the final mark.

Each time one had to login into the Learning Management System (LMS) where the units and activities were located. Each participant could work at their own pace and were given two weeks to complete the assignment and activities in each unit before they could take the test. The LMS would not allow a participant to proceed to the next unit until all the activities in the current unit were completed.

The assignment

Here is a street map of Hitown. Every day Katy drives from her home at A to work at C via B, where she picks up Kevin. She always drives the shortest route – the shortest route from A to B is a distance of 4 “blocks”; one such route is shown. She likes to drive a different shortest route every day.



Assignment questions

1. How many different shortest routes are there from A to B?
2. What is the distance (in blocks) of the shortest route from B to C?
3. How many different shortest routes are there from B to C?

4. How many different shortest routes are there from A to C via B?
5. How many different shortest routes are there from A to C (not necessarily via B, i.e. Katy does not necessarily pick up Kevin)?

The High School Course

The writer received communication via email in August 2022 that he had been entered for an online High School Problem Solving Course. Various problem had to be solved over a six-week period in a flipped classroom mode. Each unit covered knowledge and understanding of the problem-solving process. The writer was given a username and password to access the course.

A PowerPoint presentation, comprising 16 slides, formed the introduction to the course. The presentation covered similar ideas and content to that of the primary school presentation. However, the different approaches and examples discussed was not as detailed. The course involved participants logging into the LMS and working through the five units of the course, comprising more than 40 problems (of different types). The activities of one unit had to be completed first before being allowed to go into the next unit. Only when one had completed all the problems within the units could one take the test. Discussions were arranged for each Friday afternoon where participants were able to share their experiences and seek clarity on certain problems from facilitators and other participants.

Some of the examples discussed in the different units

1. Take any 3-digit number and repeat it to form a 6-digit number. Now divide by 7, then divide the answer by 11, then divide the answer by 13. What is your answer?
2. Which of the following numbers is not divisible by 5?
 - (a) $231^2 - 211^2$
 - (b) $213^2 - 212^2$
 - (c) $213^2 - 231^2$
 - (d) $213^2 - 211^2$
 - (e) $213^2 - 212^2$
3. In the game of rugby, it is possible to score 3, 5 or 7 points. Which of the following scores are possible/impossible? 27; 31; 32; 73; 61
4. There are 16 soccer teams in the Premier Soccer League. If each team plays every other team twice in the season, how many matches in total are played in the season?
5. What is the remainder of $98765432123456789 \div 10$?

Despite the demanding schedule of the writer, all activities in the units were completed. This enabled the writer to take the online test within three weeks.

Reflections from the writer's participation in the online problem-solving courses for teachers

- The LMS used in both courses was user-friendly and quite easy to negotiate. The instructions were also quite clear. One could also go back to previous units to correct mistakes and to revise.
- The courses were very comprehensive and demanding
- While there were discussion groups for both courses, most of the time one worked independently going through the various problem-solving strategies/heuristics and problems and completing the activities in each unit (flipped classroom mode).
- The problems contained a wide range of mathematical content and covered different levels of cognition.
- Participants were able to download activities which included interactive worksheets, PowerPoint slides, approaches to problem solving and other key activities.
- The writer did exceptionally well in both courses. This was probably due to his long involvement in problem solving programmes.

Findings

The findings of this paper are now written within the context of the research subsidiary questions, and consequently the research questions of this paper.

The writers' experiences in teaching mathematics through an online platform

These experiences have been well-documented in this paper with the writer showing how he used the Microsoft teams' platform to teacher/tutor mathematics learners who were participating in the South African Mathematics Team Competition. The use of Microsoft Teams, Zoom and Google Meet was self-taught, and the writer has become more comfortable in using these platforms. Since then, the writer has participated in several online education forums where the writer was able to share and receive information in mathematics and other education related matters. The writer also assisted primary teachers in one-hour Friday sessions with some ideas and techniques in mathematics problem solving. These teachers were busy with another online problem-solving course. The writer has also been able to teacher university students using the Microsoft Teams' platform.

The writer's experiences in learning mathematics through an online platform

Details of the writer's involvement in learning mathematics through an online platform (Learning Management System) are shown earlier in this paper. The writer is able to effectively use the LMS to learn more about mathematics problem solving for both primary and high school, adding to his knowledge in the subject. While the performance of the writer was exceptional, this is not the main purpose of doing the courses. The main purpose(s) was to get acquainted about using the LMS, doing and learning about problem solving in different contexts, in an online environment. In addition to the two online courses mentioned in this paper, the writer has also completed online courses on Professional Learning Communities and Gender Responsive Pedagogy (for grade R) through the Belgian organisation VVOB which does work in some developing countries. He has also completed an online Coding and Robotics course through the Department of Basic Education's LMS. The writer used the knowledge gained from these activities to

organise an online Science Symposium, attended by more than 50 teachers, in August 2021 as well as an online Language Symposium, attended by more than 40 teachers, in September 2021.

To what extent were these experiences successful?

It is important to acknowledge that these experiences came at a time when the world was going through a trying time with the COVID- 19 pandemic. The writer wanted to learn various ways of reaching teachers and learners and to support them in different programmes. These experiences were very successful, and the writer is able to continue to use these online platforms to teach and to learn in the post-COVID-19 era. The writer also belongs to several Mathematics groups on Facebook where he is an active participant and has learnt even more.

Reflection on these experiences

The writer reflects on these experiences with pride and dedication. These are outlined after the discussions in the results section of this paper. As an educationist with more than 40 years' experience, the writer has learnt even more to share with learners and teachers in different contexts. This makes the writer up-to-date and relevant and being able to support learners and teachers for some years to come.

Conclusion

As a lifelong learner, the writer has shared his experiences in online teaching and learning in effort to show that despite challenges, it is possible to make a positive difference in the lives of others and to support those in need. However, this requires hard work and dedication. It is hoped that the experiences shared in this paper may serve as a motivation to others to do something similar in their workspaces. In this way we all can contribute to developing and building our country.

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DATA-INSIGHTS INTO THE CHALLENGES FACED BY TEACHERS TEACHING EUCLIDEAN GEOMETRY THROUGH INTEGRATED ICT SOFTWARE

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This research examined the lesser-explored area of difficulties encountered by educators when incorporating Information and Communication Technology tools into Euclidean Geometry instruction, which is currently lacking in educational technology studies. In contrast to previous studies that primarily examine student outcomes, this research places emphasis on the perspective of educators and their ability to effectively teach abstract concepts in light of rapid technological advancements. Employing a Participatory Action Research approach, the investigation engages educators as collaborative researchers to gain a comprehensive understanding of Euclidean Geometry pedagogy. The findings demonstrated significant hurdles experienced by educators and learners alike due to the complex nature of Euclidean Geometry and the constantly evolving realm of technology. The study emphasises the importance of collaborative approaches and teamwork in educational settings, signalling a shift away from traditional teaching methods that focus on individualism. The research highlights an important finding regarding the limited proficiency of educators in utilising ICT tools effectively, as evaluated using the Technological Pedagogical Content Knowledge framework. This reveals a gap between their technical skills and their ability to apply them pedagogically. As a result, it is recommended that comprehensive professional development programs be implemented to address this issue, with a focus on integrating technological knowledge, pedagogical strategies, and content understanding.

Keywords: Euclidean Geometry, Integrated ICT Software, Teacher Challenges, Professional Development, Participatory Action Research

Introduction

Recent research has underscored ongoing difficulties in both teaching and understanding Euclidean geometry, with educators and students facing significant hurdles. Ogbonnaya (2021) highlighted the struggles of teachers and pre-service teachers in conveying Euclidean geometry concepts, linking students' poor performance to a lack of comprehensive training and grasp of mathematical principles. Echoing these concerns, Tachie (2020) pointed out similar obstacles in South Africa, where a deficit in mathematical concept comprehension and inadequate pedagogical training impede effective teaching of Euclidean geometry. Machisi (2021) delved into the perspectives of grade 11 students on their experiences learning Euclidean geometry, revealing issues related to teaching methods and students' attitudes towards the subject. Complementing these insights, Pingpor et al., (2023) and Bayaga et al. (2019) explored the efficacy of GeoGebra software as an educational tool, suggesting potential advancements in student comprehension of Euclidean geometry.

In our modern society, where technology has significantly impacted various industries, the field of education is no exception (Hayati et al., 2022). The continuous advancements in technology have led to significant transformations in teaching methodologies and practices,

making technology an essential tool in the classroom. Researchers have established a theoretical construct referred to as TPACK, encompassing the integration of Technological, Pedagogical, and Content Knowledge. This model aids in comprehending and employing effective pedagogy through the incorporation of technology in teaching practices (Mosia & Matabane, 2022). The purpose of this study is to gain a comprehensive understanding of the intricate relationship and interdependence between technology, pedagogy, and content knowledge (Bwalya & Rutegwa, 2023).

This encompasses recognising both the challenges and opportunities that arise from this dynamic interaction. The integration of technology into teaching is best understood through the TPACK framework, also known as Technological Pedagogical Content Knowledge. This conceptual model explores the intersecting domains of Technology Knowledge, Pedagogical Knowledge, and Content Knowledge. Effective integration of technology into teaching goes beyond just technological skills. The TPACK model emphasises the importance of not only having technological knowledge but also strong pedagogical knowledge and content-specific instructional strategies. This well-rounded proficiency in technological, pedagogical, and content knowledge is necessary for effective teaching in subjects like Euclidean Geometry, where tools such as GeoGebra, HeyMaths!, and Geometer's Sketchpad can be particularly useful (He et al., 2022). Pedagogical knowledge encompasses the abilities needed to effectively manage and organise teaching and learning activities, while content knowledge refers to a comprehensive understanding of Euclidean geometry itself. By integrating technology into teaching practices, educators can enhance student engagement, facilitate interactive and collaborative learning experiences, and provide personalised instruction tailored to individual student needs.

A primary challenge identified in this study is the difficulty that both learners and educators encounter in comprehending and effectively teaching Euclidean geometry concepts (Ogbonnaya, 2021; Tachie, 2020). This challenge is intensified by the abstract nature of Euclidean geometry() and the rapid technological advancements in ICT tools. A significant gap in knowledge and skills among educators often hinders the realisation of the full potential of ICT in enhancing geometry teaching and learning experiences. While existing literature, such as the works of Liu & Kaino (2007), has highlighted the potential of ICT in making abstract mathematical concepts more accessible to learners (Pucholt, 2020), there is a notable dearth of research focusing on the challenges of using integrated ICT software for teaching Euclidean geometry from the perspective of the educators.

The existing body of literature predominantly emphasises the effectiveness of ICT in improving student learning outcomes, often overlooking the practical challenges that educators face in integrating these technologies into their teaching practices. Studies conducted in various educational contexts, including South Africa, Nigeria, and Korea, have pointed out the inadequacies in teachers' content knowledge (Ancheta, 2020) and technical skills in using ICT software for geometry teaching. These studies, however, primarily focus on student learning outcomes, with less consideration given to the multifaceted challenges faced by teachers, especially in integrating different types of software to teach complex and abstract subjects like Euclidean geometry. Furthermore, there is a noticeable gap in research that holistically combines the three elements of TPACK – technological, pedagogical, and content knowledge – in a manner that is both theoretically robust and practically applicable.

While the challenges of keeping up with rapidly advancing software are acknowledged (Tella et al., 2007), there is a lack of detailed exploration into the specific challenges related to the integration of various software types and adapting them to effective teaching strategies. This gap highlights the need for research that not only addresses the technological aspects of ICT software but also delves into the pedagogical and content-related challenges that teachers face. The significance of this research lies in its potential to contribute substantially to educational practices and policies by deepening the understanding of the challenges teachers face in integrating ICT into geometry education. Enhancing teachers' TPACK is crucial in an era where technology is becoming increasingly integral to education (Suprpto et al., 2021; Çinar, 2022; Nuroh et al., 2022; Çoban et al., 2022). This research aims to provide insights into the challenges teachers face in using integrated ICT software for geometry instruction, offering strategies to improve their ability to design and facilitate lessons that make abstract concepts more accessible and engaging for students. By doing so, the study aims to contribute not only to the theoretical discourse but also to practical improvements in teaching methodologies. The study is designed to bridge the identified gaps in the literature by offering a comprehensive understanding of the challenges faced by teachers in instructing Euclidean geometry using integrated ICT software. The research aims to, identify and meticulously analyse the specific challenges associated with the use of integrated ICT software for teaching Euclidean geometry.

This study adopts social constructivism as a guiding framework to elucidate and delineate effective pedagogical practices. Central to this approach is the belief that augmenting TPACK for teaching Euclidean geometry with Information and Communication Technology is a collaborative endeavour. The justification for employing social constructivism lies in its foundational principles, structure, objectives, and the roles it assigns to researchers and participants, coupled with its ontological and epistemological perspectives (González-Salamanca et al., 2020; Rhodes & Calder, 2022). Social constructivism contends that reality is a collective construction shaped by human interaction. It implies that people within a community forge their perspectives of the world through mutual understanding. Thus, social constructivism provides a lens through which reality is viewed not as a pre-existing entity to be discovered but as a construct of human agency (Zimmerman, 2021; Adebola & Tsotetsi, 2022; Craig et al., 2023). Its ontological aspect can be interpreted through the concept of bricolage, suggesting that a communal reality is a mosaic of individual perceptions, mirroring the intricacies and diversities within a society. This notion is supported by Pritchard and Woollard (2010), who observe that while individual perceptions within a community may largely align, they can also diverge significantly due to varied experiences and interactions. This is reflective of complex ontology, acknowledging the existence of multiple, sometimes conflicting, realities in the formation of a collective consciousness. From an epistemological standpoint, social constructivism posits that knowledge is a product of social interaction, whereby individuals interpret and understand their environment through these interactions. Consequently, social exchanges become a conduit for knowledge creation. This perspective allows for the recognition of diverse methods of knowledge acquisition. For instance, the understanding of familial relationships may be formed through the social environment of the home while simultaneously acknowledging that other methods, such as genetic testing, offer alternative pathways to knowledge.

Applying this theoretical framework, the study conceptualises effective teaching in the realm of TPACK for Euclidean geometry using ICT as an interactive process involving both the learners and their environment. This stance is unreserved in its exploration of TPACK, viewing knowledge creation as inherently human-centric. It influences the methodology of the study, aligning with Kincheloe's (2009) assertion that knowledge development in teaching should be teacher-led, challenging the conventional top-down approach of professional development programmes. The study advocates for a reimagining of the teaching profession, transforming it into a learning space where both teachers and students are active participants in knowledge construction. This approach aligns with the principles of self-directed learning, emphasising that teachers should be empowered to engage in ICT training that resonates with their unique pedagogical needs rather than conforming to a standardised, prescriptive model. This paradigm shift aims to foster an environment where teaching transcends its traditional boundaries, becoming a dynamic interplay of learning and instruction.

Research Methods

This investigation adopted a Participatory Action Research (PAR) methodology, underpinned by the premise that educators and practitioners immersed in the milieu of Euclidean geometry pedagogy possess the most insightful perspectives for strategy development. Recognising a gap in the researcher's expertise regarding the nuances of TPACK in Euclidean geometry, this study positioned these individuals not merely as subjects but as co-researchers. Their lived experiences and practical challenges in teaching Euclidean geometry were deemed integral to addressing the study's objectives. This participatory approach aligns with the ethos of PAR, which values the knowledge and input of those directly involved in the context being studied.

The research commenced with a series of preliminary meetings aimed at convening a diverse group with shared experiences in the realm of mathematics education. These sessions served a dual purpose: to amalgamate relevant stakeholders and to mitigate any potential power imbalances within the group. The genesis of the study was an invitation extended to the principal investigator by a key educational figure, pseudonymously referred to as Mr. Phehello, to participate in an academic subject improvement initiative.

The core research team comprised a dynamic mix of four mathematics teachers, the principal investigator (serving as the academic expert), a Deputy Chief Education Specialist (DCES), and a computer programmer. Utilising the Participatory Action Research approach allowed us to leverage the collective strength, mirroring the interdisciplinary essence of the TPACK framework in the context of Euclidean geometry (Falk et al., 2019; Kholgi, 2020; Silva, 2020; Fitrianto, 2021). This collaborative effort resulted in the development of an action and implementation plan focused on the effective use of ICT software for teaching Euclidean Geometry. Data were generated through our planning meetings, lesson observations, and post-lesson reflections. This data was audio-recorded and subsequently transcribed. Our study employed discourse analysis to extract insights from this material. The use of Critical Discourse Analysis (CDA) enabled a more profound and nuanced understanding of our interactions and discourses, specifically within the context of teaching and learning Euclidean geometry.

Results

The study underscores the value of teamwork in education, particularly in the context of integrating technology and pedagogy. Chavan (2013) highlights that collaboration in educational settings can significantly enhance lesson preparation, leading to a more focused and objective-driven approach in teaching. This concept aligns with the broader educational consensus that teamwork among educators is crucial for sharing skills, knowledge, and best practices, especially in areas combining technical and pedagogical aspects.

Below is the evidence from our planning session during. This became evident during a meeting to establish a team, when I asked, “how are you currently supporting one another in teaching?” this was the response provided: Mrs Kotudi (teacher), who taught mathematics at the same school as Mr Phehello and the Grade 12 teacher, responded, “Currently, sir, what is really happening is that it is every man for himself.” And this was supported by another teacher who said: “Indeed, Ntate” (meaning “sir”). Another teacher added: “This thing is going to help us to start working together.” And another teacher supported “This is different from what we have been doing, here I am not teaching for you, we do it together”.

The feedback received during a meeting to form a group of teachers involved in the research offers valuable insight into the existing state of collaborative practices among educators. Mrs. Kotudi's statement, "Every individual works on their own," highlights the absence of collaboration and a supportive school culture. Another participant echoed this sentiment, indicating that teachers collectively recognise and experience feelings of isolation and individualism within their work. Furthermore, the discussion suggests an increasing awareness among educators about embracing a new approach. One teacher's statement, "This initiative will enable us to collaborate more effectively" demonstrates an awareness of the potential advantages that teamwork and cooperation could offer in their instructional approaches. This sentiment is further supported by another teacher's assertion, "The approach here departs from our traditional teaching methods. It signifies a shift towards collaborative learning where I am not the sole instructor, but we work together." This statement emphasises a departure from an individualistic perspective and highlights the significance of shared accountability and cooperative endeavours in education.

Classroom Observation and Discussions

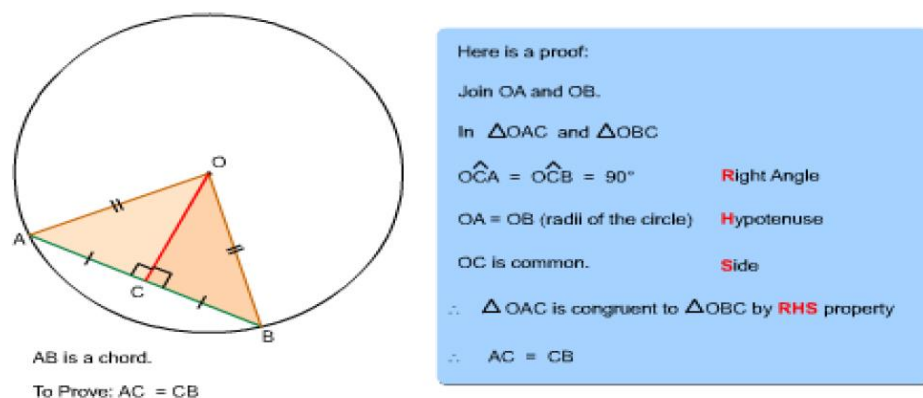


Figure 11: Proof taught in the Class

Effective lesson facilitation, grounded in social constructivist theory, offers learners opportunities for social interaction, especially when using ICT software to learn mathematical concepts (Hense & Mandl, 2012: 21). This approach implies that optimal teaching occurs when learners engage with both computer software and their peers to discover mathematical concepts. According to Gagné's conditions of learning theory, effective learning happens when a lesson takes into account learners' prior knowledge and uses ICT software to create stimulating educational scenarios (Zhang, 2022; Endriani et al., 2022). Rosli et al., (2020). further specify that well-facilitated lessons should capture learners' attention, clarify lesson objectives, stimulate learners, incorporate recall of previous knowledge, present engaging material, offer guidance, elicit performance, provide feedback, and enhance retention and transfer of knowledge. Vygotsky's theory (1978, as cited in Siyepu, 2013) also emphasises that teachers should build upon what learners can do independently, based on their prior knowledge, before introducing new concepts that require instructional support.

The CAPS document policy (DoE, 2011) mandates that teachers structure their lessons to enable learners to interact, engage, communicate, demonstrate, and record their thought processes. Contrary to these principles and good teaching practices, Mr. Mokoena's use of HeyMath! software did not facilitate opportunities for learners to geometrically represent theorem statements. He merely recited these statements and followed them with geometric representations, which contradicts the CAPS policy that advocates for interactive and participative learning (DoE, 2011: 10). During Mr. Mokoena's lesson, learners were passive, neither engaging in mathematical activities nor demonstrating their thought processes. They simply listened to HeyMath!, missing opportunities to enhance their problem-solving skills, such as interpreting problems and representing them geometrically. Ideally, Mr Mokoena could have paused before each software explanation to allow learners to analyse and share their understanding of the theorem statements, identify key mathematical concepts, and then geometrically represent these statements. Such an approach would have fostered social interaction among learners and with the teacher, aiding in the assessment of learners' prior knowledge and clarification of misconceptions.

The inadequacy in lesson presentation was further evident in the narratives surrounding HeyMath! software and Mr. Mokoena's failure to integrate his teaching with the software. For

instance, in attempting to prove that $AC = CB$, Mr Mokoena simply presented the proof without involving learners in the discovery process. This approach, where Mr Mokoena stated, “Here is the proof, join OA and OB,” denied learners the chance to explore the proof themselves. Furthermore, no explanation was provided as to why joining OA and OB was necessary, making the theorem too abstract for learners to comprehend. This method of teaching, focused on showing rather than involving learners, contradicts the learner-centred principles, which advocate for teachers to provide instructional guidance and support coherent knowledge representation. A learner's question at the end of the lesson highlighted this gap: “Ntate (sir), how do you know that you must join OA, or are we always supposed to join OA?”

Mr. Mokoena's lesson presentation, evidenced in his approach to proving congruency between two triangles, continued this pattern. He did not enhance the HeyMath! lesson with his input, missing opportunities to ensure that learners understood concepts such as “angle OCA equals angle OCB equals 90 degrees.” Although the diagram used various notations and symbols to describe the relationship between angles, Mr. Mokoena did not explain these notations to the learners. This approach is contrary to Vygotsky's views on semiotics, which emphasise the role of teachers as mediators in helping learners internalise systems of signs and symbols. Furthermore, by not encouraging learners to identify the congruent sides and angles in the triangles without prior direction, Mr. Mokoena missed a chance to promote learner-centeredness. He could have paused the lesson and complemented the HeyMath! presentation and encouraged learners to independently identify the congruent elements of the triangles $\triangle OAC$ and $\triangle OBC$, thereby fostering a more interactive and engaging learning environment (Figure 1).

Teacher knowledge for teaching with technology

The analysis of teacher preparedness and proficiency in using ICT tools through the lens of the TPACK framework reveals significant gaps in educators' ability to integrate these tools into their teaching effectively. The TPACK framework underscores the importance of an intersection of technological, pedagogical, and content knowledge, yet the findings indicate deficiencies in all three domains. In terms of technological knowledge, teachers demonstrated a basic understanding of ICT tools but often lacked depth. Classroom observations and lesson reflection revealed a pattern of uncertainty and difficulty in troubleshooting. A representative teacher's comment, "I can navigate the basic functions, but when technical issues arise, I'm at a loss", highlights this gap. This lack of in-depth technological proficiency frequently led to interruptions and a diminished ability to integrate technology seamlessly into teaching practices.

Furthermore, PK also presented challenges, particularly in effectively blending ICT tools with teaching methodologies. Many educators struggled to adapt these tools to their existing teaching styles or to develop new strategies that leveraged technology effectively. In a reflective session, teachers expressed their uncertainty about effectively incorporating technology into their teaching despite having knowledge of how to use it. This disconnection between technology and pedagogy highlights the importance of professional development that not only emphasises the technical aspects of ICT tools but also focuses on integrating them into

instructional strategies. Although the teacher had a strong understanding of content knowledge, it did not always align seamlessly with the utilisation of ICT tools.

Teachers sometimes faced difficulties when integrating technology into their subject matter teaching. Feedback from teachers highlighted challenges such as feeling unsure about how to use certain tools effectively in alignment with specific topics, like Euclidean Geometry. Further analysis of classroom interactions and lesson reflection revealed a lack of confidence and mastery over technology among teachers. Language used by teachers during lessons often indicated uncertainty, with phrases like "I'm not sure how to do this" frequently heard. Concrete evidence of these challenges was observed through instances where teachers struggled to utilise interactive software, causing disruptions in the lesson flow. It became apparent that further support and development opportunities were necessary for teachers to overcome these obstacles and integrate technology more seamlessly into their teaching practices.

Students' perceptions mirrored these findings. Interviews with students indicated that they often viewed their teachers as less adept with technology. Such perceptions can impact the teacher-student dynamic, as one student's comment illustrates: "Sometimes, it feels like we know more about the computers than our teacher does." These findings suggest a pressing need for comprehensive training and support for educators, addressing not just the technological aspect (TK) but also the integration of technology with pedagogical strategies (PK) and content (CK). Such professional development should aim to enhance teachers' skills across all TPACK components, improving their confidence and effectiveness in using ICT tools. Additionally, establishing a community of practice for educators to share experiences and strategies can foster a supportive environment for continuous professional growth in the effective use of technology in education.

Discussions of findings

This study focuses on the difficulties faced by educators and students in teaching Euclidean Geometry using integrated ICT software. It uncovers substantial obstacles, which are also discussed by Ogbonnaya (2021) and Tachie (2020). Additionally, this research sheds light on the perspective of educators, an often overlooked aspect in the discourse on technology in education. In contrast to previous studies that primarily examine student learning outcomes, this research takes a distinct approach by highlighting the perspective of educators. This aspect is often overlooked but crucial in discussing the role of technology in education. This approach differs from the studies conducted by Bayaga et al. (2019) and Machisi (2021), which focus primarily on students' understanding and learning experiences.

The current research provides new insights into the role of educators in incorporating digital tools and teaching abstract concepts like Euclidean Geometry. This contributes to our understanding of navigating the complexities of teaching in a technology-driven educational environment. The findings of this research suggest that educators face several challenges when integrating ICT software into the teaching of Euclidean Geometry. Moreover, this research emphasises the significance of collaborative effort and teamwork in educational environments, specifically when it comes to incorporating technology into teaching methods. This aligns with the wider consensus in education, as emphasised by Chavan (2013), which underscores the value of collective sharing of expertise and information. In our study, we delve into the current

state of collaboration among educators, highlighting a shift towards more cooperative and synergistic approaches.

This expands upon existing research by exploring the practical aspects of collaborative practices in educational settings, which has received less attention compared to studies focusing on teaching methodologies and knowledge levels (Pucholt, 2020; Ancheta, 2022). Through this research, we found that effective collaboration among educators not only enhances teaching methods but also improves student engagement and overall academic achievement. The study's alignment with social constructivist theory, particularly in lesson facilitation using ICT software, offers a fresh perspective when compared to the theories emphasised by Gagné and Vygotsky. While these theories advocate for interactive and participative learning, The current research concretises these theories by demonstrating their application in the context of Euclidean Geometry teaching. This aspect of our study contributes to a deeper understanding of how these theoretical frameworks can be effectively implemented in real classroom settings. This dimension is often theoretical in nature in existing literature.

A critical contribution of this study is the analysis of teacher preparedness and proficiency in using ICT tools through the TPACK framework. While Çınar (2022) and Çoban et al. (2022) explore the theoretical underpinnings and applications of TPACK in various contexts, the current study provides a detailed examination of the gaps in technological, pedagogical, and content knowledge among teachers. This finding is crucial as it not only identifies the deficiencies but also underscores the need for a holistic approach to professional development, focusing on all three domains of the TPACK framework. This aspect of our research adds depth to the existing discourse on TPACK, emphasising the practical challenges and solutions in integrating technology into teaching practices.

Conclusion

This research contributes significantly to the field of educational technology as it investigates the difficulties that educators encounter when teaching Euclidean Geometry using integrated ICT software. By shifting attention from student learning outcomes to a perspective focused on educators, this study fills an important void in existing literature. It provides fresh perspectives on the intricacies and complications associated with incorporating technology into classrooms. The emphasis placed on challenges faced by educators, such as navigating the abstract concepts of Euclidean Geometry and keeping up with rapidly evolving ICT tools, highlights the necessity for more educator-centred approaches in education research and practice.

The recognition of the importance of teamwork and collaboration in integrating technology into education signifies a shift in educational practices. Rather than solely relying on traditional individualistic methods, there is now an increased emphasis on cooperative and synergistic approaches among educators. This aligns with the prevailing consensus in education regarding the significance of shared skills, knowledge, and best practices, particularly when it involves combining technical and pedagogical aspects. Moreover, this research illustrates how social constructivist theory can be applied to use ICT software for lesson facilitation. It demonstrates the successful implementation of theoretical frameworks within real classroom settings. Such achievements are especially notable when teaching

abstract subjects like Euclidean Geometry, where incorporating technology presents both advantages and obstacles.

The research extensively examines the preparedness and competence of teachers in employing ICT tools within the framework of TPACK. It underscores the discrepancies in educators' technological, pedagogical, and content knowledge, emphasising the significance of holistic professional development initiatives that focus on enhancing not only technical skills but also integrating technology effectively with pedagogical strategies and subject-specific expertise. Overall, this study underscores the complexities and challenges that educators face when integrating ICT software in teaching Euclidean Geometry.

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TEACHER'S PERSPECTIVES ON LEARNERS' SOLUTION STRATEGIES FOR TRANSFORMATIONAL FUNCTIONS: ADVANCED SUBSIDIARY LEVEL MATHEMATICS

Ntsasa Lisema & Machaba Masilo

This study focused on the teachers' perspectives on learners' solutions strategies for transformational functions at advanced subsidiary level mathematics. Functions hence transformation of functions are a much-needed content for solid background in mathematics for pre-calculus. Learners were given tasks that strategically required the description and the performance of the transformations. Qualitative case study approach was used wherein 15 learners' responses were randomly chosen and a concerned teacher participated in semi-structured interviews. Those responses were categorised according to the set themes emanating from the errors made and analysed. The teacher interview was done to further obtain verbal evidence connected to the errors made. The results depicted that challenges emanate from the nature in which the functions are expressed, ability on algebraic manipulation and even more demanding are cases which need geometrical skills. The study therefore recommends that sequential teaching, integrated with technology, be used to instil geometrical skills and to aid conceptual understanding.

Keywords: perspectives, conceptual understanding, transformational functions, algebraic and geometrical skills, sequential teaching

Introduction

Upon successfully completing International General Certificate of Secondary Education (IGCSE), learners normally between the ages 16-19 are expected to Advanced subsidiary (AS) level. This is Cambridge International Examinations (CIE) programme aimed at advancing learners' skills and strategically prepare them for tertiary education. AS level mathematics is essential for preparing learners for university mathematics and related courses. It is normally offered to learners who obtained at least grade C (above average) in Mathematics in the Lesotho General Certificate of Secondary Education (LGCSE) or equivalent. IGCSE and LGCSE are equal by benchmark and offer equivalent content specifically on functions and the assessment focusing on similar competencies. The structure of the content is purely aimed at enhancing specialisation in future mathematics learning. LGCSE syllabus (2021) stipulates that learners should be able to describe fully and perform transformations and a combination of transformations on simple plane figures using reflection, translation, stretch and shear. This becomes a critical pre-requisite knowledge for AS level mathematics which potentially grow from transformations of shapes to that of functions. Transformation of functions has seemingly emerged as a challenge to learners in both description and performance due to algebraic and geometrical skills required to handle object and image and it is quite imperative to learn such properly. Learning function transformation is part of learning the concept of function by providing new opportunities for students to use, reflect and discover knowledge about the subject (Lage & Gaisman, 2006).

Teaching transformation of functions

The teaching of the concept of functions generally starts with a formal definition and then examples of various presentations are given to strengthen understanding (Parhizgar et al., 2022). There must be a cohesive link between geometrical objects and functions as such different forms of functions are elucidated to the learners. Going into next level of mathematics learning, AS level Mathematics syllabus (2023) stipulates that candidates should “understand and use the transformations of the graph of $y = f(x)$ given by $y = f(x) + a$, $y = f(x + a)$, $y = af(x)$, $y = f(ax)$ and simple combinations of these”. This imply that terms like translation, reflection and stretch should be mentioned in the description relating f into the image of f and such should be algebraically seen from the given function. Cooney et al. (2010) illuminated the importance of learning function transformation and propose combining and transforming functions is one of the five big ideas (the function concept, covariation and rate of change, families of functions, combining and transforming functions, and multiple representations of functions). Learners normally commit a number of conceptual errors in both performance and description of the transformed function. The misunderstanding of changing variables and applying the roles of variables in terms of object (x) and image (y) can be a major challenge for most students (Carlson & Oehrtman, 2017). Transformations of functions require algebraic and geometrical skills so that the positions of either an object or an image is obtained clearly by keeping the algebraic and geometrical procedures concepts correct. However, difficulties with algebra lie with both algebra itself and also with other areas of mathematics that students will have encountered, which are seen as prerequisites for the learning and understanding of algebra (Bush & Karp, 2013). Generally, learners have problems dealing with various forms of algebra hence transformations of functions. This relates to functions expressed in different forms like $y = ax^2 + bx + c$, $y = a(x - r_1)(x - r_2)$ and $y = a(x - h)^2 + k$. In a study on students’ thoughts of a quadratic function using a clinical interview, Celik & Guzel (2017) observed that it is necessary for students to convert the standard forms to factorised forms or the vertex forms and to relate them with the graphical properties for improving their thinking about quadratic functions. It therefore implies that conceptual understanding and connection be made on the parent function to the transformed function while registering the graphical representation of both.

Problem statement

Learners at AS level Mathematics experience various challenges among most of which are content based especially on the algebraic section. In CIE schools, there is a document called REG rating in which learners tick Red Ember Green to denote level of understanding of the topic with G indicating well understood while R shows the opposite. Transformation of functions registered most Rs and that therefore gave rise to a concern over the causative agents to this impasse. The study was geared onto finding teacher’s perspectives on learners’ solution strategies on the transformational functions.

Significance of the study

The purpose of the study was to investigate the teachers' perspectives on learners' solution strategies for transformational functions at AS level mathematics. This was done through testing learners on transformations tasks which include both the description of a transformation as well as the performance of a transformation.

This study was underpinned by the two research questions;

1. What are the algebraic and graphical errors learners do on transformational functions?
2. What teaching and learning processes can be learned from these errors on transformations of functions?

Reviewed literature

Functions form a critical content for pre-calculus courses, and it is quite imperative that learners are exposed and swiftly assisted compatibly during knowledge imparting. To this, Lage & Gaisman (2006) added that the study of function transformations has become important in pre-calculus courses at higher education institutions because it gives students new opportunities to use and reflect on the concept of function. Functions exist in different forms at different levels of mathematics learning, and it is not different at AS level. Learners are exposed to linear, quadratic and trigonometric functions both in algebraic and graphical form. Many studies reveal that secondary students have difficulties in understanding the concept of functions, and quadratic functions (Eraslan, 2008). The knowledge of functions is essential in algebra as the use of functions is intertwined with other content areas (Mutambara et al., 2019). This enables learners to algebraically cope while dealing with different forms of functions. As stated by Suzanne (2015), the topic of function is one of the main requirements as it provides the knowledge in being able to discover and seek solutions to common issues related to the real world. This conceptually mean that, the acquired knowledge is deemed usable in other aspects of mathematics and handling some life expectations. Functions are essential for pre-calculus as pre-requisite knowledge for dealing with variable and optimisation. Applying quadratic functions enables the application of mathematical thinking and reasoning which involves making the paths of decision (Center, 2012). Functions are liable to be used in various modelling processes and in constructions, and as such future mathematics users need to be equipped with a deeper compatible content.

Teaching at this level should scaffold learners into making mathematical connections into real-life situations. In any single or combined transformation, learners should possess adequate algebraic and geometrical skills to showcase deeper understanding. Most precisely, when learners give no meaning to the variables of the function, understanding is poor. The verbal translation for the conceptual understanding of functions needs to be explored since it is crucial to identify the relationship contained in algebraic expressions as claimed by Hohensee (2017). This will drastically enhance learners' visual and reasoning abilities through understanding of both algebraic and geometrical concepts as involved in the transformational functions. Fonger (2016) identified students' modes of reasoning about functions that supported students' abilities to symbolise function rules. They found that students conceived function rules as representing both relationships of correspondence and relationships of coordinated change.

Generally, a function $y = f(x)$ can be related to $y = f(x) = -af(-b(x + h)) \pm k$ using transformations: translation, reflection and stretch. The variables a, b, h and k represent distinct transformations said with a correct magnitude and direction and most crucially the order in which these transformations occur. Learners struggle with algebraic explanations that go with direction or scalar especially the horizontal movement. When teachers become researchers in their own teaching, learners' responses form critical components of classroom proceedings.

Table 1 Illustration of the Signs and Variables guiding the Transformations.

$y = f(x) = -a f(-b(x + h)) \pm k$
1. Reflection in x -axis 2. Vertical stretch if $a > 1$ Vertical compression if $0 < a < 1$ 3. Reflection in y -axis 4. Horizontal stretch if $0 < b < 1$ of factor $\frac{1}{b}$ Horizontal compression if $b > 1$ of factor $\frac{1}{b}$ 5. Horizontal shift(translation) ✓ if $+h$ then h units to the left, ✓ if $-h$ then h units to the right 6. Vertical shift of $\pm k$ units

Learners need to be guided properly into algebraic and geometric meaning connecting object to image function. Knowing or identifying the transformed form of a graph helps students to understand the relations among various representations of functions such as algebraic and graphic forms Yiğit Koyunkaya & Boz-Yaman (2023).

This embeds on the Realistic Mathematics Education Model (RME). According to Arsaythamby & Zubainur (2014) central to the philosophy of Realistic Mathematics Education is that students develop their mathematical understanding by working from contexts that make sense to them. This relates to the responses learners give as they directly depict or connect to their local sense and attested by Zakaria & Syamann (2017) that learning and mastery of concepts in mathematics is inclined and grounded in realistic or context-based settings. Elements of what is real and make sense to the learner relates to the acquisition of both vertical and horizontal mathematisation of the learnt concepts. In the context of transformational functions, ideally visual skills are specifically required in connecting geometrical position of the image point by algebraically making the correct manipulation.

Methodology

The study was intended to investigate learners' challenges in conceptual understanding of transformations of functions at AS level. The investigation focused on learners performing transformations and describing the transformations of functions. AS level schools in Lesotho remain few and one institution was used on the notion that it has about normal academic performance and a larger candidature thereby giving a good population so that sampling processes could be obtained using probability options. Qualitative case study was employed as

guided by the purpose of the study. According to Yıldırım and Şimşek (2011), the case study is used to obtain detailed information about the subject under study and to deeply understand the case from various perspectives.

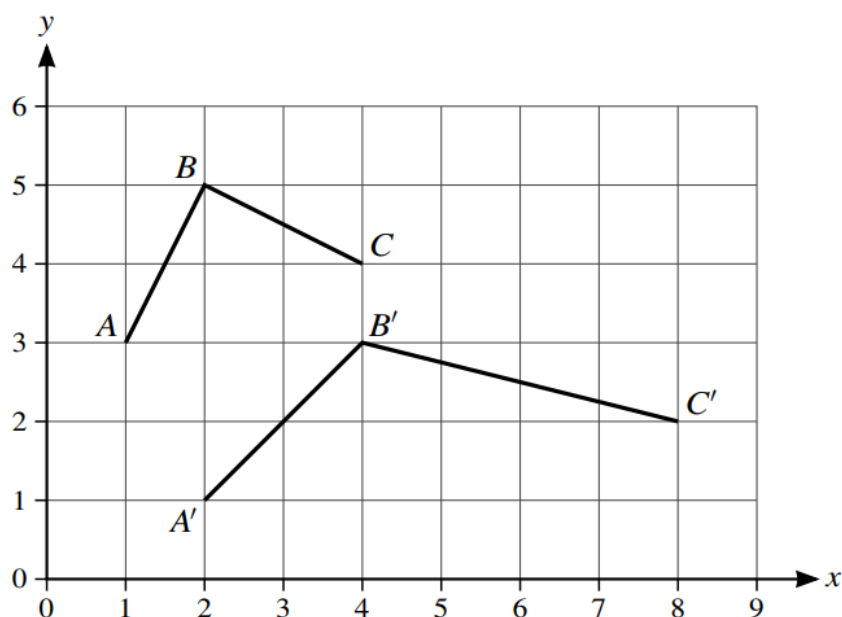
Selection of participants hence population and sampling, is critical for any study as it gives basis of data to use and analysis. Frankel and Wallen (2009) narrated that, a sample in a research study is the group on which information is obtained. A population of 38 learners enabled purposive random selection of 15 learners who participated in the study. The aim of purposeful sampling is to select information-rich cases whose study will illuminate the questions under study (Ratcliff, 2016). Learners were given a series of test items on transformations of functions along the length of the academic year. Ultimately in preparation for the final examination two questions were given, having content on the description and the performance on transformation of functions. The scripts were marked, categorised according to the errors made and then the emerging themes formed the fundamental points of discussions which led to the implications and analysis. Analysis of data entails breaking down the information gathered into elements in order to obtain responses to the research questions (Sauro, 2015). The themes were coded and may be seen in the discussions of the learners' solutions strategies included: a full description made (f_d), errors made (e_m), no attempt (n_a), correct answer (c_a), correct x value (c_x), correct y value (c_y), algebraic method used (a_u) and graphical method used (g_u). On the condition that both answers were wrong then the code is b_w . This was made to cater for learners' various solution strategies and way of thinking while dealing with transformations of functions. Learners' solutions strategies are important not only for scores but more in informing about the thinking processes, knowledge gained and the algebraic and geometrical skills. In addition, critically informing the study as guided by the research questions, algebraic and geometrical skills were integral.

The analysis of learners' solution strategies was followed by teacher less structured interview questions which purposively focused on emerging themes from learners' responses. This was deemed useful to obtain experiences from classroom experiences and crucial verbal evidence. Using less-structured interviews is an appropriate way of exploring interviewee's feelings (Plowright, 2011). As defined by DeMarrais (2004), a research interview is a process in which a researcher and participant engage in a conversation focused on questions related to a research study. The interview data were organised to get an overview of what they revealed, and test responses were grouped into manageable themes (Ratcliff, 2012). For discussion purpose, S_1 represents script 1.

Test items

1. The graph of $y = f(x)$ is transformed to graph of $y = f(2x) - 3$.
 - (a) Describe fully the two transformations that have been combined to give the resulting transformation. [3]
 - (b) The point $P(5,6)$ lies on the transformed curve $y = f(2x) - 3$.
State the coordinates of the corresponding point on the original curve $y = f(x)$. [2]
- Source 9709/12/O/N/21

2. The diagram shows the graph of $y = f(x)$, which consists of the two straight lines AB and BC . The lines $A'B'$ and $B'C'$ form the graph of $y = g(x)$, which is the result of applying a sequence of two transformations, in either order, to $y = f(x)$.



State fully the two transformations.

[4]

9709/13/M/J/23

Expected solutions

These are given to portray the level of expectation in terms of the precise wording and terminology. Under transformations, the order of events is also crucial as it will be geometrically wrong to mention translation before stretches because under combined transformations the image would land in two different positions.

Table 1: Expected Solutions

Question	Expected response
1. (a)	Horizontal or in x -axis stretch, factor $\frac{1}{2}$ Vertical shift, 3 units downwards or Translation vector $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$
(b)	(10,9)
2.	Stretch or horizontal parallel to x -axis, factor 2 Translation, vector $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

Results and discussions

As guided by the purpose of the study, the findings were analysed using qualitative data analysis methods. In which case the themes were used to make deductions.

Table 2: results on the tasks

Question	Scripts	Themes						
1. (a)		f_d	e_m	a_e	n_a			
Total	15	9	5	1	0			
(b)		c_a	x_c	y_c	b_w	a_u	g_u	n_a
Total	15	3	1	6	5	7	6	0
2.		f_d	e_m	a_e	n_a			
Total	15	2	8	3	2			

The first question was well performed by majority of the learners with few errors made on the algebraic representation. Thus, most learners were able to algebraically observe the transformations involved. However, 1(b) needed a combination of algebraic and geometrical skills in terms of locating the position of the image. This invited numerous errors from the learners hence only few were able to obtain at least a mark. This adds that, learners failed to connect the form of functions and the performance in terms of moving the given point. This is despite some trying to do it using the grid or algebraically. The expectation was that the concept of Object + Vector = Image, symbolically $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ be utilised followed by the interpretation of the stretch factor and the vertical shift. Lastly on question 2, the expectation was a full description be made regardless of the order as that was not mentioned as a precision of the response.

Sample responses

1(a)

- (a) Describe fully the two single transformations that have been combined to give the resulting transformation. [3]

P(2) has been stretched horizontally by 2 units
across the y-axis and shifted 2 units to
the right.

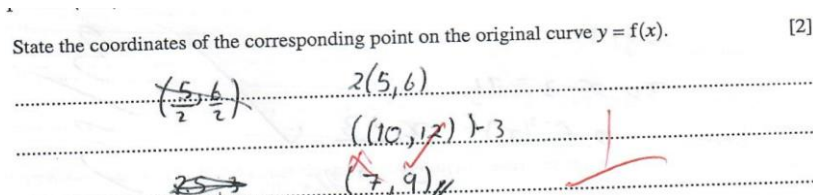
This learner could only pick that there is a horizontal stretch but was unable to use proper language under transformations and failed to connect the given numbers accordingly. In response to this solution strategy the teacher said, “on S_3 there is evidence of mix of concepts as the learner used the 2 not a scale factor but rather as a vertical shift”. The teacher went on to suggest that another confusing aspect is the wrong direction of the translations.

- (a) Describe fully the two single transformations that have been combined to give the resulting transformation. [3]

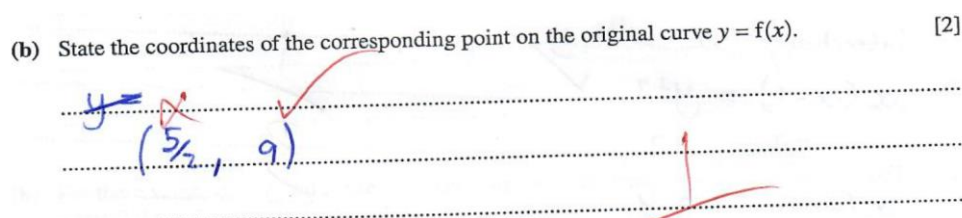
stretch in the x-axis, stretch factor $\frac{1}{2}$
translation in the y-axis $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

On S_7 it is obvious that stretch was conceptually understood, and the proper language was used. It was quite surprising however that the translation vector was missed. This is rather a basic aspect of transformation. The teacher could only say “these are not concept errors but rather concentration problems”. There is evidence of direction that the vertical shift poses another challenge which has cost learners to lose some crucial marks. Another important skill in mathematics is the ability to check own answer in a way to ensure that that all steps taken really make true connections. It is quite imperative that learners are given ample opportunity to learn from their own mistakes without any punishment as this could improve collaborative learning skills.

1(b)



There were mixed feelings by the teacher on commenting to S_{11} solution strategy but outlined that despite mistakes made there are lessons on this response. The fact that the object point $(5, 6)$ was multiplied by 2 was brilliant showing understanding of stretch however this learner missed that stretch was followed by vertical shift of 3 units which means only y will be affected as seen from $(10, 12)$. The teacher further said there are few things to correct in this case, but it is very promising. In such cases, it barely means that sequence of the transformations was not conceptually understood and thus there is a need to do repeated tasks testing the same combined transformation. It is quite essential to bear in mind the sequence during teaching and learning especially on combined transformations. This will be cohesive teaching in bonding the prior knowledge with new conceptual knowledge. Focusing on several types of sequential tasks for each function transformation as well as spending more time in the designed cycle could foster change of students' mental structures at the process and object levels (Maharaj, 2013). Mathematical tasks should be appropriately chosen to help learners build on ideas and challenge their conceptual understanding as argued by Carpenter et al (2003) that although the selection of tasks can provide a context for engaging students in examining their conceptions of the meaning of whatever object or transformation, the nature of the discussion of the mathematical ideas is critical. Teaching of combined transformations of functions requires this procedure and be tested over time to elucidate the sequence of the transformations.

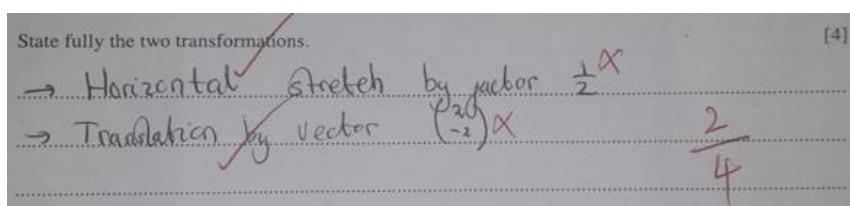


The response of S_{14} is the direct opposite to that of S_{11} and the teacher was happy that learners are picking at least one mark by obtaining one correct coordinate. Both learners contradict Parent, (2015) who noted that learners had a conceptual obstacle with the y -intercept of the

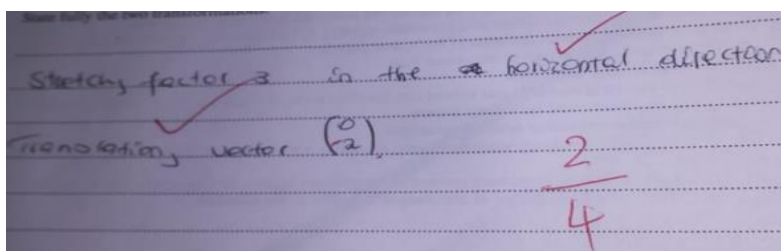
function and the y -coordinate of the vertex. The learner was able to obtain the correct y -coordinate but did the opposite when using the stretch factor 2. Looking at the errors made by both, the teacher suggested that it could be beneficial to assist S_{11} and S_{14} simultaneously as that could develop a deeper understanding of the concept due to their similar concept errors.

Question 2

This task was poorly performed by majority of the learners who made mistakes on the stretch factor or the translation vector. The only difference is that there is a diagram to use as opposed to algebraic functions only. Geometrical skills are needed to locate both Object and image properly via use of various approaches.



Key issues were picked by this learner as seen on S_4 wherein stretch and translation were given only to miss on the stretch factor and the translation vector. This shows that the diagram was not used optimally and with full understanding. The teacher admitted that most learners fail to use the already drawn diagrams and some answer without testing it on other points which moved.



Similarly, the same errors were observed on S_{13} where the types of transformations were clearly stated but failed to connect the other finer details of the transformation aspects involved. The teacher uttered that learners continue to stay behind on geometrical skills and fail to make mathematical connections on the object and image using all points. This is key under transformations especially at this level of learning. As stated by Nt'sasa (2021, p. 444), “when learners are not able to reach analysis and create levels, it mostly implies that the teaching was not compatible, and learners failed to make mathematical connections”.

Discussion and conclusion

Teachers need to be researchers in their own teaching because the way the learners respond directly correlates with individual understanding of concepts. Such responses should be used to inform better and more compatible teaching and learning processes. Learners had a major challenge on transformations where the diagram is involved than on the algebraic functions. They struggled to make mathematical connections using both geometry and algebra. This was related to the procedural understanding of the key concepts. Different forms of functions need to be used during teaching and learning processes to lay a solid foundation of the crucial

aspects. It could be predicted that given quadratic functions, this problem would be more observed due to various forms such that functions could be phrased in the following forms: $y = ax^2 + bx + c$, $y = a(x - r_1)(x - r_2)$ and $y = a(x - h)^2 + k$. Relating $y = f(x)$ to all these could be quite tedious hence will form part of the future study on this aspect. It is therefore recommended that both teaching and learning should be informed by mathematical tasks that connect with reality to aid conceptual understanding. It is further recommended that learners with similar errors be assisted together to enhance a deeper understanding. Sequential teaching is best positioned to enhance conceptual understanding in challenging aspects of mathematics.

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