Exploiting an Algebraic Identity to Derive the Quadratic Formula

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INTRODUCTION

In this article we explore an alternative derivation of the quadratic formula, i.e. the well-known formula for the solutions of the general quadratic equation $ax^2 + bx + c = 0$. The motivation for the article stems from the observation that the standard approach to deriving the formula, i.e. the method of 'completing the square', often tends to be a mechanical procedure that lacks conceptual grounding (Man, 2009; Samson, 2013). With this in mind, alternative proofs are worth exploring not only to enhance students' conceptual understanding, but also to gain fresh perspectives and broader insights into existing methods and approaches (Clarke, 1998).

THE PROOF

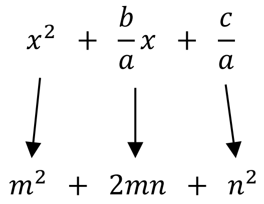
We seek solutions to the standard quadratic equation $ax^2 + bx + c = 0$, or equivalently:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{1}$$

where $a \neq 0$ and a, b, c are real. Rather than using the standard method of 'completing the square', we are going to make use of the perfect square binomial identity:

$$(m + n)^2 \equiv m^2 + 2mn + n^2 \tag{2}$$

Our strategy is to establish a connection between the three terms on the right-hand side of (2) with the three terms on the left-hand side of (1).



- Equating the first term of each expression we have $m^2 = x^2$. By taking the positive square root, and without loss of generality, we thus have $m = x$.
- Equating the second term of each expression we have $2mn = \frac{b}{a}x$, and since we already have $m = x$, simplifying and rearranging this gives $n = \frac{b}{2a}$.
- Equating the third term of each expression we have $n^2 = \frac{c}{a}$. However, since we have already established that $n = \frac{b}{2a}$ this gives $n^2 = \frac{b^2}{4a^2}$.

We are now able to express the perfect square binomial identity given in (2) in terms of a, b and x:

$$(m+n)^2 \equiv m^2 + 2mn + n^2$$

$$\Downarrow$$

$$\left(x+\frac{b}{2a}\right)^2 \equiv x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \tag{3}$$

We now add the constant term $\frac{c}{a}$ to both sides of (3):

$$\left(x+\frac{b}{2a}\right)^2 + \frac{c}{a} \equiv x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} \tag{4}$$

With a bit of rearranging we establish an equivalent algebraic expression for the quadratic expression given in (1):

$$x^2 + \frac{b}{a}x + \frac{c}{a} \equiv \left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} \tag{5}$$

Using the algebraic equivalence shown in (5) we rewrite the quadratic equation shown in (1) as:

$$\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0 \tag{6}$$

We then rearrange (6) and solve for x as follows:

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\Rightarrow \quad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2-4ac}}{2a} \tag{7}$$

Consequently (7) gives our standard quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \tag{8}$$

CONCLUDING COMMENTS

By using the perfect square binomial identity, the term $\frac{b^2}{4a^2}$ arises naturally. It is hoped that exploring this alternative derivation of the quadratic equation provides further rationale for the method of 'completing the square', and will add to students' conceptual understanding of the algebraic process involved in establishing the quadratic formula.

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